MULTIPLE SOLUTIONS FOR NEUMANN PROBLEM WITH P-LAPLACIAN

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Abstract. In this paper we prove that the Neumann problem with p-Laplacian:

\[
\begin{align*}
(\mathcal{P}) \quad &-\Delta_p u + |u|^{p-2} u = f(x, u), \text{ in } \Omega \\
&|\nabla u|^{p-2} \frac{\partial u}{\partial n} = 0, \text{ on } \partial \Omega
\end{align*}
\]

has an unbounded sequence of solutions in \(W^{1,p}(\Omega), 1 < p < \infty\), using a multiple version of the "Mountain Pass" theorem.

1. Introduction and preliminary results

Let \(\Omega\) be an open bounded subset in \(\mathbb{R}^N, N \geq 2\), with smooth boundary, \(1 < p < \infty\), \(f: \Omega \times \mathbb{R} \to \mathbb{R}\) be a Caratheodory function which satisfies the growth condition:

\[
|f(x, s)| \leq c(|s|^{q-1} + 1), \text{ a.e.}x \in \Omega, (\forall) s \in \mathbb{R},
\]

where \(c \geq 0\) is constant, \(1 < q < p^* = \begin{cases} \frac{Np}{N-p}, & \text{if } p < N \\ +\infty, & \text{if } p \geq N \end{cases}\).

We consider the Neumann problem \((\mathcal{P})\), where \(\Delta_p\) is the p-Laplacian operator defined by

\[
\Delta_p u = \text{div}(-\nabla u|^{p-2}\nabla u) \text{ and } \frac{\partial u}{\partial n} = \nabla u \cdot n
\]

We shall use the standard notation:

\[
W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega), i \in \overline{1,N} \right\}
\]

equipped with the norm

\[
||u||_{1,p} = ||u||_{0,p}^p + \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|_{0,p}^p
\]

where \(||\cdot||_{0,p}\) is the usual norm on \(L^p(\Omega)\).
We define a new equivalent norm on the space $W^{1,p}(\Omega)$:

$$
\|u\|_{1,p}^p = \|u\|_{0,p}^p + \|\nabla u\|_{0,p}^p = \int_{\Omega} |u|^p + \int_{\Omega} \left( \sum_{i=1}^{n} \left( \frac{\partial u}{\partial x_i} \right)^2 \right)^{p/2}.
$$

Then the space $(W^{1,p}(\Omega), ||| \cdot |||_{1,p})$ is separable, reflexive and uniformly convex Banach space.

The dual norm on $(W^{1,p}(\Omega), ||| \cdot |||_{1,p})^*$ is denoted by $||| \cdot |||_{*,1,p}$.

The operator $-\Delta_p$ may be seen acting from $W^{1,p}(\Omega)$ into $(W^{1,p}(\Omega))^*$ by

$$
< -\Delta_p u, v > = \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v, \text{ for all } u, v \in W^{1,p}(\Omega)
$$

**Definition 1.** A function $u \in W^{1,p}(\Omega)$ is said to be a solution for the problem $(P)$ iff

$$
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v + \int_{\Omega} |u|^{p-2} u v = \int_{\Omega} f(x,u) v, \text{ for all } v \in W^{1,p}(\Omega)
$$

If $u \in W^{1,p}(\Omega)$ and $\Delta_p u \in L^q(\Omega)$ we can speak about $|\nabla u|^{p-2} \frac{\partial u}{\partial n}$ and $|\nabla u|^{p-2} \frac{\partial u}{\partial n} \in W^{-\frac{1}{p} - \frac{1}{p'}}(\partial \Omega)$ (see e.g.[6]).

Let $\Psi : L^q(\Omega) \to \mathbb{R}$ be defined by

$$
\Psi(u) = \int_{\Omega} F(x,u), \text{ where } F(x,s) = \int_{0}^{s} f(x,\tau) d\tau.
$$

The function $F$ is Caratheodory and

$$
|F(x,s)| \leq c_1(|s|^q + 1), \text{ a.e. } x \in \Omega, (\forall) s \in \mathbb{R}
$$

where $c_1 \geq 0$ is constant.

The functional $\Psi$ is continuously Frechet differentiable on $L^q(\Omega)$ and $\Psi'(u) = N_f(u)$, for all $u \in L^q(\Omega)$, where $N_f$ is the Nemytskii operator of $f$:

$$
N_f(u)(x) = f(x,u(x)), \text{ a.e. } x \in \Omega.
$$

Let $\varphi : [0, \infty) \to \mathbb{R}$ be a normalization function defined by $\varphi(t) = t^{p-1}$ and

$$
J_{\varphi} : W^{1,p}(\Omega) \to \mathcal{P}(W^{1,p}(\Omega))^*
$$

be the duality mapping corresponding to $\varphi$.

Then $J_{\varphi} u = \partial \varphi(u)$ for all $u \in W^{1,p}(\Omega)$ (see [5]) where

$$
\phi(u) = \int_{0}^{||u||_{1,p}} \varphi(t) dt = \frac{1}{p} ||u||_{1,p}^p
$$

and $\partial \phi$ is the subdifferential of $\varphi$ in the sense of convex analysis.
The functional $\phi$ is convex continuously Fréchet differentiable on $W^{1,p}(\Omega)$ and $\phi'(u) = -\Delta_p u + |u|^{p-2}u$, for all $u \in W^{1,p}(\Omega)$.

So $J_\varphi$ is single valued and

$$J_\varphi u = \phi'(u) = -\Delta_p u + |u|^{p-2}u,$$

for all $u \in W^{1,p}(\Omega)$.

Then the Euler-Lagrange functional $\mathcal{F} : W^{1,p}(\Omega) \to \mathbb{R}$,

$$\mathcal{F}(u) = \phi(u) - \varphi(u) = \frac{1}{p}||u||_p^p - \int_\Omega F(x,u) \, dx$$

and

$$\mathcal{F}'(u) = \phi'(u) - \varphi'(u) = -\Delta_p u + |u|^{p-2}u - N_f(u).$$

If $u \in W^{1,p}(\Omega)$ is a critical point for $\mathcal{F}$, that is $\mathcal{F}'(u) = 0$, then $\Delta_p u + |u|^{p-2}u = N_f(u)$ and consequently $u$ is solution for the problem $(P)$. In order to show that the functional $\mathcal{F}$ has an unbounded sequence of critical points we use a multiple version of the "Mountain Pass" theorem (see e.g. Theorem 9.12 in [7]).

**Theorem 1.1.** Let $X$ be an infinite dimensional real Banach space and let $f \in C^1(X, \mathbb{R})$ be even, satisfy (P.S.) condition. Suppose $f(0) = 0$ and:

(i) there are constants $\rho, \alpha > 0$ such that $f( ||x||_p \geq \alpha )$.

(ii) for each finite dimensional subspace $X_1$ of $X$ the set $\{ x \in X : f(x) \geq 0 \}$ is bounded. Then $f$ possesses an unbounded sequence of critical values.

We recall that the functional $f \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition (P.S.) if for every sequence $(u_n) \subset X$ with $(f(u_n))$ bounded and $f'(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence.

Since $W^{1,p}(\Omega)$ is uniformly convex and $J_\varphi$ is single valued then $J_\varphi$ satisfies the (S+)-condition: if $u_n \to u$ (weakly in $W^{1,p}(\Omega)$) and

$$\lim_{n \to \infty} \sup \ < J_\varphi u_n, u_n - u > \leq 0,$$

then $u_n \to u$ (see e.g. [5], Proposition 2).

2. Existence result

We need the following result:

**Proposition 2.1.** Suppose the Caratheodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies:

(i) the growth condition (1)

(ii) there are numbers $\theta > p$ and $s_0 > 0$ such that

$$0 \leq \theta F(x,s) \leq sf(x,s), \text{ for a.e. } x \in \Omega, (\forall) \ |s| \geq s_0.$$

Then, if $X_1$ is a finite dimensional subspace of $W^{1,p}(\Omega)$ the set $S = \{ u \in X_1 : F(u) \geq 0 \}$ is bounded in $W^{1,p}(\Omega)$.

**Proof.** From (3) there is $\gamma \in L^\infty(\Omega)$, $\gamma > 0$ on $\Omega$ (see [5]), such that

$$F(x,s) \geq \gamma(x)|s|^{\theta}, \text{ a.e. } x \in \Omega, (\forall) \ |s| \geq s_0.$$

For $u \in W^{1,p}(\Omega)$ let us denote

$$\Omega_1(u) = \{ x \in \Omega : |u(x)| \geq s_0 \}, \Omega_2(u) = \Omega \setminus \Omega_1(u).$$
By (2) we have
\[ \left| \int_{\Omega_2(u)} F(x, u) \right| \leq \int_{\Omega_2(u)} |F(x, u)| \leq c_1 |u|^q + 1 \leq c_1 \int_{\Omega} s_0^q + \int_{\Omega} c_1 = c_1 (s_0^q + 1) \cdot \text{vol } \Omega = k_1 \]
and using (4) we have
\[ (5) \quad \mathcal{F}(u) = \frac{1}{p} ||u||_{1,p} - \int_{\Omega_1(u)} F(x, u) - \int_{\Omega_2(u)} F(x, u) \leq \frac{1}{p} ||u||_{1,p} - \int_{\Omega_1(u)} \gamma(x)|u|^\theta + k_1 = \frac{1}{p} ||u||_{1,p} - \int_{\Omega_1(u)} \gamma(x)|u|^\theta + \int_{\Omega_2(u)} \gamma(x)|u|^\theta + k_1 \leq \frac{1}{p} ||u||_{1,p} - \int_{\Omega} \gamma(x)|u|^\theta + k_2 \]
where \( k_2 = ||\gamma||_{\infty} s_0^q \cdot \text{vol } \Omega + k_1 \).

The functional \( || \cdot ||_\gamma : W^{1,p}(\Omega) \to \mathbb{R} \), defined by
\[ ||u||_\gamma = \left( \int_{\Omega} \gamma(x)|u|^\theta \right)^{\frac{1}{\theta}} \]
is a norm on \( W^{1,p}(\Omega) \).

On the finite dimensional subspace \( X_1 \) the norms \( ||| \cdot |||_{1,p} \) and \( || \cdot ||_\gamma \) being equivalent, there is a constant \( \bar{k} = \bar{k}(X_1) > 0 \) such that
\[ |||u|||_{1,p} \leq \bar{k} \left( \int_{\Omega} \gamma(x)|u|^\theta \right)^{\frac{1}{\theta}} \]
for all \( u \in X_1 \).

Consequently, by (5), on \( X_1 \) it holds:
\[ \mathcal{F}(u) \leq \frac{1}{p} \bar{k}^p \left( \int_{\Omega} \gamma(x)|u|^\theta \right)^{\frac{1}{\theta}} - \int_{\Omega} \gamma(x)|u|^\theta + k_2 = \frac{1}{p} \bar{k}^p ||u||_{1,p}^p - ||u||_{\gamma}^\theta + k_2. \]
Therefore
\[ \frac{1}{p} \bar{k}^p ||u||_{1,p}^p - ||u||_{\gamma}^\theta + k_2 \geq 0 \]
for all \( u \in S \) and since \( \theta > p \) we conclude that \( S \) is bounded in \( W^{1,p}(\Omega) \) (in the norm \( || \cdot ||_\gamma \) and so in the norm \( ||| \cdot |||_{1,p} \)).

Now, we can state

**Theorem 2.1.** Suppose the Caratheodory functions \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is odd in the second argument : \( f(x, s) = -f(x, -s) \) and satisfies:
(i) there is \( q \in (1, p^*) \) such that
\[ |f(x, s)| \leq c(|s|^{q-1} + 1), \text{ a.e. } x \in \Omega, \forall s \in \mathbb{R} \]
(ii) \( \limsup_{s \to 0} \frac{f(x,s)}{|s|^{p-2}s} < \lambda_1 \) uniformly with a.e. \( x \in \Omega \),

where \( \lambda_1 = \inf \left\{ \|v\|_{W^{1,p}_0(\Omega)}^p : v \in W^{1,p}(\Omega), v \neq 0 \right\} \).

(iii) there are constants \( \theta > p \) and \( s_0 > 0 \) such that

\[
0 < \theta F(x,s) \leq sf(x,s) \quad \text{for a.e. } x \in \Omega, \forall |s| \geq s_0
\]

Then the problem \( (P) \) has an unbounded sequence of solutions.

**Proof.** It’s enough to show that \( F \) has an unbounded sequence of critical points in \( W^{1,p}(\Omega) \).

For this we shall use the theorem 1.1.

Clearly \( F(0) = 0 \) and \( F \) is even since \( f \) is odd.

By (i), (ii) and (iii) and proposition 2.1 it results that \( F \) satisfies the (PS) condition and hypothesis (i) and (ii) of the theorem 1.1.

**References**