ON SOME GRONWALL-BIHARI-WENDORFF-TYPE INEQUALITIES

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Abstract. This paper presents certain considerations on some lemmas of Gronwall-Bihari-Wendorff type, which follow from abstract Gronwall lemma for Picard operators.

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1. Introduction

This paper presents certain considerations on some lemmas of Gronwall-Bihari-Wendorff type, which follow from abstract Gronwall lemma for Picard operators ([9], [10]). By this method certain generalizations for hyperbolic differential inequalities of Gronwall-Wendorff’s classical inequalities are presented; these inequalities involve the Riemann function for a linear hyperbolic operator.

2. Operatorial inequalities

In what follows we present some operatorial inequalities which are deduced from abstract Gronwall lemma ([9], [10], [11]).

Definition 1. ([9], [10]) Let $(X, d)$ be a metric space. An operator $f : X \to X$ is called a Picard operator if there exists $x^* \in X$ such that

(i) $F_f = \{x^*\}$
(ii) $(f^n(x_0))_{n \in \mathbb{N}}$ converges to $x^*$, for all $x_0 \in X$.

Definition 2. ([9], [10]) Let $(X, d)$ be a metric space. An operator $f : X \to X$ is said to be a weakly Picard operator if the sequence $(f^n(x_0))_{n \in \mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which may depend on $x_0$) is a fixed point of $f$.

Lemma 1. (Abstract Gronwall lemma; [10], [11]) Let $(X, d)$ be an ordered metric space and $A : X \to X$ an operator.

We suppose that:

(i) $A$ is a Picard operator
(ii) $A$ is monotone increasing.

If $x_\lambda$ is the fixed point of the operator $A$, then
(a) $x \leq A(x) \Rightarrow x \leq x_\lambda$
(b) $x \geq A(x) \Rightarrow x \geq x_\lambda$. 

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The following lemmas follow from Lemma 1.

**Lemma 2.** (Stetsenko, Shaaban [13]) Let $E$ be a semiordered Banach space. If for an element $u(v)$ we have

$$u \leq Au + f \quad (v \geq Av + f)$$

where $f$ is a fixed element and $A : E \to E$ an increasing operator.

$(U_1)$: If the equation $y = Ay + f$ has the unique solution $y^*$, which is the limit of the sequence $(y_n)_{n \in \mathbb{N}}$ defined by $y_{n+1} = Ay_n + f$, then

$$u \leq y^* \quad (v \geq y^*).$$

**Proof.** Consider the operator $B : X \to X$, $x \to Ax + f$, Because the condition $(U_1)$ is fulfilled, the operator $B$ is Picard and we have

$$u \leq B(u).$$

Then Lemma 2 follows from the abstract Gronwall lemma.

**Lemma 3.** (Zeidler [11], [14]) Let $(X, \|\cdot\|, \leq)$ be an ordered Banach space and $A : X \to X$ be a continuous, linear and positive operator, with spectral radius $r(A) < 1$. Let $x, y, g \in X$. Then

$$x \leq A(x) + g$$

and

$$y = A(y) + g$$

always implies

$$x \leq y.$$

**Proof.** Since $r(A) < 1$, the operator

$$B : X \to X, \quad x \to Ax + f$$

is a Picard operator. As $A$ is linear and positive, $A$ is increasing, and Lemma 3 follows from the abstract Gronwall lemma (Lemma 1).

**Lemma 4.** (Zima [14]) Let $X$ be a semiordered Banach space. Let $A : X \to X$, be a linearly bounded, subadditive and increasing operator such that

$$\sum_{k=0}^{\infty} \|A^k\| < \infty.$$

Let $g, x \in X$ and $x < g + Ax$.

Then

$$x < \sum_{k=0}^{\infty} A^k g.$$

**Proof.** Consider the operator $B : X \to X, x \to Ax + g$. Because $A$ is linearly bounded, subadditive, increasing operator and $\sum_{k=0}^{\infty} \|A^k\| < \infty$, the operator $B$ is
Picard. If $x^*_B$ is the fixed point of $B$ and

$$S_n x = \sum_{k=0}^{n-1} A^k g + A^n x,$$

then

$$\lim_{n \to \infty} S_n x = \sum_{k=0}^{\infty} A^k g = x^*_B$$

and $x < x^*_B$. (Here $\lim_{n \to \infty} \|A^n\| = 0$, and $x < S_n x$).

**Lemma 5.** (Martynyuk, Lakshmikantham, Leela [3]) Let $X$ be a semiordered, complete metric space. If $x_n \in X$, $x_n \leq x_{n+1}$ for all $n \geq 1$ and exists $\lim_{n \to \infty} x_n = x_0$, then $x_n \leq x_0$. Let $T : X \to X$ be an increasing operator and for a $m \in \mathbb{N}$, $T^m$ is a contraction.

If $x_0$ is unique fixed point of $T$, then $x \leq T x \Rightarrow x \leq x_0$.

**Proof.** Since $T^m$ is a contraction and $T$ is increasing operator and has a unique fixed point $x_0$, then $T$ is a Picard operator and Lemma 5 follows from Lemma 1.

3. Applications

The following inequalities follow from Lemma 1 (Abstract Gronwall lemma).

**3.1. Hyperbolic differential inequality ([4])** We consider the following hyperbolic inequality

$$\frac{\partial^2 u}{\partial x \partial y} \leq f \left( x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right), \quad (x, y) \in \overline{D},$$

and the Darboux problem

$$\frac{\partial^2 u}{\partial x \partial y} = f \left( x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right), \quad (x, y) \in \overline{D}$$

(3)

\[
\begin{cases}
  u(x, 0) = \varphi(x), & x \in [0, a] \\
  u(0, y) = \psi(y), & y \in [0, b], \quad \varphi(0) = \psi(0)
\end{cases}
\]

where $\overline{D} = [0, a] \times [0, b]$, $f \in C(\overline{D} \times \mathbb{R}^3)$, $\varphi \in C^1[0, a]$, $\psi \in C^1[0, b]$, $u \in C^1(\overline{D})$ and $\frac{\partial^2 u}{\partial x \partial y} \in C(\overline{D})$.

We have

**Theorem 1.** If

(i) $f \in C(\overline{D} \times \mathbb{R}^3)$,

(ii) $|f(x, y, u_1, u_2, u_3) - f(x, y, v_1, v_2, v_3)| \leq L f \max(|u_i - v_i|)$, $i = 1, 2, 3$,

(iii) $\varphi \in C^1[0, a]$, $\psi \in C^1[0, b]$,

(iv) $f(x, y, \ldots) : \mathbb{R}^3 \to \mathbb{R}$ is monotone increasing,

then

(a) the Darboux problem (2)+(3) has a unique solution $u^*$
(b) if $u$ is a solution of (1)+(3) then $u \leq u^\ast$.

**Proof.** We put the problem (2)+(3) as a fixed point problem. If $u$ is a solution of the problem (2)+(3), then $\left( u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right)$ is a solution of the following system

$$
\begin{align*}
\begin{cases}
\forall (x, y) & u(x, y) = \varphi(x) + \psi(y) - \varphi(0) + \int_0^x \int_0^y f(s, t, u(s, t), v(s, t), w(s, t))dsdt \\
\forall (x, y) & v(x, y) = \varphi'(x) + \int_0^x f(x, t, u(x, t), v(x, t), w(x, t))dt \\
\forall (x, y) & w(x, y) = \psi'(y) + \int_0^y f(s, y, u(s, y), v(s, y), w(s, y))ds
\end{cases}
\end{align*}
$$

or in general form

$u(x, y) = A_1(u, v, w)(x, y)$

$v(x, y) = A_2(u, v, w)(x, y)$

$w(x, y) = A_3(u, v, w)(x, y)$

$u, v, w \in C(D)$.

If $(u, v, w) \in C(D)^3$ is a solution of (4) then $u \in C^1(D)$ and $v = \frac{\partial u}{\partial x}, w = \frac{\partial u}{\partial y}$ i.e., $u$ is a solution of (2)+(3).

Let $X := C(D) \times C(D) \times C(D)$ and $\| (u, v, w) \| := \max(\max_{D} |u(x, y)|e^{-\tau(x+y)}, \max_{D} |v(x, y)|e^{-\tau(x+y)}, \max_{D} |w(x, y)|e^{-\tau(x+y)})$ $(C(D), +, \mathbb{R}, \| \cdot \|_B)$ is a Banach space.

Let $A : X \to C(D) \times C(D) \times C(D)$, we have

$$
\| A(u_1, v_1, w_1) - A(u_2, v_2, w_2) \|_B \leq \frac{L_f}{\tau} \| (u_1, v_1, w_1) - (u_2, v_2, w_2) \|_B.
$$

Thus if $\tau > 0$ is such that $L_f/\tau < 1$, then the operator $A$ is a contraction so $A$ is a Picard operator. From (iv) we have that $A$ is monotone increasing. Let $u$ be a solution of (1).

Then

$$
\left( u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \leq A \left( u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right).
$$

From Lemma 1 we have that

$$
\begin{align*}
\forall (x, y) & \frac{\partial u}{\partial x} \leq \frac{\partial u^\ast}{\partial x} \\
\forall (x, y) & \frac{\partial u}{\partial y} \leq \frac{\partial u^\ast}{\partial y}.
\end{align*}
$$

**Example 1.** (see [4], [8]) Let $a, b > 0$ and $D = [0, a] \times [0, b]$. Let $p, q, r, g \in C(D)$. We consider the following hyperbolic inequality

$$
\begin{align*}
\frac{\partial^2 u}{\partial x \partial y} + p(x, y) \frac{\partial u}{\partial x} + q(x, y) \frac{\partial u}{\partial y} + r(x, y)u \leq g(x, y), \quad (x, y) \in D
\end{align*}
$$
and the Darboux problem

\[
(2') \quad \frac{\partial^2 u}{\partial x \partial y} + p(x, y) \frac{\partial u}{\partial x} + q(x, y) \frac{\partial u}{\partial y} + r(x, y)u = g(x, y), \quad (x, y) \in \mathcal{D}
\]

\[
(3') \quad \begin{cases} 
  u(x, 0) = \varphi(x), & x \in [0, a] \\
  u(0, y) = \psi(y), & y \in [0, b], \quad \varphi(0) = \psi(0),
\end{cases}
\]

where \( \varphi \in C^1[0, a] \) and \( \psi \in C^1[0, b] \).

We suppose that \( p \leq 0, q \leq 0 \) and \( r \leq 0 \).

Then the Darboux problem \((2') + (3')\) has a unique solution \( u^* \).

If \( u \) is a solution of \((1') + (3')\) then \( u \leq u^* \). In this case

\[
u^*(x, y) = v(0, 0; x, y)\varphi(0) + \int_0^x v(s, 0; x, y)(\varphi'(s) + q(s, 0)\varphi(s))ds + \int_0^y v(0, t; x, y)(\psi'(y) + p(0, t)\psi(t))dt + \int_\mathcal{D} v(s, t; x, y)g(s, t)dsdt
\]

where \( v \) is the Riemann function.

**Example 2.** ([4]) We consider the inequalities

(i) \( \frac{\partial^2 u}{\partial x \partial y} + p(y) \frac{\partial u}{\partial x} \leq g(x, y) \)

and

(ii) \( \frac{\partial^2 u}{\partial x \partial y} + q(x) \frac{\partial u}{\partial y} \leq g(x, y) \).

Then the Riemann functions are

\[
v = \exp \left( \int_0^y p(t)dt \right) \text{ and respectively } v = \exp \left( \int_0^x q(s)ds \right).
\]

### 3.2. Wendorff-type inequality

The following inequality follows from Lemma 2 ([13]).

**Theorem 2.** Let \( u, v \in C(\mathbb{R}_+^2, \mathbb{R}_+) \) and \( c \in \mathbb{R}_+^* \).

If \( u(x, y) \) verifies the inequality

\[
u(x, y) = c + \int_{x_0}^x \int_{y_0}^y v(s, t)u(s, t)dsdt, \quad x \geq x_0, \; y \geq y_0
\]

and \( v(x, y) \) is monotone increasing, and if \( u^* \) is the unique solution of equation

\[
\frac{\partial u}{\partial x} = \left( \int_{y_0}^y v(x, t)dt \right) u(x, y)
\]

then \( u(x, y) \leq u^*(x, y) \), where

\[
u^*(x, y) = c \cdot \exp \left( \int_{x_0}^x \int_{y_0}^y v(s, t)dsdt \right).
\]

Then \( u(t) \leq u^*(t) \), where \( u^*(t) \) is the solution of corresponding Bernoulli’s equation.
Proof. In this case the operator $A$ is defined by

$$A = \int_{x_0}^{x} \int_{y_0}^{y} v(s, t)u(s, t)dsdt.$$ 

References


