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A COMBINED METHOD FOR DIFFERENTIAL EQUATIONS

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Let be the equation

$$P(x) \equiv f(x) - L(x) = 0, \tag{1}$$

where $L \in Hom(X,Y)$, $f : X \longrightarrow Y$ is continuous and Gâteaux differentiable in a later specified subset of X, the X and Y being some particularized PLC spaces. We write formally $P'(x) = (f'(x) - L) \in Hom(X,Y)$, and $[u,v;P] = ([u,v;f] - L) \in Hom(X,Y)$, where a divided difference $[u,v;f]$ in the knots $u,v \in X$ means a linear and continuous mapping of X into Y with $[u,v;f](u-v) = f(u) - f(v)$.

As approximations of the solution of the equation (1) we use two monotonic sequences. The increasing sequence is given by the formula:

$$(f'(y_n) - L)(x_{n+1}) = (L - f)(x_n);\ (n = 1,2,...), \tag{2}$$

where $y_n = \alpha_n x_{n-1} + (1-\alpha_n)x_n$, with $\alpha_n \in [0,1]$.

The decreasing sequence is obtained by the formula:

$$([x_0,w_n;f] - L)(w_{n+1}) = ([x_0,w_n;f] - f)(x_n),\ (n = 0,1,\ldots). \tag{3}$$

After we state our main theorem, we use it:

- to approximate the solution of Cauchy's problems for first order ODE;
- to solve numerically two-point boundary value problems;
- to solve numerically Dirichlet problems for elliptic equations.

1. The basic theorem

Theorem 1 (Goldner and Trîmbiţaş, 1998). *Let X be a locally full PLC space, Y a regular and locally full PLC space, and $D \subseteq X$ a convex subset. Let us suppose the points $x_0, w_0 \in intD$ with $x_0 \le w_0$, the continuous Gâteaux differentiable mapping with a positive second order divided difference on the (o)-interval $[x_0,w_0]$ $f : intD \longrightarrow Y$, and $L \in Hom(X,Y)$ satisfy the following conditions:*

(i) *there exists the compact and positive mapping L^{-1};*
(ii) $L(x_0) \le f(x_0),\ L(w_0) \ge f(w_0)$*;*
(iii) *there exists a linear and continuous mapping $g \in \mathcal{L}(X,Y)$ such that for all $x \in [x_0,w_0]$ we have $f'(x) \ge g(x)$ and $\Gamma = L - g$ has a positive and continuous inverse;*

(iv) *for all* u, v *in* $[x_0, w_0]$ *there is a mapping* $[u, v; P]^{-1}$*, negative and continuous. Then*

(j) *the equation (1) has a unique solution* $x^* \in [x_0, w_0]$*;*

(jj) *for all* $n \in \mathbb{N}$ *there are the iterates* $(x_n), (w_n)$ *given by (2),(3);*

(jjj) *for all* $n \in \mathbb{N}$ *we have* $x_0 \leq x_1 \leq \ldots \leq x_n \leq x^* \leq w_n \leq \ldots \leq w_1 \leq w_0$*;*

(jv) $\lim_{n\to\infty} x_n = \lim_{n\to\infty} w_n = x^*$.

Proof. See [7]. □

2. Cauchy's problem

We apply the Theorem 1 to the Cauchy's problem

$$\begin{cases} x'(t) = \varphi(t, x(t)), \ t \in]0, 1]; \\ x(0) = 0. \end{cases} \tag{4}$$

Theorem 2. *[Goldner and Trîmbiţaş, 1998]Let* $\varphi : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function, partially derivable and convex with respect to the second variable* x*. If there exists* $x_0, w_0 \in C^1([0, 1])$ *with* $x_0(t) \leq w_0(t)$ *for all* $t \in [0, 1]$*,* $x_0(0) = w_0(0) = 0$*, such that for all* $t \in [0, 1]$ *we have* $x_0^{'}(t) \leq \varphi(t, x_0(t))$*,* $w_0^{'}(t) \geq \varphi(t, w_0(t))$ *then:*

(j) *it exists the increasing sequence* (x_n) *and the decreasing sequence* (w_n) *in* $C^1([0, 1])$ *given by*

$$x_{n+1}(t) = \exp\left(\int_0^t \frac{\partial\varphi\left(s, y_n(s)\right)}{\partial x} ds\right) \cdot \tag{5}$$
$$\cdot \int_0^t \varphi\left(s, x_n(s) - \frac{\partial\varphi\left(s, y_n(s)\right)}{\partial x} x_n(s)\right) \exp\left(-\int_0^s \frac{\partial\varphi\left(z, y_n(z)\right)}{\partial x} dz\right) ds,$$

where $y_n(s) = \alpha_n x_{n-1}(s) + (1 - \alpha_n) x_n(s)$*, for all* $n = 1, 2, \ldots,$ $s \in [0, 1]$*, and* (α_n) *a sequence with* $\alpha_n \in [0, 1]$.

$$w_{n+1}(t) = \exp\left(\int_0^t \left[x_0(s), w_n(s); \varphi\right](x) ds\right) \cdot \tag{6}$$
$$\cdot \int_0^t \varphi\left(s, x_n(s) - \left[x_0(s), w_n(s); \varphi\right](x) \cdot w_n(s)\right)$$
$$\cdot \exp\left(-\int_0^s \left[x_0(z), w_n(z); \varphi\right](x) dz\right) ds$$

for all $n = 1, 2, \ldots,$ *and* $t \in [0, 1]$*, where*

$$\left[u(s), v(s); \varphi\right](x) = \begin{cases} \frac{\varphi(s, u(s)) - \varphi(s, v(s))}{u(s) - v(s)}, & \text{if } s \in \{t \in [0, 1] | u(t) \neq v(t)\} \\ \frac{\partial\varphi(s, u(s))}{\partial x}, & \text{if } s \in \{t \in [0, 1] | u(t) = v(t)\} \end{cases};$$

(jj) *the sequences* $(x_n(t))$ *and* $(w_n(t))$ *are convergent in the topology of the uniform convergence in* $C([0, 1])$ *to a function* $x^* \in C^1([0, 1])$ *and for all* $t \in [0, 1]$ *and* $n = 0, 1, 2, \ldots$ *we have* $x_n(t) \leq x^*(t) \leq w_n(t)$*;*

(jjj) x^* *is the unique solution of the problem (4) with* $x_0(t) \leq x^*(t) \leq w_0(t)$ *for all* $t \in [0, 1]$.

3. The two-point boundary value problem

Let us consider the differential equation

$$x^{(2n)}(t)+\varphi(t,x(t))=0,\ \ t\in]0,1[, \tag{7}$$

with the homogeneous boundary conditions

$$x^{(j)}(0)=x^{(j)}(1)=0;\ \ (j=\overline{0,r-1}). \tag{8}$$

Theorem 3 (Goldner and Trîmbiţaş, 1999)**.** *Let $\varphi:[0,1]\times\mathbb{R}\mapsto\mathbb{R}$ be a continuous function with respect to both variables, convex with respect to x, and having a continuous partial derivative with respect to the second variable. Let us suppose there exist the functions x_0, $w_0\in C^{2r}\,(]0,1[)\cap C^{r-1}([0,1])$ verifying the inequalities $x_0(t)\le w_0(t)$ for all $t\in[0,1]$, $x_0^{(2n)}(t)\ge-\varphi(t,x_0(t)$, $w_0^{(2n)}(t)\le-\varphi(t,w_0(t))$ for each $t\in]0,1[$, and satisfying (8). If the differential operators generated by the differential expressions*

$$\left(L_x(h)\right)(t)=-h^{(2r)}(t)-\frac{\partial\varphi\left(t,x(t)\right)}{\partial x}\cdot h(t),\ \ t\in]0,1[, \tag{9}$$

$$\left(L_{u,v}\left(h\right)\right)(t)=-h^{(2r)}(t)-\left[u(t),v(t);\varphi\right]_x\cdot h(t),\ \ t\in]0,1[, \tag{10}$$

with the boundary conditions (8) for h have a unique and positive Green's function for all u,v and x in the (o)-interval $[x_0,w_0]$, then:

(j) *there exist the increasing sequence (x_n) and the decreasing sequence (w_n) of functions in $C^{2r}\,(]0,1[)\cap C^{r-1}([0,1])$ given by*

$$\left(L_{y_n}(x_{n+1})\right)(t)=\varphi(t,x_n(t))-\frac{\partial\varphi\left(t,y_n(t)\right)}{\partial x}x_n(t);\ n=0,1,\ldots, \tag{11}$$

where $y_0=x_0$, $y_n=\alpha_nx_{n-1}+(1-\alpha_n)x_n$, for $n=1,2,\ldots$, with $\alpha_n\in[0,1]$, and

$$\left(L_{x_0,w_n}\left(w_{n+1}\right)\right)(t)=\varphi(t,w_n(t))-\left[x_0(t),w_n(t);\varphi\right]w_n(t);\ n=0,1,\ldots, \tag{12}$$

for each $t\in]0,1[$, the unknown functions x_{n+1} and w_{n+1} satisfying the boundary conditions (8);

(jj) *the sequences (x_n) and (w_n) converge in the topology of uniform convergence in $C([0,1])$ to the same limit $x^*\in C^{2r}\,(]0,1[)\cap C^{r-1}([0,1])$, and for each $t\in]0,1[$, and $n=0,1,\ldots$, we have $x_n(t)\le x^*(t)\le w_n(t)$;*

(jjj) *the function x^* is the unique solution of (7)–(8) verifying $x_0(t)\le x^*(t)\le w_0(t)$ for each $t\in]0,1[$.*

Proof. See [8] □

4. The Dirichlet problem

$$\sum_{i,j=1}^{m}a_{ij}(x)\frac{\partial^2u(x)}{\partial x^i\partial x^j}+\varphi(x,u(x))=0,\quad x=(x^i)_{i=\overline{1,m}}\in\Omega \tag{13}$$

$$u(x)=0,\quad x\in\partial\Omega, \tag{14}$$

where $\Omega \subset \mathbb{R}^m$ open bounded, $L : C^2(\Omega) \cap C^1(\bar{\Omega}) \longrightarrow L^2(\bar{\Omega})$ given by $L(h) = -\sum_{i,j=1}^{m} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x^i \partial x^j}$ is uniformly elliptic, $\partial\Omega$ continuous and piecewise indefinitely derivable, $a_{ij} : \bar{\Omega} \to \mathbb{R}$, and $\varphi : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ indefinitely derivable.

Theorem 4 (Goldner and Trîmbiţaş, 2001)**.** *Let $\varphi : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ a continuous function with respect to all variables, convex with respect to u, and having continuous partial derivative with respect to u. Let us suppose there exist the functions $u_0, w_0 \in C^2(\Omega) \cap C^1(\bar{\Omega})$ verifying the inequalities $u_0(x) \le w_0(x)$ for all $x \in \Omega$, $(L(u_0))(x) \le \varphi(x, u_0(x))$, $(L(w_0))(x) \ge \varphi(x, w_0(x))$ for each $x \in \Omega$, and satisfying (14). If the differential operator generated by the differential expressions*

$$(L_u(h))(x) = -\sum_{i,j=1}^{m} a_{ij}(x)\frac{\partial^2 h}{\partial x^i \partial x^j} - \frac{\partial\varphi(x, u(x))}{\partial u} h(x), \quad x \in \Omega \tag{15}$$

$$(L_{v,w}(h))(x) = -\sum_{i,j=1}^{m} a_{ij}(x)\frac{\partial^2 h}{\partial x^i \partial x^j} - [v(x), w(x); \varphi]_{(u)}\, h(x), \quad x \in \Omega \tag{16}$$

with the boundary condition (14) for h have a unique and positive Green's function for all u, v, w in (o)-interval $[u_0, w_0]$, then:

(j) *there exist the increasing sequence (u_n) and the decreasing sequence (w_n) of functions in $C^2(\Omega) \cap C^1(\bar{\Omega})$ given by*

$$(L_{y_n}(u_{n+1}))(x) = \varphi(x, u_n(x)) - \frac{\partial\varphi(x, y_n(x))}{\partial u} u_n(x); \quad n = 0, 1, \ldots \tag{17}$$

where $y_0 = u_0$, $y_n = \alpha_n u_{n-1} + (1 - \alpha_n) u_n$, for $n = 1, 2, \ldots$, with $\alpha_n \in [0, 1]$, and

$$(L_{w_0,w_n}(w_{n+1}))(x) = \varphi(x, w_n(x))\, [w_0(x), w_n(x); \varphi]_{(u)}\, w_n(x);\ n = 0, 1, \ldots \tag{18}$$

for each $x \in \Omega$, the unknown functions u_{n+1} and w_{n+1} satisfying the boundary condition (14);

(jj) *the sequences (u_n) and (w_n) converge in the topology of uniform convergence in $C(\bar{\Omega})$ to the same limit $u^* \in C^2(\Omega) \cap C^1(\bar{\Omega})$, and for each $x \in \Omega$, and $n = 0, 1, \ldots$, we have $u_n(x) \le u^*(x) \le w_n(x)$;*

(jjj) *the function u^* is the unique solution of (13)–(14) verifying $u_0(x) \le u^*(x) \le w_0(x)$, for each $x \in \Omega$.*

Proof. See [9]. □

5. Numerical examples

For the two-point boundary value problem we consider the equation

$$x''(t) + x^3(t) + \frac{4 - (t - t^2)^3}{(t + 1)^3} = 0, \quad t \in]0, 1[\tag{19}$$

with the boundary conditions

$$x(0) = x(1) = 0. \tag{20}$$

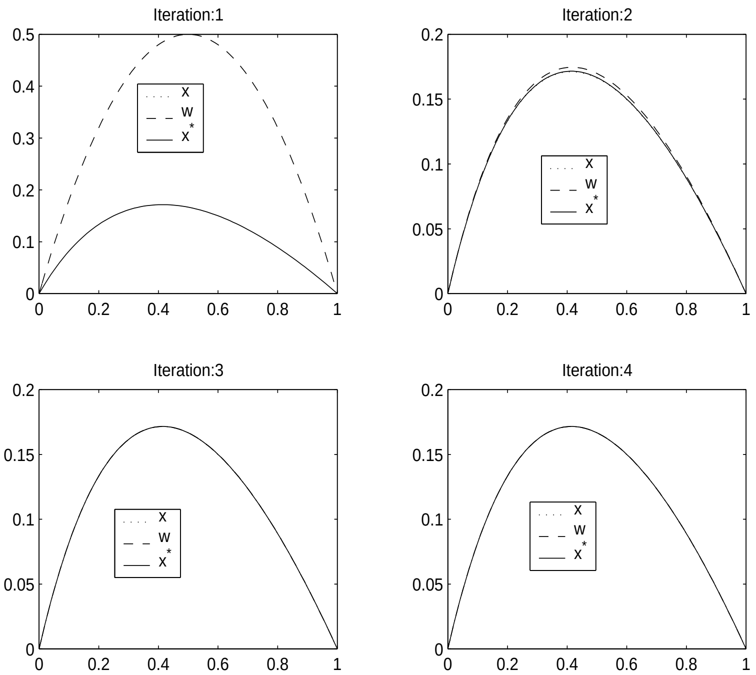


FIGURE 1. Iterations plot

The exact solution is

$$x^*(t) = \frac{t - t^2}{t + 1}.$$

Initial approximations are $x_0 = 0$, $w_0(t) = 2(t - t^2)$; $\alpha_n = \frac{1}{n+1}$; $\varepsilon = 10^{-5}$

We used an uniform grid, where $N = 100$.

For Dirichlet problem

$$U = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u^2 + x^2y^2 - x^2y - xy^2 + xy = 0, \forall (x, y) \in]0, 1[\times]0, 1[\tag{21}$$

$$u(x, 0) = u(x, 1) = u(0, y) = u(1, y) = 0, \quad \forall (x, y) \in [0, 1] \times [0, 1] \tag{22}$$

Initial approximation $u_0(x, y) = 0 \leq w_0(x, y) = x^2y^2 - x^2y - xy^2 + xy$.

For $\varepsilon = 10^{-6}$, 2 iterations are needed in order to achieve the desired tolerance.

The solutions u and w appear in the figure 3.

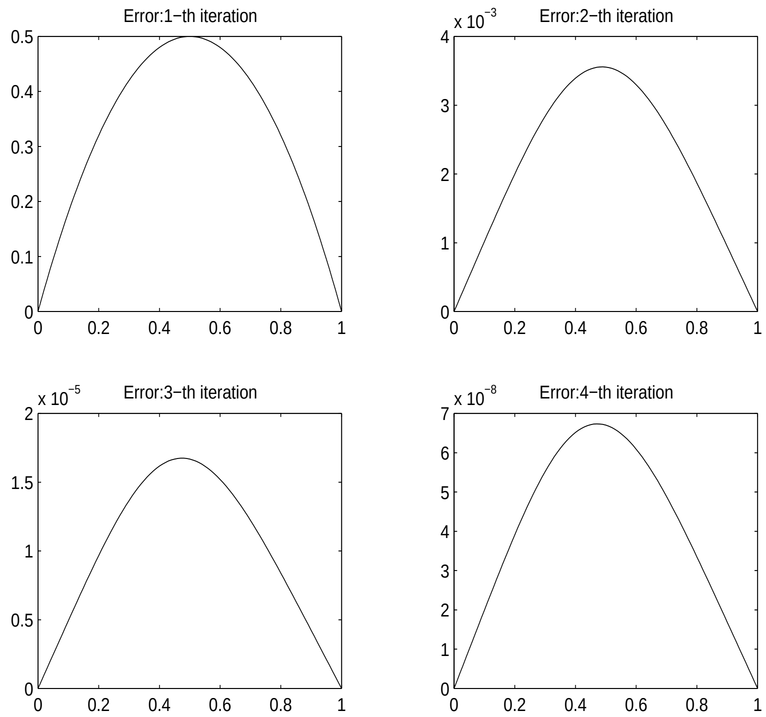
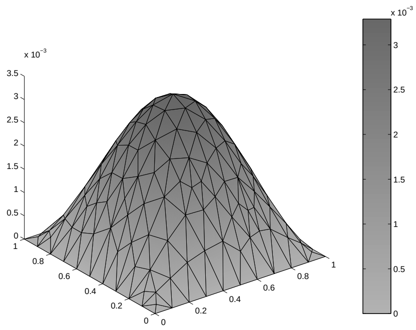
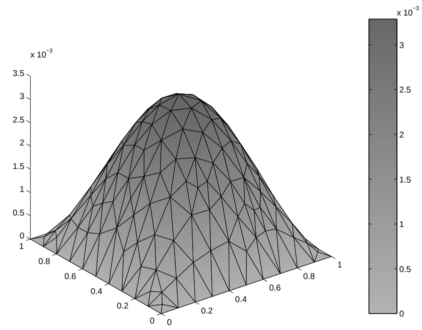


FIGURE 2. Error plot



(a) u_n



(b) w_n

FIGURE 3. The graph of u_n (left) and w_n for $n = 2$

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