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ON SOME GRONWALL-BIHARI-WENDORFF-TYPE INEQUALITIES

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Abstract. This paper presents certain considerations on some lemmas of Gronwall-Bihari-Wendorff type, which follow from abstract Gronwall lemma for Picard operators.
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1. Introduction

This paper presents certain considerations on some lemmas of Gronwall-Bihari-Wendorff type, which follow from abstract Gronwall lemma for Picard operators ([9], [10]). By this method certain generalizations for hyperbolic differential inequalities of Gronwall-Wendorff's classical inequalities are presented; these inequalities involve the Riemann function for a linear hyperbolic operator.

2. Operatorial inequalities

In what follows we present some operatorial inequalities which are deduced from abstract Gronwall lemma ([9], [10], [11]).

Definition 1. ([9], [10]) Let (X, d) be a metric space. An operator $f : X \to X$ is called a Picard operator if there exists $x^* \in X$ such that

(i) $F_f = \{x^*\}$

(ii) $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x^*, for all $x_0 \in X$.

Definition 2. ([9], [10]) Let (X, d) be a metric space. An operator $f : X \to X$ is said to be a weakly Picard operator if the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of f.

Lemma 1. (Abstract Gronwall lemma; [10], [11]) *Let (X, d) be an ordered metric space and $A : X \to X$ an operator.*

We suppose that:

(i) A is a Picard operator

(ii) A is monotone increasing.

If x^*_A is the fixed point of the operator A, then

(a) $x \le A(x) \Rightarrow x \le x^*_A$

(b) $x \ge A(x) \Rightarrow x \ge x^*_A$.

The following lemmas follow from Lemma 1.

Lemma 2. (Stetsenko, Shaaban [13]) *Let E be a semiordered Banach space. If for an element $u(v)$ we have*

$$u \leq Au + f \quad (v \geq Av + f)$$

where f is a fixed element and $A : E \to E$ an increasing operator.

(U_1): If the equation $y = Ay + f$ has the unique solution y^*, which is the limit of the sequence $(y_n)_{n\in\mathbb{N}}$ defined by $y_{n+1} = Ay_n + f$, then

$$u \leq y^* \quad (v \geq y^*).$$

Proof. Consider the operator $B : X \to X$, $x \to Ax + f$, Because the condition (U_1) is fulfilled, the operator B is Picard and we have

$$u \leq B(u).$$

Then Lemma 2 follows from the abstract Gronwall lemma.

Lemma 3. (Zeidler [11], [14]) *Let $(X, \|\cdot\|, \leq)$ be an ordered Banach space and $A : X \to X$ be a continuous, linear and positive operator, with spectral radius $r(A) < 1$. Let $x, y, g \in X$. Then*

$$x \leq A(x) + g$$

and

$$y = A(y) + g$$

always implies

$$x \leq y.$$

Proof. Since $r(A) < 1$, the operator

$$B : X \to X, \quad x \to A(x) + g$$

is a Picard operator. As A is linear and positive, A is increasing, and Lemma 3 follows from the abstract Gronwall lemma (Lemma 1).

Lemma 4. (Zima [14]) *Let X be a semiordered Banach space. Let $A : X \to X$, be a linearly bounded, subadditive and increasing operator such that*

$$\sum_{k=0}^{\infty} \|A^k\| < \infty.$$

Let $g, x \in X$ and $x < g + Ax$.

Then

$$x < \sum_{k=0}^{\infty} A^k g.$$

Proof. Consider the operator $B : X \to X$, $x \to Ax + g$. Because A is linearly bounded, subadditive, increasing operator and $\sum_{k=0}^{\infty} \|A^k\| < \infty$, the operator B is

Picard. If x_B^* is the fixed point of B and

$$S_n x = \sum_{k=0}^{n-1} A^k g + A^n x,$$

then

$$\lim_{n\to\infty} S_n x = \sum_{k=0}^{\infty} A^k g = x_B^*$$

and $x < x_B^*$. (Here $\lim\limits_{n\to\infty} \|A^n\| = 0$, and $x < S_n x$).

Lemma 5. (Martynyuk, Lakshmikantham, Leela [3]) *Let X be a semiordered, complet metric space. If $x_n \in X$, $x_n \le x_{n+1}$ for all $n \ge 1$ and exists $\lim\limits_{n\to\infty} x_n = x_0$, then $x_n \le x_0$. Let $T : X \to X$ be an increasing operator and for a $m \in \mathbb{N}$, T^m is a contraction.*

If x_0 is unique fixed point of T, then

$$x \le Tx \Rightarrow x \le x_0.$$

Proof. Since T^m is a contraction and T is increasing operator and has a unique fixed point x_0, then T is a Picard operator and Lemma 5 follows from Lemma 1.

3. Applications

The following inequalities follow from Lemma 1 (Abstract Gronwall lemma).

3.1. Hyperbolic differential inequality ([4]) We consider the following hyperbolic inequality

$$\frac{\partial^2 u}{\partial x \partial y} \le f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right), \quad (x,y) \in \overline{D}, \tag{1}$$

and the Darboux problem

$$\frac{\partial^2 u}{\partial x \partial y} = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right), \quad (x,y) \in \overline{D} \tag{2}$$

$$\begin{cases} u(x,0) = \varphi(x), & x \in [0,a] \\ u(0,y) = \psi(y), & y \in [0,b],\ \varphi(0) = \psi(0) \end{cases} \tag{3}$$

where $\overline{D} = [0,a] \times [0,b]$, $f \in C(\overline{D} \times \mathbb{R}^3)$, $\varphi \in C^1[0,a]$, $\psi \in C^1[0,b]$, $u \in C^1(\overline{D})$ and $\dfrac{\partial^2 u}{\partial x \partial y} \in C(\overline{D})$.

We have

Theorem 1. *If*

(i) $f \in C(\overline{D} \times \mathbb{R}^3)$,

(ii) $|f(x,y,u_1,u_2,u_3) - f(x,y,v_1,v_2,v_3)| \le L_f \max(|u_i - v_i|),\ i = 1,2,3,$

(iii) $\varphi \in C^1[0,a],\ \psi \in C^1[0,b]$,

(iv) $f(x,y,\dots) : \mathbb{R}^3 \to \mathbb{R}$ *is monotone increasing,*

then

(a) the Darboux problem (2)+(3) has a unique solution u^*

(b) if u is a solution of (1)+(3) then $u \leq u^*$.

Proof. We put the problem (2)+(3) as a fixed point problem. If u is a solution of the problem (2)+(3), then $\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$ is a solution of the following system

$$(4) \quad \begin{cases} u(x,y) = \varphi(x) + \psi(y) - \varphi(0) + \int_0^x \int_0^y f(s,t,u(s,t),v(s,t),w(s,t))dsdt \\ v(x,y) = \varphi'(x) + \int_0^y f(x,t,u)x,t),v(x,t),w(x,t))dt \\ w(x,y) = \psi'(y) + \int_0^x f(s,y,u(s,y),v(s,y),w(s,y))ds \end{cases}$$

or in general form

$$\begin{aligned} u(x,y) &= A_1(u,v,w)(x,y) \\ v(x,y) &= A_2(u,v,w)(x,y) \\ w(x,y) &= A_3(u,v,w)(x,y) \end{aligned}$$

$u, v, w \in C(\overline{D})$.

If $(u,v,w) \in C(\overline{D})^3$ is a solution of (4) then $u \in C^1(\overline{D})$ and $v = \frac{\partial u}{\partial x}$, $w = \frac{\partial u}{\partial y}$ i.e., u is a solution of (2)+(3).

Let $X := C(\overline{D}) \times C(\overline{D}) \times C(\overline{D})$ and

$$\|(u,v,w)\| := \max(\max_{\overline{D}} |u(x,y)|e^{-\tau(x+y)}, \max_{\overline{D}} |v(x,y)|e^{-\tau(x+y)}, \max_{\overline{D}} |w(x,y)|e^{-\tau(x+y)})$$

$(C(\overline{D}, +, \mathbb{R}, \| \cdot \|_B)$ is a Banach space.

Let $A : X \to X$, $(u,v,w) \to (A_1(u,v,w), A_2(u,v,w), A_3(u,v,w))$, we have

$$\|A(u_1,v_1,w_1) - A(u_2,v_2,w_2)\|_B \leq \frac{L_f}{\tau} \|(u_1,v_1,w_1) - (u_2,v_2,w_2)\|_B.$$

Thus if $\tau > 0$ is such that $L_f/\tau < 1$, then the operator A is a contraction so A is a Picard operator. From (iv) we have that A is monotone increasing. let u be a solution of (1).

Then

$$\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \leq A\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right).$$

From Lemma 1 we have that

$$\begin{aligned} u &\leq u^* \\ \frac{\partial u}{\partial x} &\leq \frac{\partial u^*}{\partial x} \\ \frac{\partial u}{\partial y} &\leq \frac{\partial u^*}{\partial y}. \end{aligned}$$

Example 1. (see [4], [8]) Let $a, b > 0$ and $\overline{D} = [0,a] \times [0,b]$. Let $p, q, r, g \in C(\overline{D})$. We consider the following hyperbolic inequality

$$(1') \quad \frac{\partial^2 u}{\partial x \partial y} + p(x,y)\frac{\partial u}{\partial x} + q(x,y)\frac{\partial u}{\partial y} + r(x,y)u \leq g(x,y), \quad (x,y) \in \overline{D}$$

and the Darboux problem

$$(2') \qquad \frac{\partial^2 u}{\partial x \partial y} + p(x,y)\frac{\partial u}{\partial x} + q(x,y)\frac{\partial u}{\partial y} + r(x,y)u = g(x,y), \quad (x,y) \in \overline{D}$$

$$(3') \qquad \begin{cases} u(x,0) = \varphi(x), & x \in [0,a] \\ u(0,y) = \psi(y), & y \in [0,b],\ \varphi(0) = \psi(0), \end{cases}$$

where $\varphi \in C^1[0,a]$ and $\psi \in C^1[0,b]$.

We suppose that $p \leq 0,\ q \leq 0$ and $r \leq 0$.

Then the Darboux problem $(2') + (3')$ has a unique solution u^*.

If u is a solution of $(1') + (3')$ then $u \leq u^*$. In this case

$$u^*(x,y) = v(0,0;x,y)\varphi(0) + \int_0^x v(s,0;x,y)(\varphi'(s) + q(s,0)\varphi(s))ds +$$

$$+ \int_0^y v(0,t;x,y)(\psi'(y) + p(0,t)\psi(t))dt + \iint_{\overline{D}} v(s,t;x,y)g(s,t)dsdt$$

where v is the Riemann function.

Example 2. ([4]) We consider the inequalities

(i) $\dfrac{\partial^2 u}{\partial x \partial y} + p(y)\dfrac{\partial u}{\partial x} \leq g(x,y)$

and

(ii) $\dfrac{\partial^2 u}{\partial x \partial y} + q(x)\dfrac{\partial u}{\partial y} \leq g(x,y)$.

Then the Riemann functions are

$$v = \exp\left(\int_0^y p(t)dt\right) \text{ and respectively } v = \exp\left(\int_0^x q(s)ds\right).$$

3.2. Wendorff-type inequality. The following inequality follows from Lemma 2 ([13]).

Theorem 2. *Let $u, v \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ and $c \in \mathbb{R}_+^*$.*

If $u(x,y)$ verifies the inequality

$$u(x,y) \leq c + \int_{x_0}^x \int_{y_0}^y v(s,t)u(s,t)dsdt, \quad x \geq x_0,\ y \geq y_0$$

and $v(x,y)$ is monotone increasing, and if u^* is the unique solution of equation

$$\frac{\partial u}{\partial x} = \left(\int_{y_0}^y v(x,t)dt\right) u(x,y)$$

then $u(x,y) \leq u^*(x,y)$, where

$$u^*(x,y) = c \cdot \exp\left(\int_{x_0}^x \int_{y_0}^y v(s,t)dsdt\right).$$

Then $u(t) \leq u^*(t)$, where $u^*(t)$ is the solution of corresponding Bernoulli's equation.

Proof. In this case the operator A is defined by
$$A = \int_{x_0}^{x} \int_{y_0}^{y} v(s,t)u(s,t)dsdt.$$

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