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GRONWALL TYPE INEQUALITIES VIA SUBCONVEX SEQUENCES

SZILÁRD ANDRÁS

Department of Applied Mathematics
Babeş-Bolyai University, Cluj-Napoca, Romania
E-mail: andrasz@math.ubbcluj.ro

Abstract. The sequence $(a_n)_{n\geq 1}$ is subconvex if there exists a natural number $p \geq 1$ such that $a_{n+p} \leq \sum_{i=0}^{p-1} \alpha_i \cdot a_{n+i}$, for all $n \geq 1$, where $\alpha_i \in (0,1)$, for $i = \overline{0, p-1}$ and $\sum_{i=0}^{p-1} \alpha_i \leq 1$. In the first part of this note we prove an abstract Gronwall type inequality which is a generalization of theorem 4.1. from [12]. In the second part we give some applications and in the third part we give discrete analogous for one of the applications.

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1. An abstract Gronwall inequality

The sequence $(a_n)_{n\geq 1}$ is subconvex of order p if $a_{n+p} \leq \sum_{i=0}^{p-1} \alpha_i \cdot a_{n+i}$, for all $n \geq 1$, where $a_i \in (0,1)$, for $i = \overline{0, p-1}$ and $\sum_{i=0}^{p-1} \alpha_i \leq 1$. A sequence $(a_n)_{n\geq 1}$ is subconvex if there exists $p \geq 1$ such that the sequence is subconvex of order p. The sequence $(a_n)_{n\geq 1}$ is a convex sequence if there exists a natural number $p \geq 1$ such that $a_{n+p} = \sum_{j=0}^{p-1} \alpha_j \cdot a_{n+j}, \forall\, n \geq 1$, where $\alpha_i \in (0,1)$, for $i = \overline{0, p-1}$ and $\sum_{i=0}^{p-1} \alpha_i = 1$. In [1] the author proved the following theorem:

Theorem 1.1. *a) Every positive subconvex sequence is convergent.*

b) The limit of the convex sequence $(a_n)_{n\geq 1}$ *which satisfies the relations* $a_{n+p} = \sum_{j=0}^{p-1} \alpha_j \cdot a_{n+j},\ \forall\ n \geq 1,$ *where* $\alpha_i \in (0,1)$ *for* $i = \overline{0, p-1}$ *and* $\sum_{i=0}^{p-1} \alpha_i = 1,$ *is*

$$\lim_{n\to\infty} a_n = \frac{\lim\limits_{n\to\infty} c_n}{\sum\limits_{j=0}^{p-1} \beta_j} = \frac{\sum\limits_{j=0}^{p-1} \beta_j \cdot a_{j+1}}{\sum\limits_{j=0}^{p-1} \beta_j},$$

where $\beta_k = \sum_{j=0}^{k} \alpha_j$*, for* $0 \leq k \leq p-1.$

These properties were used to prove some fixed point theorems from [4]. In this section we generalize the following theorem given by Rus [12]:

Theorem 1.2. *If* X *is an ordered metric space and* $A : X \to X$ *an increasing weakly Picard operator, then we have the following implications:*

a) If $x \in X$ *and* $x \leq Ax$*, then* $x \leq A^\infty x$*;*

b) If $x \in X$ *and* $x \geq Ax$*, then* $x \geq A^\infty x$*,*

where $A^\infty x = \lim\limits_{n\to\infty} x_n$ *and* $x_{n+1} = Ax_n$ *with* $x_0 = x$*.*

Our main theorem is:

Theorem 1.3. *If* X *is an ordered metric space and* $A : X \to X$ *an increasing weakly Picard operator, then we have the following implications:*

a) If $x \in X$ *and* $x \leq \sum_{i=0}^{p-1} \alpha_i \cdot A^{i+1}x$*, then* $x \leq A^\infty x$*;*

b) If $x \in X$ *and* $x \geq \sum_{i=0}^{p-1} \alpha_i \cdot A^{i+1}x$*, then* $x \geq A^\infty x$*,*

where $A^\infty x$ *is defined as in theorem (1.2) and* $\alpha_i \in (0,1)$*, for* $i = \overline{0, p-1}$ *with* $\sum_{i=0}^{p-1} \alpha_i = 1.$

Proof. We have the following inequalities:

$$A^k x \leq \sum_{i=0}^{p-1} \alpha_i \cdot A^{k+i+1} x,$$

for $k \in \mathbb{N}$. Define the sequence $(a_n)_{n\geq -p+1}$ with the properties $a_k = 0$ for $k \in \{-p+1, -p+2, \ldots, -1\},\ a_0 = 1$ and $a_{n+p} = \sum_{j=0}^{p-1} \alpha_j \cdot a_{n+j},\ \forall\ n \geq -p+1$. By multiplying the above inequalities with a_k for $k = \overline{-p+1, n}$ and adding term by term the obtained inequalities, we deduce

$$x \le \sum_{i=1}^{p} \gamma_i \cdot A^{n+p+i} x,$$

where $\gamma_i = \sum_{k=i}^{p-1} \alpha_k \cdot a_{n+p+i-k}$. The right hand part is convergent to $A^{\infty} x \cdot l \cdot \sum_{i=0}^{p-1} \beta_i$, where $\beta_i = \sum_{k=i}^{p-1} \alpha_k$ and l is the limit of the sequence $(a_n)_{n \ge -p+1}$. Due to theorem (1.1) this limit exists and is equal to

$$\frac{\sum_{j=-p+1}^{0} \beta_j \cdot \alpha_{j+1}}{\sum_{j=0}^{p-1} \beta_j} = \frac{1}{\sum_{j=0}^{p-1} \beta_j},$$

so the assertion of theorem (1.3) follows. □

Remark 1.1. *An alternative solution is the following:*

The operator $\sum_{ii=0}^{p-1} \alpha_i \cdot A^{i+1} x$ *is also a weakly Picard operator and for fixed* x *the sequences of succesive approximation* $x_{n+1} = Ax_n$ *with* $x_0 = x$ *and* $y_{n+1} = \sum_{i=0}^{p-1} \alpha_i \cdot A^{i+1} y_n$ *with* $y_0 = x$*, has the same limit so the abstract Gronwall inequality of theorem (4.1) from [12] implies the required inequality.*

Remark 1.2. *If* $\alpha_1 = 1$ *and* $\alpha_i = 0$ *for* $i = \overline{2, p-1}$ *we obtain theorem (4.1) from [12] (the abstract Gronwall inequality).*

Remark 1.3. *Theorem (1.2) is different from theorem (4.1) of [12] because the inequality* $x \le \sum_{i=0}^{p-1} \alpha_i \cdot A^{i+1} x$ *doesn't imply the inequality* $x \le Ax$.

2. Applications

Theorem 2.1. *If* $K : [a, b] \times \mathbb{R} \to \mathbb{R}$ *is a continuous and positive function,* $\alpha, \beta, \alpha_1, \alpha_2$ *are positive constants and* $\alpha_1 + \alpha_2 = 1$ *then the inequality*

$$y(x) \le \alpha + \alpha_1 \beta \int_a^x K(x, s) y(s) ds + \alpha_2 \beta^2 \int_a^x K_2(x, s) y(s) ds + \alpha_2 \alpha \beta \int_a^x K(x, s) ds$$

implies $y(x) \leq y^*(x),\ \forall\ x \in [a,b]$, *where* $K_2(x,s) = \int\limits_s^x K(x,t)K(t,s)dt$ *and* y^* *is the unique continuous solution of the equation* $y(x) = \alpha + \beta \int\limits_a^x K(x,s)y(s)ds$.

Proof. We consider the space of continuous functions $X = C[a,b]$ and the operator $A : X \to X$ defined by $(Ay)(x) = \alpha + \beta \int\limits_a^x K(x,s)y(s)ds$. Due to the given conditions this operator is an increasing Picard operator and

$$\alpha_1 \cdot Ax + \alpha_2 \cdot A^2 x = \alpha_1 \cdot \left(\alpha + \beta \int\limits_a^x K(x,s)y(s)ds \right) +$$

$$+\alpha_2 \left(\alpha + \beta \int\limits_a^x K(x,s) \left(\alpha + \beta \int\limits_a^x K(s,t)y(t)dt \right) ds \right) =$$

$$= \alpha + \alpha_1\beta \int\limits_a^x K(x,s)y(s)ds + \alpha_2\beta^2 \int\limits_a^x K_2(x,s)y(s)ds + \alpha_2\alpha\beta \int\limits_a^x K(x,s)ds.$$

From theorem(1.3) we deduce the required inequality. □

Theorem 2.2. *If* $K_{1,2} : [a,b] \times \mathbb{R} \to \mathbb{R}$ *are continuous and positive functions, and they satisfy the conditions of theorem (2) from[1],* $\alpha, \beta, \alpha_1, \alpha_2$ *are positive constants and* $\alpha_1 + \alpha_2 2 = 1$ *then the inequality*

$$y(x) \leq \alpha + \alpha_1\beta \left(\int\limits_a^x K(x,s)y(s)ds + \int\limits_a^b K_2(x,s)y(s)ds \right) +$$

$$+\beta\alpha\alpha_2 \left(\int\limits_a^x K(x,s)ds + \int\limits_a^b K_2(x,s)ds \right) +$$

$$+\alpha_2\beta^2 \left(\int\limits_a^x K_1^{(2)}(x,s)y(s)ds + \int\limits_a^b K_2^{(2)}(x,s)ds \right)$$

implies $y(x) \leq y^*(x),\ \forall\ x \in [a,b]$, *where*

$$K_1^{(2)}(x,s) = \int\limits_s^x K_1(x,t)K_1(t,s)dt + \int\limits_a^b K_2(x,t)K_1(x,t)dt,$$

$$K_2^{(2)}(x,s) = \int\limits_a^x K_1(x,t)K_2(t,s)dt + \int\limits_a^b K_2(x,t)K_2(x,t)dt$$

and $y^*(x)$ is the unique solution of the equation

$$y(x) = \alpha + \beta \int_a^x K_1(x,s)y(s)ds + \beta \int_a^b K_2(x,s)y(s)ds.$$

Proof. Consider the operator $A : X \to X$ defined by

$$(Ay)(x) = \alpha + \beta \int_a^x K_1(x,s)y(s)ds + \beta \int_a^b K_2(x,s)y(s)ds.$$

Due to the given conditions this operator is an increasing Picard operator and

$$\alpha_1 \cdot Ax + \alpha_2 \cdot A^2 x = \alpha_1 \cdot \left(\alpha + \beta \int_a^x K_1(x,s)y(s)ds + \beta \int_a^b K_2(x,s)y(s)ds \right) +$$

$$+\alpha_2 \left(\alpha + \alpha\beta \left(\int_a^x K_1(x,s)ds + \int_a^b K_2(x,s)y(s)ds \right) + \right.$$

$$\left. +\beta^2 \left(\int_a^x K_1^{(2)}(x,s)y(s)ds + \int_a^b K_2^{(2)}(x,s)y(s)ds \right) \right) =$$

$$= \alpha + \alpha_1\beta \left(\int_a^x K(x,s)y(s)ds + \int_a^b K_2(x,s)y(s)ds \right) +$$

$$+\beta\alpha\alpha_2 \left(\int_a^x K(x,s)ds + \int_a^b K_2(x,s)ds \right) +$$

$$+\alpha_2\beta^2 \left(\int_a^x K_1^{(2)}(x,s)y(s)ds + \int_a^b K_2^{(2)}(x,s)ds \right),$$

where

$$K_1^{(2)}(x,s) = \int_s^x K_1(x,t)K_1(t,s)dt + \int_a^b K_2(x,t)K_1(x,t)dt,$$

$$K_2^{(2)}(x,s) = \int_a^x K_1(x,t)K_2(t,s)dt + \int_a^b K_2(x,t)K_2(x,t)dt.$$

□

3. A discrete analogous

Theorem 3.1. *If the terms of the sequences* $(a_k)_{k\geq 1}$ *and* $(b_k)_{k\geq 1}$ *are positive numbers and they satisfy the following inequality:*

$$a_n \leq \alpha + \frac{1}{2}\sum_{j=1}^{n-1} b_j a_j + \frac{a}{2}\sum_{j=1}^{n-1} b_j + \frac{1}{2}\sum_{k=1}^{n-1}\sum_{j=k}^{n-1} b_j b_k a_k$$

then we have $a_n \leq \alpha \prod_{k=1}^{n-1}\left(1 + b_k + \frac{b_k^2}{2}\right)$.

Proof. From the given inequality we have $a_1 \leq \alpha$ and $a_2 \leq \alpha\left(1 + b_1 + \frac{b_1^2}{2}\right)$. For $n = 3$ we have

$$a_3 \leq \alpha + \frac{b_1 a_1}{2} + \frac{b_2 a_1}{2} + \alpha\frac{b_1}{2} + \alpha\frac{b_2}{2} + \frac{b_1^2 a_1}{2} + \frac{b_1 b_2 a_1}{2} + \frac{b_2^2 a_1}{2} \leq$$
$$\leq \alpha\left(1 + b_1 + \frac{b_1^2}{2}\right)\left(1 + b_2 + \frac{b_2^2}{2}\right).$$

The general case follows by induction on n as the above case. □

Remark 3.1. *This inequality is a discrete analogous of theorem (2.1) for* $\alpha_1 = \alpha_2 = \frac{1}{2}$. *We give this case for the simplicity of the proof. The general case can also be treated.*

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