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DATA DEPENDENCE OF THE FIXED POINTS OF SOME PICARD OPERATORS

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Abstract. The purpose of this paper is to give a data dependence theorem of the fixed point for a Picard operator. We also study the data dependence of the unique solution for a nonlinear integral equation.

Key words: fixed point, Picard operator, data dependence, integral equation.
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1 Introduction

Let X be a nonempty set and $f : X \to X$ be an operator. We denote by F_f the fixed points set of f.

Definition 1.1 (Rus [2], [3]) *Let (X, d) be a metric space. An operator $f : X \to X$ is a* Picard operator (*briefly* P. o.) *iff there exists $x^* \in X$ such that $F_f = \{x^*\}$ and $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x^*, for all $x_0 \in X$.*

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. We consider the following conditions:

(i_φ) φ is monotone increasing;

(ii_φ) $\varphi(t) < t$, for each $t > 0$;

(iii_φ) φ is right continuous;

(iv_φ) $\lim_{t\to+\infty}(t - \varphi(t)) = +\infty$.

We denote by ϕ the class of all functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy the conditions (i_φ), (ii_φ) and (iii_φ) and we denote by Φ the class of all continuous functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy the conditions (i_φ), (ii_φ) and (iv_φ).

For a function $\varphi \in \Phi$ and for $\eta > 0$ we put

$$\varphi_\eta := \sup \{ \ t \mid t \in \mathbb{R}_+, \ t - \varphi(t) \le \eta \ \}.$$

It is clear that $\varphi_\eta \to 0$, as $\eta \searrow 0$.

Example 1.1 *Let* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$*, defined by* $\varphi(t) = at$*, for each* $t \in \mathbb{R}_+$*, where* $a \in]0,1[$*. Then* $\varphi \in \Phi$*.*

Example 1.2 *Let* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$*, defined by* $\varphi(t) = t/(t+1)$*, for each* $t \in \mathbb{R}_+$*. Then* $\varphi \in \Phi$*.*

Example 1.3 ([1]) *Let* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$*, defined by* $\varphi(t) = \ln(t+1)$*, for each* $t \in \mathbb{R}_+$*. Then* $\varphi \in \Phi$*.*

We denote by $\mathcal{C}([a,b],\mathbb{R}^n)$ the Banach space of the continuous functions $u : [a,b] \to \mathbb{R}^n$, equipped with Cebyshev's norm $\|u\|_C = \sup_{t\in[a,b]} \|u(t)\|$.

2 Data dependence of the fixed point for some Picard operators

Theorem 2.1 (Rus [3]) *Let* (X,d) *be a complete metric space,* $\varphi \in \phi$ *and* $f : X \to X$ *be an operator. We suppose that*

$$d(f(x), f(y)) \le \varphi(d(x,y)),$$

for each $x, y \in X$*.*

Then f *is P. o..*

Further on we shall give a data dependence theorem for the unique fixed point of such a Picard operator.

Theorem 2.2 *Let* (X,d) *be a complete metric space,* $\varphi_1, \varphi_2 \in \Phi$ *and* $f_1, f_2 : X \to X$ *be two operators. We suppose that*:

(i) $d(f_i(x), f_i(y)) \le \varphi_i(d(x,y))$*, for each* $x, y \in X$*,* $i \in \{1,2\}$*;*

(ii) *there exists* $\eta > 0$ *such that*

$$d(f_1(x), f_2(x)) \le \eta,$$

for each $x \in X$*.*

Then

$$d(x_1^*, x_2^*) \le \min\{\varphi_{1\eta}, \varphi_{2\eta}\},$$

where x_i^* *is the unique fixed point of* f_i*,* $i \in \{1,2\}$*.*

Proof. Let $i, j \in \{1,2\}$, with $i \ne j$. We have

$$d(x_i^*, x_j^*) = d(f_i(x_i^*), f_j(x_j^*)) \le$$

$$\le d(f_i(x_i^*), f_i(x_j^*)) + d(f_i(x_j^*), f_j(x_j^*)) \le \varphi_i(d(x_i^*, x_j^*)) + \eta$$

and from this we get that

$$d(x_i^*, x_j^*) - \varphi_i(d(x_i^*, x_j^*)) \le \eta.$$

Hence $d(x_i^*, x_j^*) \le \varphi_{i\eta}$.

Therefore $d(x_1^*, x_2^*) \le \min\{\varphi_{1\eta}, \varphi_{2\eta}\}$. ■

3 Applications

In what follows we shall prove again the Theorem 2 given by Constantin in [1], using the Theorem 2.1. Moreover, the conditions imposed on the function φ are weaker than those from the Theorem 2 in [1].

Theorem 3.1 *We consider the integral equation*

$$x(t) = \int_a^b K(t, s, x(s))\ ds + f(t),\ t \in [a, b] \tag{3.1}$$

and we suppose that:

(i) $f \in \mathcal{C}([a, b], \mathbb{R}^n)$;

(ii) $K : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous*;

(iii) *there exists* $\varphi \in \phi$ *such that*

$$\|K(t, s, u) - K(t, s, v)\| \le l(t, s)\ \varphi(\|u - v\|),$$

for each $t, s \in [a, b]$ *and for every* $u, v \in \mathbb{R}^n$, *where* $l(t, \cdot) \in L^1[a, b]$, *for each* $t \in [a, b]$ *and* $\sup_{t\in[a,b]} \int_a^b l(t, s)\ ds \le 1$.

Then the integral equation (3.1) *has a unique solution in* $\mathcal{C}([a, b], \mathbb{R}^n)$.

Proof. We consider the operator $A : \mathcal{C}([a, b], \mathbb{R}^n) \to \mathcal{C}([a, b], \mathbb{R}^n)$, defined by

$$A(x)(t) = \int_a^b K(t, s, x(s))\ ds + f(t),\ t \in [a, b].$$

Let $x, y \in \mathcal{C}([a, b], \mathbb{R}^n)$. We have

$$\|A(x)(t) - A(y)(t)\| \le \int_a^b \|K(t, s, x(s)) - K(t, s, y(s))\|\ ds \le$$

$$\le \int_a^b l(t, s)\ \varphi(\|x(s) - y(s)\|)\ ds \le \varphi(\|x - y\|_C)\ \int_a^b l(t, s)\ ds,$$

for each $t \in [a, b]$. It follows that

$$\|A(x) - A(y)\|_C \le \varphi(\|x - y\|_C).$$

From the Theorem 2.1 we get that $F_A = \{x^*\}$. This x^* is the unique solution of the integral equation (3.1). ■

Remark 3.1 *Let* $x_0 \in \mathcal{C}([a, b], \mathbb{R}^n)$. *Taking into account the fact that the operator* A *defined in the proof of the Theorem* 3.1 *is a P. o., it follows that the sequence* $(x_n)_{n\in\mathbb{N}}$, *defined by*

$$x_{n+1}(t) = \int_a^b K(t, s, x_n(s))\ ds + f(t),\ t \in [a, b],$$

for each $n \in \mathbb{N}$, *converges uniformly to the unique solution* x^* *of the integral equation* (3.1).

In the Theorem 2 from [1], Constantin presents a qualitative result relative to the stability of the unique solution of the integral equation (3.1) to small perturbations of the free term.

Further on we shall prove a data dependence theorem on K and f, of the unique solution x^* of the integral equation (3.1).

Theorem 3.2 *We consider the integral equations*

$$x(t) = \int_a^b K_1(t, s, x(s))\ ds + f_1(t),\ \ t \in [a, b] \tag{3.2}$$

and

$$x(t) = \int_a^b K_2(t, s, x(s))\ ds + f_2(t),\ \ t \in [a, b]. \tag{3.3}$$

We suppose that:

(i) $f_1, f_2 \in \mathcal{C}([a, b], \mathbb{R}^n)$;

(ii) $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ *are continuous*;

(iii) *there exist* $\varphi_1, \varphi_2 \in \Phi$ *such that*

$$\|K_i(t, s, u) - K_i(t, s, v)\| \le l_i(t, s)\ \varphi_i(\|u - v\|),$$

for each $t, s \in [a, b]$ *and for every* $u, v \in \mathbb{R}^n$, *where* $l_i(t, \cdot) \in L^1[a, b]$, *for each* $t \in [a, b]$ *and* $\sup_{t\in[a,b]} \int_a^b l_i(t, s)\ ds \le 1$, $i \in \{1, 2\}$;

(iv) *there exists* $\eta_1 > 0$ *such that*

$$\|K_1(t, s, u) - K_2(t, s, u)\| \le \eta_1,$$

for each $t, s \in [a, b]$ *and for every* $u \in \mathbb{R}^n$;

(v) *there exists* $\eta_2 > 0$ *such that*

$$\|f_1(t) - f_2(t)\| \le \eta_2,$$

for each $t \in [a, b]$.

Then

$$d(x^*_A, x^*_B) \le \min\ \{\varphi_{1\eta}, \varphi_{2\eta}\},$$

where x^*_A *is the unique solution of the integral equation* (3.2), x^*_B *is the unique solution of the integral equation* (3.3) *and* $\eta = \eta_1\ (b - a) + \eta_2$.

Proof. We consider the operators $A, B : \mathcal{C}([a, b], \mathbb{R}^n) \to \mathcal{C}([a, b], \mathbb{R}^n)$, defined by

$$A(x)(t) = \int_a^b K_1(t, s, x(s))\ ds + f_1(t),\ \ t \in [a, b]$$

and

$$B(x)(t) = \int_a^b K_2(t, s, x(s))\ ds + f_2(t),\ t \in [a, b].$$

From the proof of the Theorem 3.1 we have that

$$\|A(x) - A(y)\|_C \leq \varphi_1(\|x - y\|_C),$$

for each $x, y \in \mathcal{C}([a, b], \mathbb{R}^n)$ and

$$\|B(x) - B(y)\|_C \leq \varphi_2(\|x - y\|_C),$$

for each $x, y \in \mathcal{C}([a, b], \mathbb{R}^n)$.

We also get that $F_A = \{x_A^*\}$ and $F_B = \{x_B^*\}$.

Let $x \in \mathcal{C}([a, b], \mathbb{R}^n)$ and $t \in [a, b]$. Then, we have

$$\|A(x)(t) - B(x)(t)\| \leq \int_a^b \|K_1(t, s, x(s)) - K_2(t, s, x(s))\|\ ds + \|f_1(t) - f_2(t)\| \leq$$

$$\leq \eta_1\ (b - a) + \eta_2 = \eta.$$

Using the Theorem 2.2, it follows the conclusion. ■

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