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MONOTONE ITERATIONS FOR THE INITIAL VALUE PROBLEM

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Abstract. In this paper we present an application of the monotone iteration technique to an evolution equation.
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1. Introduction

Consider the initial value problem (IVP)

$$\begin{cases} u' + Au(t) = f(t, u(t)), \ t \in [0, T] \\ u(0) = u_0 \end{cases} \tag{1}$$

where $u_0 \in E$, E is an ordered Banach space, $f : [0, T] \times E \to E$ is continuous and $A : D(A) \subset E \to E$ is the generator of a positive semigroup of linear operators (see [17, 23]).
In some cases, the study of (1) is easier in the form of an operator equation,

$$Lu = N(u), \ u \in \mathcal{D}(L), \tag{2}$$

where $L : \mathcal{D}(L) \subset Y \to X$ and $N : Y \to X$ are two operators between the ordered Banach spaces Y and X.
The monotone iterative technique can be ilustrated using (2) as follows.
Let $\alpha \in \mathcal{D}(L)$ be a *lower solution* of (2) (i.e. $L\alpha \leq N(\alpha)$) and let us consider the sequence $(\alpha_n)_{n\geq 0}$ given by $\alpha_0 = \alpha$ and

$$L\alpha_{n+1} = N(\alpha_n), \ n \geq 0. \tag{3}$$

If $Lu \leq Lv$ implies $N(u) \leq N(v)$ and L is surjective, this sequence is well defined and the following inequalities hold.

$$L\alpha \leq N(\alpha) = L\alpha_1 \leq N(\alpha_1) = L\alpha_2 \leq ....$$

The problem is to assure that the sequence of monotone iterations $(L\alpha_n)_{n\geq 0}$ is convergent (or only a subsequence) to some Lu^* and $Lu^* = N(u^*)$.
This holds, for example, if there exists un upper solution $\beta \in \mathcal{D}(L)$ (i.e. $N(\beta) \leq L\beta$) with $L\alpha \leq L\beta$ and $N \circ L^{-1} : [L\alpha, \ L\beta] \to [L\alpha, \ L\beta]$ is continuous and compact.

In this paper we present an application of this technique to the evolution equation (1). We work with mild solutions and, also, mild lower and upper solutions.
There are some generalizations in ordered sets of the monotone iterative technique and we present them here as abstract results given by R. Lemmert [32] and S. Carl and S. Heikkila [10]. Using one of these abstract results we obtained an existence theorem for an implicit evolution equation.
This subject is an old one. Let us remember only few other names with contributions in this field: M.A. Krasnosel'skii [18], H. Amann [2] Ladde, Lakshmikantham and Vatsala [19] R. Precup [24, 25], A. Buică [7, 8], S. Carl and S. Heikkilla [9, 10] X. Liu, S. Sivaloganathan and S. Zhang [22].

The approximation for the solution u^* of $Lu = N(u)$ with the sequence (α_n) given by $L\alpha_{n+1} = N(\alpha_n)$ is at most linear, i.e.

$$||u^* - \alpha_{n+1}|| \leq c||u^* - \alpha_n||.$$

The generalized quasilinearization method offer monotone sequences that converge quadratically to the solution. Such a sequence, (α_n) is given by
$\alpha_0 = \alpha$ and

$$L\alpha_{n+1} = N(\alpha_n) + M(\alpha_n)(\alpha_{n+1} - \alpha_n),\ n \geq 0.$$

Under some additional assumptions on L, N, M one can prove that

$$||u^* - \alpha_{n+1}|| \leq c||u^* - \alpha_n||^2.$$

This method combines the quasilinearization method of R. Bellman and R. Kalaba [21] and the Newton's method as it is stated in [14, 3, 13]. It was applied to many kind of problems related to differential or integral equations: V. Lakshmikanthan and A.S. Vatsala [21], R. Precup [26, 27], S.G. Deo and C. McGloin Knoll [12], S. Carl and V. Lakshmikantham [11], B. Ahmad, J. Nieto and N. Shahzad [1].
In this paper we give an abstract theorem which provide approximations for the solution of the operator equation $Lu = N(u)$, where L is linear and N could be nonlinear. We replace the differentiability condition for the nonlinear part with a metric condition. We show how some results form [21] regarding the initial value problem for an ODE and for an n-th order system of ODE are obtained as consequences of our abstract result.
When one apply the monotone iterative techniques to differential equations, there is need to use of differential inequalities. This is one of the reasons for we present some here.

This paper is organized as follows.
1. Introduction.
2. Differential inequalities.
3. Monotone linear (iterative) approximations.
4. Monotone quadratic (iterative) approximations.

2. Differential inequalities

Some of the main tools used in the monotone iterative techique are differential inequalities.

We call a linear differential inequality (LDI) the following implication.

$$\begin{gathered} u'(t) + Au(t) \leq v'(t) + Av(t),\ t \in [0,T] \text{ and} \\ u(0) \leq v(0) \\ \text{imply } u(t) \leq v(t),\ t \in [0,T]. \end{gathered}$$

We call a nonlinear differential inequality (NDI) the following implication.

$$\begin{gathered} u'(t) + Au(t) - f(t,u(t)) \leq v'(t) + Av(t) - f(t,v(t)),\ t \in [0,T] \text{ and} \\ u(0) \leq v(0) \\ \text{imply } u(t) \leq v(t),\ t \in [0,T]. \end{gathered}$$

We present here some differential inequalities collected from P. Volkmann [31]), A. Buică [5, 6], G. Herzog and R. Lemmert [16].

Proposition 2.1. *Let $u, v \in C^1[0,T]$ and $A \in \mathbb{R}$. Then (LDI) holds.*
Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous. We assume that f is or locally Lipschitz or increasing or decreasing. Then (NDI) holds.

We say that $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is *quasimonotone increasing* (qmi) with respect to x if $u \leq v$ and $u_k = v_k$ imply $f_k(t,u) \leq f_k(t,v)$.
If there exists $\omega > 0$ such that $f(t,\cdot) + \omega Id$ is increasing then f is qmi.

Proposition 2.2. *Let $u, v \in C^1([0,T],\mathbb{R}^n)$ and $A = (-a_{ij}) \in \mathcal{M}_n(\mathbb{R})$. Then (LDI) holds if and only if $a_{ij} \geq 0$ when $i \neq j$.*
Let $A = (-a_{ij}) \in \mathcal{M}_n(\mathbb{R})$ be such that $a_{ij} \geq 0$ when $i \neq j$. Let $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous, locally Lipschitz and qmi. Then (NDI) holds.

Proposition 2.3. *Let $u, v \in C^1([0,T], E)$ and $A : \mathcal{D}(A) \to E$ be a linear operator. Then (LDI) holds if and only if $(-A)$ is the generator of a positive semigroup of operators.*
Let $A : \mathcal{D}(A) \to E$ be a linear operator which generates a positive semigroup of contractions. Let $f : [0,T] \times E \to E$ be continuous, Lipschitz and such that there exists $\omega > 0$ such that $f(t,\cdot) + \omega Id$ is increasing. Then (NLI) holds.

Let us notice that these kind of results assure that

$$\alpha(t) \leq u^*(t) \leq \beta(t),\ t \in [0,T],$$

where α, β are lower and upper solutions, and u^* is a solution of the IVP. If the existence of some ordered lower and upper solutions is guaranteed, then the hypothesis for the nonlinear part can be relaxed in order to assure that the solution is between α and β. We say that $\alpha \in C^1([0,T]; E)$ is a lower solution of IVP (1) if

$$\begin{gathered} \alpha'(t) + A\alpha(t) \leq f(t,\alpha(t)),\ t \in [0,T] \\ \alpha(0) \leq u_0 \end{gathered}$$

Analoguosly we define the upper solution.

Let α and β be ordered (i.e. $\alpha \leq \beta$) lower and upper solutions for the IVP (1). Let us denote

$$D_{\alpha,\beta} = \{(t,u) \in [0,T] \times E \;:\; \alpha(t) \leq u \leq \beta(t)\}.$$

Proposition 2.4. *Let $\alpha, \beta \in C^1([0,T], E)$ be ordered lower and upper solutions for the IVP (1) and $A : \mathcal{D}(A) \to E$ be a linear operator which generates a positive semigroup of contractions. Let $f : [0,T] \times E \to E$ be continuous, locally Lipschitz and such that $f(t,\cdot) + \omega Id$ is increasing in $D_{\alpha,\beta}$ for some $\omega > 0$. Then the unique mild solution, denoted u^*, of (1) is defined on $[0,T]$ and $\alpha(t) \leq u^*(t) \leq \beta(t)$, for every $t \in [0,T]$.*

Remark. Let Ω be a bounded domain of $\mathbb{R}^n$ and $\tilde{f} : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function which has a continuous partial derivative with respect to the last variable. Let us define $f : [0,T] \times C(\Omega) \to C(\Omega)$, $f(t,u)(x) = \tilde{f}(t,x,u(x))$ and consider $\alpha, \beta : [0,T] \to C(\Omega)$ continuous with $\alpha \leq \beta$. Then $f(t,\cdot) + \omega Id$ is increasing in $D_{\alpha,\beta}$, where ω is such that $\frac{\partial \tilde{f}}{\partial u} \geq -\omega$ in $[\alpha(t), \beta(t)]$ for every $t \in [0,T]$.

In the end of this section we intend to illustrate how differential inequalities can be used to prove stability of the steady-stae solution.
X. Liu, S. Sivaloganathan and S. Zhang studied in [22] the following model for the growth of populations of two species.

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u &= u(a_1 - b_1 u + c_1 v) \\ \frac{\partial v}{\partial t} - \Delta v &= v(a_2 + b_2 u - c_2 v), \quad \text{a.e. } (0,\infty) \times \Omega \\ u(0,x) &= u_0(x),\ v(0,x) = v_0(x),\ \overline{\Omega} \\ \frac{\partial u}{\partial \gamma}(t,x) &= 0,\ \frac{\partial v}{\partial \gamma} = 0,\ (0,\infty) \times \partial\Omega. \end{cases}$$

Since the linear part generates a semigroup of linear contractions and the nonlinear part is locally Lipschitz, the local existence is assured. Let

$$\eta_1 = (a_1 c_2 + a_2 c_1)/(b_1 c_2 - b_2 c_1) \text{ and } \eta_2 = (a_1 b_2 + a_2 b_1)/b_1 c_2 - b_2 c_1).$$

We have that (η_1, η_2) is a steady-state solution for this system, and $(\eta_1 - \varepsilon, \eta_2 - \varepsilon)$ and $(\eta_1 + \varepsilon, \eta_2 + \varepsilon)$ are ordered lower and upper solutions. The hypothesis of Proposition 2.4 are fullfiled. Then, any solution with initial values between the lower and upper solutions will remain between these values for all $t > 0$. Thus the following result holds.

Theorem 2.1. [22] *If $b_1 c_2 > b_2 c_1$, $b_1 \geq c_1$, $c_2 \geq b_2$ and $a_1, a_2 \geq 0$ then the steady-state solution (η_1, η_2) is stable.*

3. Monotone linear iterative approximations

In the introduction we have presented the main ideas of the monotone iterative technique. Let us now state precisely some results.
Let K be a cone in the ordered Banach space X, that is, a closed, convex subset of X such that $K \cap -K = \{0\}$, where 0 denotes the null element of X. The cone K induces the order relation in X defined by $u \leq v$, $u, v \in X$ if and only if $v - u \in K$. For $\alpha \leq \beta$ *the order interval* $[\alpha, \beta]$ is the set of all $u \in X$ such that $\alpha \leq u \leq \beta$. The

cone K is said to be *regular* if any monotone increasing sequence contained in an order interval is convergent.

Theorem 3.1. *([2]) Let X be an ordered metric space, $\alpha_), \beta_0 \in X$. If $N : [\alpha_0, \beta_0] \to [\alpha_0, \beta_0]$ is a monotone increasing and compact operator, then the sequence (α_n), given by $\alpha_0 = \alpha$, $\alpha_{n+1} = N(\alpha_n)$, $n \geq 0$ is monotone increasing and converges to the minimal fixed point of N.*

Theorem 3.2. *(Lemmert [32]) Let X be an ordered set, $\alpha \in X$, $N : X \to X$ be increasing and $\alpha \leq N(\alpha)$. We assume that every chain in $\{Nu : \alpha \leq u\}$ has a supremum. Then N has a fixed point.*

Theorem 3.3. *(Carl-Heikkila [10]) Let W be a nonempty set, X be an ordered set, $\alpha \in W$. Let $L, N : X \to X$ be such that $L\alpha \leq N(\alpha)$, and $Lu \leq Lv$ implies $N(u) \leq N(v)$. We assume that $L(W)$ is an ordered metric space and each (u_n) in W with (Lu_n) and $(N(u_n))$ monotone increasing, is such that $(N(u_n))$ converges in $L(W)$. Then there exists u_*, a solution of $Lu = N(u)$.*

Corollary 3.1. *Let X be an ordered Banach space with a normal cone K, $\alpha, \beta \in X$. Let $N : [\alpha, \beta] \to [\alpha, \beta]$ be increasing. If K is regular or $N([\alpha, \beta])$ is compact in X then N has extremal fixed points.*

Using Theorem 3.1 we can prove the existence of extremal mild solutions and of monotone iterative approximations for the IVP (1).

We say that $u \in C([0, T]; E)$ is a *mild solution* of IVP (1) if u is a solution of the integral equation

$$u(t) = e^{-At} u_0 + \int_0^t e^{-A(t-s)} f(s, u(s)) ds. \tag{4}$$

A mild lower solution is a function $\alpha \in C([0, T]; E)$ which satisfy (4) with $\leq$ instead of $=$.

Theorem 3.4. *Let us assume that the following hypotheses are fullfiled.*
(H1) There exist α a mild lower solution and β a mild upper solution of the IVP (1);
(H2) $(-A)$ is the generator of a positive linear semigroup of compact operators;
(H3) $f + \omega Id$ is increasing in $D_{\alpha,\beta}$.
Then there exist extremal mild solutions $\alpha \leq u_* \leq u^* \leq \beta$ and monotone sequences (α_n), (β_n) which converge to u_* and u^*, respectively.
The sequence (α_n) is given by

$$\alpha_{n+1} = S(t)u_0 + \int_0^t S(t-s)[f(s, \alpha_n) + \omega \alpha_n] ds,$$

where $S(t) = e^{-\omega t} e^{-At}$.

Using Theorem 3.3 we have studied the IVP for an evolution equation in the implicit form,

$$\begin{cases} u' + Au - f(t,u) = G(t,u,u' + Au - f(t,u)), t \in (0,T) \\ u(0) = u_0. \end{cases}$$

Let us list first some hypothesis and state some preliminary results and definitions.

(A1) $(-A)$ is the generator of a semigroup of linear positive contractions.
(F2) $f : (0,T) \times E \to E$ is Carathéodory and there exists $\omega > 0$ such that $f(t,\cdot) + \omega Id$ is increasing.
(F3) there exists $a \in L^1(0,T)$ such that $|f(t,u) - f(t,v)| \leq a(t)|u-v|$ for all $u, v \in E$.
(G4) G is sup-measurable and increasing.

$\widetilde{L} : D(L) \subset C([0,T];E) \to L^1(0,T;E) \times E$

$$\widetilde{L}u = (w, w_0) \text{ iff}$$
$$u(t) = S(t)w_0 + \int_0^t S(t-s)[w(s)f(s,u(s)) + \omega u(s)]ds.$$

$Nu(t) = G(t, u(t), Lu(t))$ and $\widetilde{N}u(t) = (Nu(t), u_0)$.

Lemma. *$\widetilde{L}u \leq \widetilde{L}v$ implies $u \leq v$.*

We say that $\alpha \in D(L)$ is *a mild lower solution* of the IVP (5) if
$\widetilde{L}\alpha(t) \leq (G(t, \alpha(t), L\alpha(t)), u_0)$, $t \in (0,T)$.

Lemma. *$\widetilde{L}, \widetilde{N} : W \to L^1(0,T;E) \times E$ are well-defined and $\widetilde{L}u \leq \widetilde{L}v$ implies $\widetilde{N}u \leq \widetilde{N}v$.*

We also consider now the following hypothesis.
(H5) There exists $\alpha, \beta \in D(L)$ mild lower and upper solutions of the IVP (5), such that $\tilde{L}\alpha \leq \tilde{L}\beta$.

The main result is the following.

Theorem 3.5. [7] *Let us assume that the hypotheses (A1),(F2),(F3), (G4),(H5) are fullfiled and that the ordered Banach space E has regular cone. Then the IVP (5) has extremal mild solutions in $W = \{u \in D(L) : \widetilde{L}\alpha \leq \widetilde{L}u \leq \widetilde{L}\beta\}$ and they are monotone increasing with respect to f, g and u_0.*

4. Quadratic approximations

The aim of this section is to present the quasilinearization method in an abstract setting. We do not assume differentiability for the nonlinear part. We replace it by a metric condition. We also present some results regarding the initial value problem from [21] as consequences of our abstract result.

Let $(X, ||\cdot||_X, \leq)$ and $(Y, ||\cdot||_Y, \leq)$ be two ordered Banach spaces where Y is a subset of X. Let $L : D(L) \subset Y \to X$ be a linear operator and $N : Y \subset (X, ||\cdot||_X) \to (X, ||\cdot||_X)$ be a continuous operator. Let $\alpha_0, \beta_0 \in D(L)$ be such that

$$L\alpha_0 \leq N(\alpha_0),\ N(\beta_0) \leq L\beta_0,\ \alpha_0 \leq \beta_0.$$

Let $M : [\alpha_0, \beta_0] \subset (Y, ||\cdot||_X) \to \mathcal{L}(Y, X)$ be a uniformly continuous operator, where $\mathcal{L}(Y, X)$ is the set of all continuous and linear operators from $(Y, ||\cdot||_X)$ to $(X, ||\cdot||_X)$.

We list some hypotheses.

(L1) For every $\alpha \in [\alpha_0, \beta_0]$ and $v \in X$, the equation $(L - M(\alpha))\, u = v$ has a unique solution, $u \in D(L)$ denoted by $u = S(v, \alpha)$.

If (L1) is fullfiled then we can consider the operator $S : X \times [\alpha_0, \beta_0] \to Y$.

(L2) $v \leq v^*$ implies $S(v, \alpha) \leq S(v^*, \alpha)$.

(L3) There exists $c_1 > 0$ such that

$$||S(v^*, \alpha_0) - S(v, \alpha_0)||_Y \leq c_1 ||v^* - v||,$$

for all $v, v^* \in X$ and $\alpha_0 \in [\alpha, \beta]$.

(L4) Let (α_n) be a sequence from Y such that (α_n) and $(L\alpha_n)$ converge in X to u^* and v^*, respectively. Then (α_n) is convergent in Y and $v^* = Lu^*$.

(N5) $N(u) \leq N(v) - M(u)(v - u)$, for all $\alpha_0 \leq u \leq v \leq \beta_0$.

(N6) There exists $c_2 > 0$ with $||N(v) - N(u) - M(u)(v - u)|| \leq c_2 ||v - u||^2$, for all $\alpha_0 \leq u \leq v \leq \beta_0$.

Theorem 4.1. *Let us suppose that the hypotheses (L1)-(L4),(N5) and (N6) are fullfiled, the order cone of X is regular, and that the equation $Lu = N(u)$ has at most one solution.*
Then the sequence $(\alpha_n)_{n\geq 0}$ given by the iterative relation

$$L\alpha_{n+1} = N(\alpha_n) + M(\alpha_n)\,(\alpha_{n+1} - \alpha_n)\,,\ n \geq 0 \tag{5}$$

is monotone increasing and converges in Y to the unique solution of $Lu = N(u)$. The convergence is quadratic.

Proof. The sequence (α_n) is well defined and its elements are from Y, as it is assured by hypothesis (L1).
Now we show that

$$\alpha_0 \leq \alpha_1 \leq \beta_0.$$

We have that

$$L\alpha_1 = N(\alpha_0) + M(\alpha_0)\,(\alpha_1 - \alpha_0)\,.$$

Then, using that $L\alpha_0 \leq N(\alpha_0)$, we obtain

$$\begin{aligned}(L - M(\alpha_0))\,\alpha_0 &\leq N(\alpha_0) - M(\alpha_0)\alpha_0 = \\ &= (L - M(\alpha_0))\,\alpha_1.\end{aligned}$$

Hypothesis (L2) assures that $\alpha_0 \leq \alpha_1$.
We also have

$$\begin{aligned} L\alpha_1 - M(\alpha_0)\alpha_1 \quad &= N\alpha_0 - M(\alpha_0)\alpha_0 \leq \text{ (using (N5)} \\ &\leq N\beta_0 - M(\alpha_0)\,(\beta_0 - \alpha_0) - M(\alpha_0)\alpha_0 \leq \\ &\leq L\beta_0 - M(\alpha_0)\beta_0. \end{aligned}$$

Hypothesis (L2) assures that $\alpha_1 \leq \beta_0$.

We prove now that

$$L\alpha_1 \leq N\alpha_1.$$

This follows by the following relations.

$$\begin{aligned} L\alpha_1 - M(\alpha_0)\alpha_1 \quad &= N\alpha_0 - M(\alpha_0)\alpha_0 \leq \text{ (using (N5)} \\ &\leq N\alpha_1 - M(\alpha_0)(\alpha_1 - \alpha_0) - M(\alpha_0)\alpha_0. \end{aligned}$$

Assume now that for some $n > 1$, $L\alpha_n \leq N\alpha_n$ and $\alpha_0 \leq \alpha_n \leq \beta_0$. Similar to the proof for $n = 1$, it can be shown that $L\alpha_{n+1} \leq N\alpha_{n+1}$ and $\alpha_0 \leq \alpha_{n+1} \leq \beta_0$.
So by induction we obtain

$$\alpha_0 \leq \alpha_1 \leq ... \leq \alpha_n \leq ... \leq \beta_0. \tag{6}$$

The convergence in the norm $||\cdot||_X$ of the sequence $(\alpha_n)_{n\geq 0}$ is assured by the regularity of the order cone K of the Banach space X. Let us denote by u^* the limit of this sequence, which satisfies

$$\alpha_n \leq u^*.$$

Using the continuity of N and the uniform continuity of M, the right hand side of (5) converges in X to $N(u^*)$. Thus, the sequence (α_n) from Y is such that (α_n) and $(L\alpha_n)$ converge in the norm of X to u^* and $N(u^*)$, respectively. Using (L5) we deduce that (α_n) converges in Y to u^* and $Lu^* = N(u^*)$.

Finally, to prove quadratic convergence, we let

$$p_n = u^* - \alpha_n.$$

$$\begin{aligned} Lp_{n+1} \quad &= Lu^* - L\alpha_{n+1} \\ &= Nu^* - N\alpha_n - M(\alpha_n)(\alpha_{n+1} - \alpha_n) = \\ &= Nu^* - N\alpha_n - M(\alpha_n)p_n + M(\alpha_n)p_{n+1} \end{aligned}$$

If we write $v^* = Lu^* + M(\alpha_n)u^*$ and notice that $u^* = S(v^*, \alpha_n)$ and $\alpha_{n+1} = S(v_n, \alpha_n)$, we deduce the following inequalities on the base of (L3) and (N6).

$$\begin{aligned} ||p_{n+1}||_Y \quad &= ||u^* - \alpha_{n+1}||_Y \leq c_1||Lp_{n+1} - M(\alpha_n)p_{n+1}||_X \\ &= c_1||Nu^* - N\alpha_n - M(\alpha_n)(u^* - \alpha_n)||_X \\ &\leq c_1c_2||p_n||_y^2 \end{aligned}$$

The proof is therefore complete. □

Let us consider the initial value problem for a first order ODE,

$$u' = f(t,u),\ t \in [0,T],\ u(0) = u_0, \tag{7}$$

where $f \in C\left([0,T] \times \mathbb{R}; \mathbb{R}\right)$.

Corollary 4.1. [21] *Assume that* $\alpha_0, \beta_0 \in C^1[0,T]$ *are such that*

$$\begin{gathered} \alpha_0' \le f(t,\alpha_0), \\ f(t,\beta_0) \le \beta_0' \text{ and} \\ \alpha_0(t) \le \beta_0(t),\ t \in [0,T] \end{gathered}$$

and that the derivatives f_u, f_{uu} *exist, are continuous, and*

$$f_{uu} \ge 0 \text{ in } \Omega,$$

where $\Omega = \{(t,u) \in [0,T] \times \mathbb{R}\ :\ \alpha_0(t) \le u \le \beta_0(t),\ t \in [0,T]\}$.
Then the sequence $(\alpha_n)_{n \ge 0}$ *given by*

$$\begin{aligned} &\alpha_{n+1}' = f(t,\alpha_n) + f_u(t,\alpha_n)(\alpha_{n+1} - \alpha_n) \\ &\alpha_{n+1}(0) = u_0 \end{aligned}$$

converge uniformly to the unique solution of (7) and the convergence is quadratic.

Remark. The proof of this theorem follows from Theorem 4.1 by choosing the spaces and the operators as follows. $Y = C[0,T]$ with the supremum norm, $X = L^1(0,T)$ with L^1-norm and natural order relations.
$D(L) = \left\{u \in C^1(0,T)\ :\ u(0) = u_0\right\}$, $Lu = u'$, $N(u)(t) = f(t,u(t))$ and $M(\alpha)u(t) = f_u(t,\alpha(t)) \cdot u(t)$.

Let us consider the initial value problem for an n-th order system of ODE,

$$u' = f(t,u),\ t \in [0,T],\ u(0) = u_0, \tag{8}$$

where $f \in C\left([0,T] \times \mathbb{R}^n; \mathbb{R}^n\right)$.

Corollary 4.2. [21] *Assume that* $\alpha_0, \beta_0 \in C^1\left([0,T]; \mathbb{R}^n\right)$ *are such that*

$$\begin{gathered} \alpha_0' \le f(t,\alpha_0), \\ f(t,\beta_0) \le \beta_0' \text{ and} \\ \alpha_0(t) \le \beta_0(t),\ t \in [0,T] \end{gathered}$$

and that
(i) $f \in C\left([0,T] \times \mathbb{R}^n, \mathbb{R}^n\right)$ *is quasimonotone increasing in* u*;*

(ii) f_u, f_{uu} *exist and are continuous satisfying*

$$f_{uu} \ge 0 \text{ on } \Omega$$

where $\Omega = \{(t,u) \in \mathbb{R}^n\ :\ \alpha_0(t) \le u \le \beta_0(t),\ t \in [0,T]\}$*;*

(iii) $a_{i,j}(t,\alpha_0) \ge 0$ *for* $i \ne j$ *where* $A(t,\alpha_0) = [a_{i,j}]$ *is an* $n \times n$ *matrix given by* $A(t,\alpha_0) = f_u(t,\alpha_0(t))$.

Then the sequence $(\alpha_n)_{n\geq 0}$ *given by*

$$\begin{aligned} &\alpha'_{n+1} = f(t,\alpha_n) + f_u(t,\alpha_n)(\alpha_{n+1}-\alpha_n) \\ &\alpha_{n+1}(0) = u_0 \end{aligned}$$

converge uniformly to the unique solution of (8) and the convergence is quadratic.

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