GRONWALL TYPE INEQUALITIES VIA SUBCONVEX SEQUENCES

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Abstract. The sequence \((a_n)_{n \geq 1}\) is subconvex if there exists a natural number \(p \geq 1\) such that
\[
a_{n+p} \leq \sum_{i=0}^{p-1} \alpha_i \cdot a_{n+i}, \quad \text{for all } n \geq 1,\]
where \(\alpha_i \in (0, 1)\), for \(i = 0, p-1\) and \(\sum_{i=0}^{p-1} \alpha_i \leq 1\). In the first part of this note we prove an abstract Gronwall inequality which is a generalization of theorem 4.1. from [12]. In the second part we give some applications and in the third part we give discrete analogous for one of the applications.

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1. An abstract Gronwall inequality

The sequence \((a_n)_{n \geq 1}\) is subconvex of order \(p\) if \(a_{n+p} \leq \sum_{i=0}^{p-1} \alpha_i \cdot a_{n+i}\), for all \(n \geq 1\), where \(\alpha_i \in (0, 1)\), for \(i = 0, p-1\) and \(\sum_{i=0}^{p-1} \alpha_i \leq 1\). A sequence \((a_n)_{n \geq 1}\) is subconvex if there exists \(p \geq 1\) such that the sequence is subconvex of order \(p\). The sequence \((a_n)_{n \geq 1}\) is a convex sequence if there exists a natural number \(p \geq 1\) such that \(a_{n+p} = \sum_{j=0}^{p-1} \alpha_j \cdot a_{n+j}, \forall n \geq 1\), where \(\alpha_i \in (0, 1)\), for \(i = 0, p-1\) and \(\sum_{i=0}^{p-1} \alpha_i = 1\). In [1] the author proved the following theorem:

Theorem 1.1. a) Every positive subconvex sequence is convergent.
b) The limit of the convex sequence \((a_n)_{n \geq 1}\) which satisfies the relations

\[
a_{n+p} = \sum_{j=0}^{p-1} \alpha_j \cdot a_{n+j}, \quad \forall \ n \geq 1,
\]

where \(\alpha_i \in (0,1)\) for \(i = 0, p-1\) and \(\sum_{i=0}^{p-1} \alpha_i = 1\), is

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = \frac{\sum_{j=0}^{p-1} \beta_j \cdot a_{j+1}}{\sum_{j=0}^{p-1} \beta_j},
\]

where \(\beta_k = \sum_{j=0}^{k} \alpha_j\), for \(0 \leq k \leq p - 1\).

These properties were used to prove some fixed point theorems from [4]. In this section we generalize the following theorem given by Rus [12]:

**Theorem 1.2.** If \(X\) is an ordered metric space and \(A : X \to X\) an increasing weakly Picard operator, then we have the following implications:

a) If \(x \in X\) and \(x \leq Ax\), then \(x \leq A^\infty x\);

b) If \(x \in X\) and \(x \geq Ax\), then \(x \geq A^\infty x\),

where \(A^\infty x = \lim_{n \to \infty} x_n\) and \(x_{n+1} = Ax_n\) with \(x_0 = x\).

Our main theorem is:

**Theorem 1.3.** If \(X\) is an ordered metric space and \(A : X \to X\) an increasing weakly Picard operator, then we have the following implications:

a) If \(x \in X\) and \(x \leq \sum_{i=0}^{p-1} \alpha_i \cdot A^{i+1} x\), then \(x \leq A^\infty x\);

b) If \(x \in X\) and \(x \geq \sum_{i=0}^{p-1} \alpha_i \cdot A^{i+1} x\), then \(x \geq A^\infty x\),

where \(A^\infty x\) is defined as in theorem (1.2) and \(\alpha_i \in (0,1)\), for \(i = 0, p-1\) with \(\sum_{i=0}^{p-1} \alpha_i = 1\).

**Proof.** We have the following inequalities:

\[
A^k x = \sum_{i=0}^{p-1} \alpha_i \cdot A^{k+i} x,
\]

for \(k \in \mathbb{N}\). Define the sequence \((a_n)_{n \geq -p+1}\) with the properties \(a_k = 0\) for \(k \in \{-p+1, -p+2, \ldots, -1\}\), \(a_0 = 1\) and \(a_{n+p} = \sum_{j=0}^{p-1} \alpha_j \cdot a_{n+j}, \forall n \geq -p + 1\). By multiplying the above inequalities with \(a_k\) for \(k = -p + 1, n\) and adding term by term
the obtained inequalities, we deduce
\[ x \leq \sum_{i=1}^{p} \gamma_i \cdot A^{n+p+i}x, \]
where \( \gamma_i = \sum_{k=1}^{p} \alpha_k \cdot a_{n+p+i-k} \). The right hand part is convergent to \( A^\infty x \cdot l \cdot \sum_{i=0}^{p-1} \beta_i \), where \( \beta_i = \sum_{k=1}^{p} \alpha_k \) and \( l \) is the limit of the sequence \((a_n)_{n \geq -p+1}\). Due to theorem (1.1) this limit exists and is equal to
\[ \sum_{j=-p+1}^{0} \beta_j \cdot \alpha_{j+1} = \frac{1}{p-1}, \]
so the assertion of theorem (1.3) follows. \( \square \)

**Remark 1.1.** An alternative solution is the following:

The operator \( \sum_{i=0}^{p-1} \alpha_i \cdot A^{i+1}x \) is also a weakly Picard operator and for fixed \( x \) the sequences of successive approximation \( x_{n+1} = Ax_n \) with \( x_0 = x \) and \( y_{n+1} = \sum_{i=0}^{p-1} \alpha_i \cdot A^{i+1}y_n \) with \( y_0 = x \), has the same limit so the abstract Gronwall inequality of theorem (4.1) from [12] implies the required inequality.

**Remark 1.2.** If \( \alpha_1 = 1 \) and \( \alpha_i = 0 \) for \( i = 2, p-1 \) we obtain theorem (4.1) from [12] (the abstract Gronwall inequality).

**Remark 1.3.** Theorem (1.2) is different from theorem (4.1) of [12] because the inequality \( x \leq \sum_{i=0}^{p-1} \alpha_i \cdot A^{i+1}x \) doesn’t imply the inequality \( x \leq Ax \).

### 2. Applications

**Theorem 2.1.** If \( K : [a, b] \times \mathbb{R} \to \mathbb{R} \) is a continuous and positive function, \( \alpha, \beta, \alpha_1, \alpha_2 \) are positive constants and \( \alpha_1 + \alpha_2 = 1 \) then the inequality

\[ y(x) \leq \alpha + \alpha_1\beta \int_{a}^{x} K(x, s)y(s)ds + \alpha_2\beta^2 \int_{a}^{x} K_2(x, s)y(s)ds + \alpha_2\alpha\beta \int_{a}^{x} K(x, s)ds \]
implies \( y(x) \leq y^*(x) \), \( \forall \, x \in [a, b] \), where \( K_2(x, s) = \int_s^x K(x, t)K(t, s)dt \) and \( y^* \) is the unique continuous solution of the equation \( y(x) = \alpha + \beta \int_a^x K(x, s)y(s)ds \).

Proof. We consider the space of continuous functions \( X = C[a, b] \) and the operator \( A : X \to X \) defined by \((Ay)(x) = \alpha + \beta \int_a^x K(x, s)y(s)ds\). Due to the given conditions this operator is an increasing Picard operator and

\[
\begin{align*}
\alpha_1 \cdot Ax + \alpha_2 \cdot A^2x &= \alpha_1 \cdot \left( \alpha + \beta \int_a^x K(x, s)y(s)ds \right) + \\
&+ \alpha_2 \left( \alpha + \beta \int_a^x K(x, s) \left( \alpha + \beta \int_a^x K(s, t)y(t)dt \right) ds \right) = \\
&= \alpha + \alpha_1 \beta \int_a^x K(x, s)y(s)ds + \alpha_2 \beta^2 \int_a^x K_2(x, s)y(s)ds + \alpha_2 \alpha \beta \int_a^x K(x, s)ds.
\end{align*}
\]

From theorem (1.3) we deduce the required inequality. \( \Box \)

**Theorem 2.2.** If \( K_{1,2} : [a, b] \times \mathbb{R} \to \mathbb{R} \) are continuous and positive functions, and they satisfy the conditions of theorem (2) from [1], \( \alpha, \beta, \alpha_1, \alpha_2 \) are positive constants and \( \alpha_1 + \alpha_2^2 = 1 \) then the inequality

\[
y(x) \leq \alpha + \alpha_1 \beta \left( \int_a^x K(x, s)y(s)ds + \int_a^b K_2(x, s)y(s)ds \right) + \\
+ \beta \alpha \alpha_2 \left( \int_a^x K(x, s)ds + \int_a^b K_2(x, s)ds \right) + \\
+ \alpha_2 \beta^2 \left( \int_a^x K_1^{(2)}(x, s)y(s)ds + \int_a^b K_2^{(2)}(x, s)ds \right)
\]

implies \( y(x) \leq y^*(x) \), \( \forall \, x \in [a, b] \), where

\[
\begin{align*}
K_1^{(2)}(x, s) &= \int_s^x K_1(x, t)K_1(t, s)dt + \int_a^b K_2(x, t)K_1(x, t)dt, \\
K_2^{(2)}(x, s) &= \int_s^x K_1(x, t)K_2(t, s)dt + \int_a^b K_2(x, t)K_2(x, t)dt
\end{align*}
\]
and \( y^*(x) \) is the unique solution of the equation

\[
y(x) = \alpha + \beta \int_a^x K_1(x, s)y(s)ds + \beta \int_a^b K_2(x, s)y(s)ds.
\]

**Proof.** Consider the operator \( A : X \rightarrow X \) defined by

\[
(Ay)(x) = \alpha + \beta \int_a^x K_1(x, s)y(s)ds + \beta \int_a^b K_2(x, s)y(s)ds.
\]

Due to the given conditions this operator is an increasing Picard operator and

\[
\alpha_1 \cdot Ax + \alpha_2 \cdot A^2 x = \alpha_1 \cdot \left( \alpha + \beta \int_a^x K_1(x, s)y(s)ds + \beta \int_a^b K_2(x, s)y(s)ds \right) +
\]

\[
+ \alpha_2 \left( \alpha + \alpha \beta \left( \int_a^x K_1(x, s)ds + \beta \int_a^b K_2(x, s)y(s)ds \right) \right) +
\]

\[
+ \beta^2 \left( \int_a^x K_1^{(2)}(x, s)y(s)ds + \beta \int_a^b K_2^{(2)}(x, s)y(s)ds \right) =
\]

\[
= \alpha + \alpha_1 \beta \left( \int_a^x K(x, s)y(s)ds + \beta \int_a^b K_2(x, s)y(s)ds \right) +
\]

\[
+ \beta \alpha \alpha_2 \left( \int_a^x K(x, s)ds + \beta \int_a^b K_2(x, s)ds \right) +
\]

\[
+ \alpha_2 \beta^2 \left( \int_a^x K_1^{(2)}(x, s)y(s)ds + \beta \int_a^b K_2^{(2)}(x, s)ds \right),
\]

where

\[
K_1^{(2)}(x, s) = \int_s^x K_1(x, t)K_1(t, s)dt + \int_a^b K_2(x, t)K_1(x, t)dt,
\]

\[
K_2^{(2)}(x, s) = \int_a^x K_1(x, t)K_2(t, s)dt + \int_a^b K_2(x, t)K_2(x, t)dt.
\]
3. A discrete analogous

Theorem 3.1. If the terms of the sequences \((a_k)_{k \geq 1}\) and \((b_k)_{k \geq 1}\) are positive numbers and they satisfy the following inequality:

\[
a_n \leq \alpha + \frac{1}{2} \sum_{j=1}^{n-1} b_j a_j + \frac{\alpha}{2} \sum_{j=1}^{n-1} b_j + \frac{1}{2} \sum_{k=1}^{n-1} \sum_{j=k}^{n-1} b_j b_k a_k
\]

then we have \(a_n \leq \alpha \prod_{k=1}^{n-1} \left(1 + b_k + \frac{b_k^2}{2}\right)\).

Proof. From the given inequality we have \(a_1 \leq \alpha\) and \(a_2 \leq \alpha \left(1 + b_1 + \frac{b_1^2}{2}\right)\). For \(n = 3\) we have

\[
a_3 \leq \alpha + \frac{b_1 a_1}{2} + \frac{b_2 a_1}{2} + \frac{b_1}{2} + \frac{b_2}{2} + \frac{b_1^2 a_1}{2} + \frac{b_2^2 a_1}{2} + \frac{b_1 b_2 a_1}{2} + \frac{b_2^2 a_1}{2} \\
\leq \alpha \left(1 + b_1 + \frac{b_1^2}{2}\right) \left(1 + b_2 + \frac{b_2^2}{2}\right).
\]

The general case follows by induction on \(n\) as the above case. \(\Box\)

Remark 3.1. This inequality is a discrete analogous of theorem (2.1) for \(\alpha_1 = \alpha_2 = \frac{1}{2}\). We give this case for the simplicity of the proof. The general case can also be treated.

References