Seminar on Fixed Point Theory Cluj-Napoca, Volume 3, 2002, 189-196
http://www.math.ubbcluj.ro/~nodeacj/journal.htm

ON OSCILLATION OF SOLUTIONS OF A LOSS LESS TRANSMISSION LINE MODEL

DAFINKA TZ. ANGELOVA

Higher School of Civil Engineering
Department of Mathematics and Informatics
1373 Sofia, 32 Suhodolska str.,Bulgaria
e-mail: angelova@vsu.bg

Abstract. Let us consider the following equation:

$$\dot{x}(t) - \alpha\dot{x}(t-h) + \beta x(t) + \alpha\gamma x(t-h) = 0. \tag{1}$$

Our main purpose is to obtain necessary and sufficient conditions for oscillation of all solutions of (1) where α, β, γ and $h > 0$.
Keywords: oscillatory function, characteristic equation, equilibrium solution.
AMS Subject Classification: 34C25.

1. Introduction

Electrical networks containing loss less transmission lines arise, for instance, in high speed computers, where such lines are used to interconnect switching circuits (see [2], [11]). Since these networks can exhibit oscillatory behavior, it is naturally to seek conditions guaranteeing oscillations of all solutions of the modelling equation.

In this paper we consider a particular circuit consisting of a loss less transmission line terminated by lumped circuits, one of which is nonlinear, with a volt-ampere characteristic $i = g(\nu)$. This circuit is described in [1,8] and the modelling partial differential network equations

$$L\frac{\partial i}{\partial t} = -\frac{\partial \nu}{\partial x}, \quad C\frac{\partial \nu}{\partial t} = -\frac{\partial i}{\partial x}, \quad 0 < x < 1, \quad t > 0$$

with boundary conditions

$$E - \nu(0,t) - Ri(0,t) = 0, C\frac{\partial \nu(1,t)}{\partial t} = i(1,t) - f(\nu(1,t))$$

are derived and reduced to a single difference-differential equation of neutral type

$$C\left[\nu_1'(t) + K\nu_1'(t-h)\right] + \left(\frac{1}{z} - g\right)\nu_1(t) - K\left(\frac{1}{z} + g\right)\nu_1(t-h) =$$
$$= -f(\nu_1(t)) - Kf(\nu_1(t-h) \qquad (NDDE)$$

where $\nu_1 = \nu(x,t)$ at $x = 1$, $\nu_0 = \nu(x,t)$ at $x = 0$, $z = \sqrt{\frac{L}{C}}, h = 2\sqrt{LC}, K = \frac{R-z}{R+z}, g = \frac{df(\nu_0)}{d\nu_0}$.

As in the theory of ordinary differential equations it is important to analyze the characteristic equation of the system linearized around the equilibrium solution. In our case the linearized equation is

$$L(\nu_1) \equiv C[\dot{\nu}_1(t) + K\dot{\nu}_1(t-h)] + \left(\frac{1}{z} - g\right)\nu_1(t) - K\left(\frac{1}{z} + g\right)\nu_1(t-h) = 0 (LDDE)$$

and the characteristic equation, obtained by substituting $\nu_1 = e^{\lambda t}$, λ is complex, into $L(\nu_1) = 0$ is $q(\lambda) \equiv \lambda - \alpha\lambda e^{-\lambda h} + \beta + \alpha\gamma e^{-\lambda h} = 0$ (CE)

where $\alpha = -K > 0, \beta = \frac{1-gz}{Cz} > 0, \gamma = \frac{1+gz}{Cz} > 0$. Denoting $\nu_1 = x$ we study (LDDE) in the form

We note such conditions (in terms of the characteristic equation) have been obtained in [3]-[7], [9], [10], [12] and in the papers cited there. The cited authors consider neutral equations different from and not including the Brayton's equation (1).

2. Preliminaries

We say the function $x(t)$ is a solution of equation (1) provided there exists $t_0 \in R$ such that $x \in C([t_0 - h, \infty), R), x(t) - \alpha x(t-h)$ is continuously differentiable for $t \geq t_0$ and (1) holds for $t \geq t_0$. A continuous function is said to be oscillatory if it has arbitrarily large zeros; otherwise it is said to be non-oscillatory. A non-oscillatory two times differentiable function $y(t)$ is said to be I_0 - function if $yy' < 0, yy'' > 0, \lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = 0$ and it is said to be I_∞ - function if $yy' > 0, yy'' > 0, \lim_{t\to\infty} |y(t)| = \lim_{t\to\infty} |y'(t)| = \infty$.

In this section we establish some lemmas which will be used in the proof of our main result.

Lemma 1 Let $x(t)$ be a solution of (1) and $a, b, c \in R$. Then each one of the functions $x(t-a), \int_{t-b}^{t-c} x(s)ds, \int_t^\infty x(s)ds$ is a solution of (1).

The conclusion is a direct consequence of the linearity and homogeneous of (1).

Lemma 2 Let $x(t)$ be a non-oscillatory solution of (1). Then there exists a non-oscillatory solution $\omega(t)$ of (1) which is either I_0-function or I_∞-function.

Proof. Without loss of generality $x(t)$ can be considered eventually positive. Set

$$u(t) = x(t) - \alpha x(t-h) + \beta \int_{t-h}^t x(s)ds \tag{2}$$

and $\omega(t) = u(t) - \alpha u(t-h) + \beta \int_{t-h}^t u(s)ds$. By Lemma 1 $u(t)$ and $\omega(t)$ are solutions of (1). Since

$$\dot{u}(t) = \dot{x}(t) - \alpha\dot{x}(t-h) + \beta x(t) - \beta x(t-h) = -(\alpha\gamma + \beta)x(t-h) < 0 \tag{3}$$

then $u(t)$ is decreasing and thus either $\lim_{t\to\infty} u(t) = -\infty$ or $\lim_{t\to\infty} u(t) = l \in R$. But

$$\dot{\omega}(t) = \dot{u}(t) - \alpha\dot{u}(t-h) + \beta u(t) - \beta u(t-h) = -(\alpha\gamma+\beta)u(t-h) \tag{4}$$

and

$$\ddot{\omega}(t) = -(\alpha\gamma+\beta)\dot{u}(t-h) > 0. \tag{5}$$

In the case when $\lim_{t\to\infty} u(t) = -\infty$ it follows that $\lim_{t\to\infty} \dot{\omega}(t) = \infty$ and hence $\omega(t) > 0$, i.e. $\omega\dot{\omega} > 0$. In view of (5) we have $\omega\ddot{\omega} > 0$ and thus $\omega(t)$ is an I_∞-function.

In the case when $\lim_{t\to\infty} u(t) = l \neq 0$ via (4) we get $\lim_{t\to\infty} [u(t)-\alpha u(t-h)]' = -(\alpha\gamma+\beta)l$ and then $l(1-\alpha) = \lim_{t\to\infty} [u(t)-\alpha u(t-h)] = \left\{ \begin{array}{rl} \infty, & l<0 \\ -\infty, & l>0 \end{array} \right.$, which is a contradiction. Thus $\lim_{t\to\infty} u(t) = 0$.But $\dot{u}(t) < 0$ and then $u(t) > 0$ and $\dot{\omega}(t) < 0$. By definition of $\omega(t)$ it follows $\lim_{t\to\infty} \omega(t) = 0$ and by (5) we obtain that $\omega(t)$ is an I_0-function. The proof is complete.

Lemma 3 Let $x(t)$ be a non-oscillatory solution of (1). Then:

a) If $x(t)$ is an I_0-function then there exists a non-oscillatory solution $u(t)$ of (1) which is an I_0-function and $\overset{+}{\bigwedge}(u) = \{\lambda > 0 : \dot{u} + \lambda u \leq 0\} \neq \emptyset$;

b) If $x(t)$ is an I_∞-function then there exists a non-oscillatory solution $\nu(t)$ of (1) which is an I_∞-function and $\overset{-}{\bigwedge}(\nu) = \{\lambda > 0 : -\dot{\nu} + \lambda\nu \leq 0\} \neq \emptyset$.

Proof. Without any loss of generality $x(t)$ can be considered eventually positive.

a) Define $u(t)$ by (2). According to Lemma 1 $u(t)$ is a solution of (1) and from the proof of Lemma 2 $u(t)$ is an I_0-function. From (2) we get

$u(t) \leq x(t) + h\beta x(t-h) < (1+h\beta)x(t-h)$

and using (3) we obtain

$$0 = \dot{u}(t)+(\beta+\alpha\gamma)x(t-h) > \dot{u}(t)+\frac{\beta+\alpha\gamma}{1+h\beta}u(t), \text{ i.e. } 0 < \frac{\beta+\alpha\gamma}{1+h\beta} \in \overset{+}{\bigwedge}(u) \Rightarrow \overset{+}{\bigwedge}(u) \neq \emptyset.$$

b) Define $\nu(t) = -u(t)$. According to Lemma 1 $\nu(t)$ is a solution of (1). In view of (3) we have $\dot{\nu}(t) = (\beta+\alpha\gamma)x(t-h) > 0, \ddot{\nu}(t) = (\beta+\alpha\gamma)\dot{x}(t-h) > 0$ and hence, $\nu(t)$ is an I_∞-function. From (2) we have

$\nu(t) < \alpha x(t-h)$ and using (3) we get

$$0 = -\dot{\nu}(t) + (\beta+\alpha\gamma)x(t-h) > -\dot{\nu}(t) + \frac{\beta+\alpha\gamma}{\alpha}\nu(t), \text{ i.e. } 0 < \frac{\beta+\alpha\gamma}{\alpha} \in \overset{-}{\bigwedge}(\nu) \Rightarrow$$

$\overset{-}{\bigwedge}(\nu) \neq \emptyset$ which proves the lemma.

Lemma 4 Let $x(t)$ be a non-oscillatory solution of (1). Then:

a) If $x(t)$ is an I_0-function then $\bigwedge^+(x) \neq \emptyset$ and $x(t) > \alpha x(t-h)$;

b) If $x(t)$ is an I_∞-function then $\bigwedge^-(x) \neq \emptyset$ and $x(t) < (\alpha+1)x(t-h)$.

Proof. Without loss of generality $x(t)$ can be considered eventually positive.

a) From (1) we have $0 > \dot{x}(t) + \beta x(t) + \alpha\gamma x(t-h) \geq \dot{x}(t) + (\beta + \alpha\gamma)x(t)$ and thus $\overset{+}{\bigwedge}(x) \neq \emptyset (\lambda = \beta + \alpha\gamma)$. Again from (1) we have $0 > \dot{x}(t) - \alpha\dot{x}(t-h)$ and integrating from t to ∞ we get $-\alpha x(t-h) > -x(t)$ since $\lim_{t\to\infty} x(t) = 0$. Then $x(t) > \alpha x(t-h)$.

b) Now from (1) we have $0 > -\alpha\dot{x}(t-h) + \beta x(t) \geq -\alpha\dot{x}(t) + \beta x(t)$ and thus $\overset{-}{\bigwedge}(x) \neq \emptyset (\lambda = \dfrac{\beta}{\alpha} > 0)$. Again from (1) we have $0 > \dot{x}(t) - \alpha\dot{x}(t-h)$ and integrating from $t-h$ to t we get $\alpha x(t-h) - \alpha x(t-2h) > x(t) - x(t-h)$, i.e. $x(t) < (\alpha+1)x(t-h)$. This proves the lemma.

Lemma 5 a) If $x(t)$ is an I_0-function and $x(t) > Mx(t-h)$ for some $M, h > 0$ then the solution $\lambda_0 > 0$ of the equation $e^{-\lambda_0 h} = M$ is an upper bound of $\bigwedge^{+}(x)$;

b) If $x(t)$ is an I_∞-function and $x(t) < Mx(t-h)$ for some $M, h > 0$ then the solution $\lambda_0 > 0$ of the equation $e^{\lambda_0 h} = M$ is an upper bound of $\bigwedge^{-}(x)$.

The proof is the same as the proof of Lemma 5 [7].

3. Main results

THEOREM 1 Necessary and sufficient condition for existence at least one non-oscillatory solution of equation (1) is that its characteristic equation (CE) has a real root.

Proof. Sufficiency. Assume (CE) has a real root λ. Then we directly check that $x(t) = e^{\lambda t}$ is a solution of (1), which is non-oscillatory.

Necessity. Assume, conversely, that there exists a non-oscillatory solution of equation (1) and, for the sake of contradiction, (CE) has no real roots. Then $q(\lambda) \neq 0$ for any $\lambda \in R$. But $\lim_{\lambda\to\infty} q(\lambda) = \infty$ thus $\inf_{\lambda \in R} q(\lambda) = m > 0$ i.e. $q(\lambda) \geq m$ and $q(-\lambda) \geq m$ which is equivalent to

$$-\lambda + \alpha\lambda e^{-h\lambda} - \beta - \alpha\gamma e^{-h\lambda} \leq -m \tag{6}$$

and

$$\lambda - \alpha\lambda e^{h\lambda} - \beta - \alpha\gamma e^{h\lambda} \leq -m \tag{7}$$

According to Lemma 2 (1) also has a non-oscillatory solution $x(t)$ which is either I_0-function or I_∞-function. Without any loss of generality we can suppose $x(t) > 0$ for $t \geq t_1 \geq t_0$. We consider two cases:

a) The case where $x(t)$ is an I_0-function. According to Lemma 4 $\overset{+}{\bigwedge}(x) \neq \emptyset$ and $x(t) > \alpha x(t-h)$. Then according to Lemma 5 $\overset{+}{\bigwedge}(x)$ is bounded above by $\lambda_0 > 0$ non-depending on x. Let $\overline{\lambda} \in \overset{+}{\bigwedge}(x)$ and consider the function

$$u(t) = x(t) - \alpha x(t-h) + \int_t^\infty x(s)ds. \tag{8}$$

According to Lemma 1 $u(t)$ is a solution of (1) and according to Lemma 3 $u(t)$ is an I_0-function. Denote by $\varphi(t) = e^{\lambda t}x(t)$ for $\lambda \in \overset{+}{\bigwedge}(x)$ with $\lambda \geq \overline{\lambda}$. We will show $\lambda + m_1 \in \overset{+}{\bigwedge}(u)$ where $m_1 = m/N_1$ for $N_1 = 1 + \dfrac{1}{\lambda} > 0$. Since $\lambda \in \overset{+}{\bigwedge}(x)$ we obtain $\varphi(t)$ is non-increasing. Integrating $\dot{x}(t) + \lambda x(t) \leq 0$ from t to ∞ and using $\lim\limits_{t\to\infty} x(t) = 0$ we get $-x(t) + \lambda \displaystyle\int_t^\infty x(s)ds \leq 0$.

On the one hand side via (8), (1), definition of $\varphi(t)$ and (7) we get

$$\dot{u}(t) + \lambda u(t) = -\alpha\gamma x(t-h) - \beta x(t) - x(t) + \lambda \left[x(t) - \alpha x(t-h) + \int_t^\infty x(s)ds \right] \leq$$
$$\leq e^{-\lambda t}\varphi(t)\left[-\alpha\gamma e^{\lambda h} - \beta + \lambda - \lambda\alpha e^{\lambda h}\right] \leq -mx(t).$$

On the other hand side from (8) in view of definition of $\varphi(t)$ we have

$$u(t) \leq e^{-\lambda t} - \frac{1}{\lambda}\left[e^{-\lambda s}\varphi(s)|_t^\infty - \int_t^\infty e^{-\lambda s}\dot{\varphi}(s)ds \right] <$$
$$< e^{-\lambda t}\varphi(t)\left(1 + \frac{1}{\lambda}\right) < N_1 x(t).$$

Then

$$\dot{u}(t) + (\lambda + m_1)u(t) \leq -mx(t) + m_1 N_1 x(t) = (-m + m_1 N_1)x(t) = 0$$

i.e. $\quad \dot{u}(t) + (\lambda + m_1)u(t) \leq 0$ and hence $\lambda + m_1 \in \overset{+}{\bigwedge}(u)$.

Now set $x_0 \equiv x, x_1 = Tx_0 = u, x_2 = Tx_1, \ldots, x_n = Tx_{n-1}, n \in N$. Thus $\overset{+}{\bigwedge}(x) \equiv \overset{+}{\bigwedge}(x_0)$ and for $\lambda \in \overset{+}{\bigwedge}(x_0)$ we have $\lambda + m_1 \in \overset{+}{\bigwedge}(x_1)$ and after n steps we obtain

$\lambda + nm_1 \in \overset{+}{\bigwedge}(x_n), n \in N$,

which is a contradiction since λ_0 is a common upper bound for all $\overset{+}{\bigwedge}(x_n)$. This completes the proof in case a).

b) The case where $x(t)$ is an I_∞-function. According to Lemma 4 $\overset{-}{\bigwedge}(x) \neq \emptyset$ and $x(t) < (\alpha+1)x(t-h)$. Then according to Lemma 5 $\overset{-}{\bigwedge}(x)$ is bounded above by $\lambda_0 > 0$ non-depending on x. Let $\underline{\lambda} \in \overset{-}{\bigwedge}(x)$. Consider the function

$$\nu(t) = -x(t) + \alpha x(t-h) + \int_{t_0}^{t-h} x(s)ds, t \geq t_1 \geq t_0 + h. \tag{9}$$

According to Lemma 1 $\nu(t)$ is a solution of (1) and according to Lemma 3 $\nu(t)$ is an

I_∞-function. Denote by $\varphi(t) = e^{-\lambda t}x(t)$, for $\lambda \in \bar{\bigwedge}(x)$ and $\lambda \geq \underline{\lambda}$. We will show $\lambda + m_2 \in \bar{\bigwedge}(\nu)$ where $m_2 = \dfrac{m}{N_2}$ for $N_2 = \left(\alpha + \dfrac{1}{\underline{\lambda}}\right) e^{-\underline{\lambda} h}$.

Since $\lambda \in \bar{\bigwedge}(x)$ we obtain $\varphi(t)$ is non-decreasing. Integrating $-\dot{x}(t) + \lambda x(t) \leq 0$ from t_0 to $t-h$ we get $0 \geq -x(t-h) + x(t_0) + \lambda \int_{t_0}^{t-h} x(s)ds \geq -x(t-h) + \lambda \int_{t_0}^{t-h} x(s)ds$.

On the one hand side

$$\lambda \int_{t_0}^{t-h} e^{\lambda s}\varphi(s)ds = \int_{t_0}^{t-h} \varphi(s)de^{\lambda s} =$$

$$= e^{\lambda s}\varphi(s) \Big|_{t_0}^{t-h} - \int_{t_0}^{t-h} e^{\lambda s}\dot{\varphi}(s)ds < e^{\lambda t - \lambda h}\varphi(t-h)$$

and then

$\nu(t) = -e^{\lambda t}\varphi(t) + \alpha e^{\lambda t - \lambda h}\varphi(t-h) + \int_{t_0}^{t-h} e^{\lambda s}\varphi(s)ds \leq e^{\lambda t}\varphi(t-h)\left(\alpha e^{-\lambda h} + \dfrac{e^{-\lambda h}}{\lambda}\right) \leq N_2 e^{\lambda t}\varphi(t-h)$.

On the other hand side

$$-\dot{\nu}(t) + \lambda\nu(t) = -\alpha\gamma x(t-h) - \beta x(t) - x(t-h) - \lambda\left[x(t) - \alpha x(t-h) - \int_{t_0}^{t-h} x(s)ds\right] \leq$$

$$\leq -(\alpha\gamma - \alpha\lambda)e^{\lambda t - \lambda h}\varphi(t-h) - (\beta + \lambda)e^{\lambda t}\varphi(t) \leq$$

$$\leq e^{\lambda t}\varphi(t-h)\left[-\alpha\gamma e^{-\lambda h} - \beta - \lambda + \lambda\alpha e^{-\lambda h}\right] \leq$$

$$\leq -me^{\lambda t}\varphi(t-h)$$

via (9), (1), definition of $\varphi(t)$ and (6). Then

$$-\dot{\nu}(t) + (\lambda + m_2)\nu(t) \leq -me^{\lambda t}\varphi(t-h) + m_2N_2e^{\lambda t}\varphi(t-h) = (-m + m_2N_2)e^{\lambda t}\varphi(t-h) = 0$$

i.e. $-\dot{\nu}(t) + (\lambda + m_2)\nu(t) \leq 0$ and hence $\lambda + m_2 \in \bar{\bigwedge}(\nu)$. As in the case a) we are led to contradiction.

The proof of the theorem is complete.

THEOREM 2 Necessary and sufficient condition for oscillation of all solutions of equation (1) is that its characteristic equation (CE) has no real roots.

Proof. Sufficiency. Assume (CE) has no real roots. If there exists a non-oscillatory solution of equation (1) then by Theorem1 we directly see (CE) has a real root, which is a contradiction.

Necessity. Let all solutions of equation (1) be oscillatory. Assume, conversely, that the characteristic equation (CE) has a real root. Then $x(t) = e^{\lambda t}$ is a solution of (1), which is non-oscillatory. This contradiction proves the theorem.

4. Conclusions and examples

Now we obtain sufficient conditions in terms of the coefficients and arguments only for the oscillation of solutions of (1). The advantage of working with such conditions rather than the characteristic equation (CE) is that the said conditions are explicit, while determining whether or not a real root to (CE) exists is a problem in itself.

COROLLARY 1 If $\beta + \alpha\gamma \geq \dfrac{1}{h} \ln \dfrac{1}{\alpha}$ then (1) has no non-oscillatory solution, which is an I_0-function.

Proof. Assume, conversely, that there exists a non-oscillatory solution $x(t)$ which is an I_0-function. Without any loss of generality we suppose $x(t) > 0$ for $t \geq t_1 \geq t_0$. According to the proof Lemma 4 $x(t) > \alpha x(t-h)$ and according to Lemma 5 $\lambda_0 = \dfrac{1}{h} \ln \dfrac{1}{\alpha}$ is an upper bound of the set $\overset{+}{\bigwedge}(x)$. Then $\lambda > \lambda_0$ for any $\lambda \in \overset{+}{\bigwedge}(x)$ which contradicts to the condition of Corollary 1 for $\lambda = \beta + \alpha\gamma \in \overset{+}{\bigwedge}(x)$. The corollary is just proved.

COROLLARY 2 If $\dfrac{\beta}{\alpha} \geq \dfrac{1}{h} \ln(1+\alpha)$ then (1) has no non-oscillatory solution, which is an I_∞-function.

The proof is similar to the proof of Corollary 1 and we omit it.

REMARK. It is easy to see that if there exists a bounded non-oscillatory solution of (1) this solution is an I_0-function and if there exists a unbounded non-oscillatory solution of (1) this solution is an I_∞-function. Consequently, if there do not exist non-oscillatory solutions of (1), which are either I_∞ - or I_0- functions, then all solutions of (1) oscillate.

EXAMPLE. Consider equation (LDDE) in the case where $z = 25$ ohms, $C = 10$ p.f., $g = .01$ mho., $h = 500$ n.s., $R = 5.4$ (3.5) ohms. Then $\beta = .003, \gamma = .005, \alpha = .648(.754)$.

Since $h(\beta + \alpha\gamma) = 3.12(3.36)$ and $\ln \dfrac{1}{\alpha} = .3186(.2820)$ the condition of Corollary 1 holds and according to this corollary equation (1) has no non-oscillatory solutions, which are I_0-functions. But $\dfrac{h\beta}{\alpha} = 2.3148(1.9893)$ and $\ln(1+\alpha) = .4994(.5618)$ and according to Corollary 2 equation (1) has no non-oscillatory solutions, which are I_∞-functions. So, according to Remark, all solutions of equation (1) are oscillatory, which confirmed the Brayton's result [3].

References

[1] R. K. Brayton, Nonlinear oscillations in a distributed network, Quart. Appl. Math. 24, No.4 (1967), 289-301.

[2] R. K. Brayton and R. A. Willoghby, On the numerical integration of a symmetric system of difference-differential equation of neutral type, J. Math. Anal. Appl., 18 (1967), 182-189.

[3] K. Farrell, Necessary and sufficient conditions for oscillation of neutral equations with real coefficients. J. Math. Anal. Appl., 140 (1989), 251-261.

[4] M. K. Grammatikopoulos, E. A. Grove and G. Ladas, Oscillations of first order neutral delay differential equations, J. Math. Anal. Appl., 120 (1986), 510-520.

[5] M. K. Grammatikopoulos, Y. G. Sficas and I. P. Stavroulakis, Necessary and sufficient conditions for oscillations of neutral equations with several coefficients. J. Differential Equations, 76, No.2 (1988), 294-311.

[6] M. K. Grammatikopoulos and I. P. Stavroulakis, Necessary and sufficient conditions for oscillations of neutral equations with deviating arguments, J. London Math. Soc. (2) 41 (1990), 244-260.

[7] E. A. Grove, G. Ladas and A. Meimaridou, A necessary and sufficient condition for the oscillation of neutral equations, J. Math. Anal. Appl., 126 (1987), 341-354.

[8] V.B.Kolmanovskij and V.R.Nosov, Stability and periodic conditions of controlled systems with retarded action, Nauka, Moskva, 1981 (in Rusiian).

[9] L. Liao and A.Chen, Necessary and sufficient conditions for oscillations of neutral delay differential equations, Ann. Differential Equations, 12, No.3 (1996), 297-303.

[10] Y.G. Sficas and I. P. Stavroulakis, Necessary and sufficient conditions for oscillations of neutral differential equations, J. Math. Anal. Appl., 123 (1987), 494-507.

[11] M. Slemrod and E. F. Infante, Asymptotic stability criteria for linear systems of difference-differential equations of neutral type and their discrete analogous, J. Math. Anal. Appl., 38 (1972), 399-415.

[12] Y. Zhou, Necessary and sufficient conditions for oscillation of neutral equations, J. Changshai Univ. Electr. Power Nat. Sci., ed.10, No.4 (1995), 352-356.