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GLOBAL ASYMPTOTIC STABILITY FOR SOME DIFFERENCE EQUATIONS VIA FIXED POINT TECHNIQUE

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Abstract. In this paper we study the global asymptotic stability for some difference equations using the fixed point theory and Picard operators.
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1 Introduction

In the last 25 years the Difference Equations Theory has been developed very much. An important problem in the study of difference equations is the problem of the global stability and the oscillatory nature of the solutions. Many results in this area has been obtained by C.W. Clark [1], R. DeValt, G. Dial, V.L. Kocic, G. Ladas [2], E. Janowski, V.L. Kocic, G. Ladas [3], V.L. Kocic and G. Ladas [4], G. Papaschinopoulos, C.J. Schinas [7] and many other. The most used technique to obtain the behavior of the solutions is the study of semicycles.

In this paper we present another technique to obtain the global asymptotic stability for the solutions of difference equations using the fixed point theory and Picard operators from the point of view of a result obtained by I. A Rus in [8]. Also, we present some examples of difference equations for which we can apply this technique.

2 Global asymptotic stability and Picard operators

First we begin with some basic definitions.

Let X be a nonempty set and $A : X \to X$ an operator. We note by:

$P(X) := \{Y \subset X \mid Y \neq \emptyset\}$

$F_A := \{x \in X \mid A(x) = x\}$ - the fixed points set of A.

Definition 2.1 *(I.A. Rus [11]). Let (X,d) be a metric space. An operator $A : X \to X$ is a (uniformly) Picard operator if there exists $x^* \in X$ such that:*

(a) $F_A = \{x^*\}$,

(b) the sequence $(A^n(x))_{n\in\mathbb{N}}$ *converges (uniformly) to* x^**, for all* $x \in X$.

For examples of Picard operators see I.A. Rus [10], [12], [11], [13].

In this paper we study the existence, uniqueness and global asymptotic stability for the k order nonlinear difference equation

$$x_{n+k} = f\left(x_n, \ldots, x_{n+k-1}\right), \qquad n \in \mathbb{N}, \tag{1}$$

with initial values $x_0, \ldots, x_{k-1} \in X$, where (X, d) is a metric space and $f : X^k \to X$.

Definition 2.2 *An element* $x^* \in X$ *is a fixed point of the operator* f *iff*

$$x^* = f(x^*, \ldots, x^*).$$

Definition 2.3 *If* $x^* \in X$ *is a fixed point for* f *then the constant sequence generated by* x^* *is an equilibrium solution for the difference equation (1).*

Definition 2.4 *An equilibrium solution* $x^* \in X$ *is a global asymptotically stable relatively to the set X if for all solutions* $(x_n)_{n\in\mathbb{N}} \subset X$ *for the difference equation (1) we have:*

$$x_n \to x^* \quad for \quad n \to \infty.$$

Remark 2.1 *If the difference equation (1) has a global asymptotically stable equilibrium solution then this equilibrium solution is unique.*

We are interested to find conditions on operator f implying that the difference equation (1) has a global asymptotically stable equilibrium solution. We can consider the operator

$$\begin{gathered} A_f : X^k \to X^k, \\ (x_0, \ldots, x_{k-1}) \longmapsto (x_1, \ldots, x_{k-1}, f\left(x_0, \ldots, x_{k-1}\right)) \end{gathered} \tag{2}$$

and we have the following relation between the global asymptotic stability and Picard operators.

Theorem 2.1 *(I.A. Rus [8]) The difference equation (1) has an unique global asymptotically stable equilibrium solution relatively to the set X if and only if the operator* A_f*, defined by (2), is a Picard operator.*

3 Fixed point theorems

In this section of the paper we give a fixed point result which ensure that the operator A_f, defined by (2), is a Picard operator.

Theorem 3.1 *(I.A. Rus [9], M.A. Şerban [15]) Let* (X, d) *be a complete metric space,* $f : X^k \to X$ *and* $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ *such that:*

(i) φ *is a (c)-comparison function k dimensional;*

(ii) for all $x_{0,} \ldots, x_{k-1}, x_k \in X$ *we have*

$$d\left(f\left(x_0, \ldots, x_{k-1}\right), f\left(x_1, \ldots, x_k\right)\right) \leq \varphi\left(d\left(x_0, x_1\right), \ldots, d\left(x_{k-1}, x_k\right)\right);$$

(iii) for all $r \in \mathbb{R}_+$ *we have*

$$\varphi\left(r, 0, \ldots, 0\right) + \varphi\left(0, r, 0, \ldots, 0\right) + \ldots + \varphi\left(0, \ldots, 0, r\right) \leq \varphi\left(r, \ldots, r\right).$$

Then:

(a) the operator A_f*, defined by (2), is a Picard operator;*

(b) we have the estimation

$$d\left(x_n, x^*\right) \leq k \cdot \sum_{i=0}^{\infty} \psi^{\left[\frac{n}{k}\right]}\left(d_0\right) \cdot \frac{\alpha^{\left[\frac{n}{k}\right]}}{1-\alpha},$$

where $\left(x_n\right)_{n\in\mathbb{N}}$ *is any solution of (1),* $d_0 = \max\limits_{i=\overline{0,k-1}} \left\{d\left(x_i, x_{i+1}\right)\right\}$ *and* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$

$$\psi(r) = \varphi\left(r, \ldots, r\right).$$

Theorem 3.2 *(M.A. Şerban [15]) Let* (X, d) *be a complete metric space and* $f : X^k \to X$ *such that:*

(i) there exist $q_i \in \mathbb{R}_+$*,* $i = \overline{1,k}$*, with* $\alpha = \sum\limits_{i=1}^{k} q_i < 1$ *such that*

$$d\left(f\left(\overline{x}\right), f\left(\overline{y}\right)\right) \leq \sum_{i=1}^{k} q_i d\left(x_i, y_i\right),$$

for all $\overline{x} = (x_1, \ldots, x_k)$*,* $\overline{y} = (y_1, \ldots, y_k) \in X^k$*.*

Then:

(a) the operator A_f*, defined by (2), is a Picard operator;*

(b) we have the estimation

$$d\left(x_n, x^*\right) \leq k \cdot d_0 \cdot \frac{\alpha^{\left[\frac{n}{k}\right]}}{1-\alpha},$$

where $\left(x_n\right)_{n\in\mathbb{N}}$ *is any solution of (1) and* $d_0 = \max\limits_{i=\overline{0,k-1}} \left\{d\left(x_i, x_{i+1}\right)\right\}$*;*

(c) operator f is continuous in $(x^*, \ldots, x^*) \in X^k$*.*

4 Applications to some difference equations

As a first example we consider the difference equation

$$x_{n+k} = a \cdot \max\{x_n, \ldots, x_{n+k-1}\} + b \tag{3}$$

where $a, b \in \mathbb{R}_+$ and we give a simplified proof of global asymptotic stability for the equilibrium solution of difference equation (3), than the proof of H. Sedaghat [14], based on Theorem 2.1.

Lemma 4.1 *For $x_0, \ldots, x_k \in \mathbb{R}_+$ we have the inequality:*

$$|\max\{x_0, \ldots, x_{k-1}\} - \max\{x_1, \ldots, x_k\}| \leq \max\{|x_0 - x_1|, \ldots, |x_{k-1} - x_k|\}.$$

Proof. This inequality means the there exits $j \in \{0, \ldots, k-1\}$ such that

$$|\max\{x_0, \ldots, x_{k-1}\} - \max\{x_1, \ldots, x_k\}| \leq |x_j - x_{j+1}|.$$

We denote by

$$m_1 = \max\{x_0, \ldots, x_{k-1}\},$$
$$m_2 = \max\{x_1, \ldots, x_k\}$$

and we suppose that for all $j \in \{0, \ldots, k-1\}$ we have

$$|x_j - x_{j+1}| < |m_1 - m_2|, \tag{4}$$

which implies that $m_1 \neq m_2$. Thus we have the following cases:

Case 1. $m_1 > m_2 = \max\{\max\{x_1, \ldots, x_{k-1}\}, x_k\}$ then $m_1 > \max\{x_1, \ldots, x_{k-1}\}$ which implies that $m_1 = x_0$ and therefore from (4), we obtain:

$$x_0 - x_1 < x_0 - m_2 \leq x_0 - x_1,$$

which is a contradiction.

Case 2. $m_2 > m_1 = \max\{x_0, \max\{x_1, \ldots, x_{k-1}\}\}$ then $m_2 > \max\{x_1, \ldots, x_{k-1}\}$ which implies that $m_2 = x_k$ and therefore from (4), we obtain:

$$x_k - x_{k-1} < x_k - m_1 \leq x_k - x_{k-1},$$

which is a new contradiction and thus the lemma is proved.□

Theorem 4.1 *Let $a \in [0; 1[$ and $b \in \mathbb{R}_+$ then*

$$x^* = \frac{b}{1-a}$$

is the unique global asymptotically stable equilibrium solution relative to $\mathbb{R}_+$ for the difference equation (3) and we have the estimation

$$|x_n - x^*| \leq k \cdot \frac{a^{\left[\frac{n}{k}\right]}}{1-a} \cdot \max\{|x_0 - x_1|, \ldots, |x_{k-1} - x_k|\},$$

for any $x_0, \ldots, x_{k-1} \in \mathbb{R}_+$.

Proof. The fact that $x^* = \frac{b}{1-a}$ is the unique equilibrium for the difference equation (3) can be obtained solving the equation:

$$x = ax + b.$$

The uniqueness and global asymptotic stability is obtained from Theorem 3.2 considering the complete metric space $X = \left(\mathbb{R}_+, d_{|\cdot|}\right)$ and

$$f : X^k \to X,$$
$$f\left(x_0, \ldots, x_{k-1}\right) = a \cdot \max\left\{x_0, \ldots, x_{k-1}\right\} + b.$$

Using Lemma 4.1 we deduce

$$\left|f\left(x_0, \ldots, x_{k-1}\right) - f\left(x_1, \ldots, x_k\right)\right| = a\left|\max\left\{x_0, \ldots, x_{k-1}\right\} - \max\left\{x_1, \ldots, x_k\right\}\right| \leq$$

$$\leq a \cdot \max\left\{\left|x_0 - x_1\right|, \ldots, \left|x_{k-1} - x_k\right|\right\}$$

and, therefore, all the conditions of Theorem 3.2 are satisfied.□

The second example is a difference equation which arise from the study of the population dynamics

$$x_{n+1} - ax_n + bx_{n-k} = 0, \tag{5}$$

equation which was studied by C.W Clark [1], S.A. Kuruklis [5], S.A. Levin and R.M. May [6].

Theorem 4.2 *Let $a, b \in \mathbb{R}$ such that*

$$|a| + |b| < 1$$

then the difference equation (5) has a unique global asymptotically stable equilibrium solution relative to $\mathbb{R}$ and we have the estimation:

$$\left|x_n - x^*\right| \leq (k+1) \cdot \frac{(|a| + |b|)^{\left[\frac{n}{k}\right]}}{1 - |a| - |b|} \cdot \max\left\{\left|x_0 - x_1\right|, \ldots, \left|x_k - x_{k+1}\right|\right\},$$

for all $x_0, \ldots, x_k \in \mathbb{R}$.

Proof. We consider the Banach space $X = (\mathbb{R}, |\cdot|)$ and

$$f : X^{k+1} \to X,$$

$$f\left(x_0, \ldots, x_k\right) = ax_k - bx_0.$$

For all $(x_0, \ldots, x_k)$, $(y_0, \ldots, y_k) \in X^{k+1}$ we have:

$$\left|f\left(x_0, \ldots, x_k\right) - f\left(y_0, \ldots, y_k\right)\right| = |a| \cdot \left|x_k - y_k\right| + |b| \cdot \left|x_0 - y_0\right|.$$

Because $|a| + |b| < 1$ we are, again, in conditions of the Theorem 3.2 which implies that operator A_f, defined by (2), is a Picard operator and therefore from the Theorem 2.1 we deduce the global asymptotic stability for the unique equilibrium solution.□

For the third example we consider the nonlinear difference equation system:

$$\begin{cases} x_{n+1} = & A + \frac{y_n}{x_{n-p}} \\ y_{n+1} = & A + \frac{x_n}{y_{n-q}} \end{cases} \tag{6}$$

where $p, q \in \mathbb{N}$. This system was studied by G. Papaschinopoulos and C.J. Schinas in [7] proving the global asymptotic stability for the unique equilibrium solution for $A > 1$. From the point of view of Theorem 3.2 and Theorem 2.1 we obtain the following result.

Theorem 4.3 *For* $A > \frac{3+\sqrt{5}}{3} = 1.74535\,5993$ *then*

$$z^* = (1 + A, 1 + A)$$

is the unique global asymptotically stable equilibrium solution relative to $X = [A; B] \times [A; B]$ *for the difference equations system (6) where*

$$B \in \left[\frac{A^2}{A-1}; A^2 - A\right[$$

and we have the estimations:

$$\delta(z_n, z^*) \le \frac{\alpha^{\left[\frac{n}{p+1}\right]}}{1-\alpha} \cdot \max\{\delta(z_0, z_1), \ldots, \delta(z_p, z_{p+1})\}, \qquad for \quad p \ge q,$$

$$\delta(z_n, z^*) \le \frac{\alpha^{\left[\frac{n}{q+1}\right]}}{1-\alpha} \cdot \max\{\delta(z_0, z_1), \ldots, \delta(z_q, z_{q+1})\}, \qquad for \quad q > p,$$

where $\alpha = \frac{1}{A} + \frac{B}{A^2}$ *and* δ *is the metric on the space* $X = [A; B] \times [A; B]$ *defined by*

$$\delta(z_1, z_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

for all $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in X$.

Proof. It is obvious that the metric space (X, δ) is complete. We have to study two cases:

Case 1. $q \le p$

In this case we define

$$f = (f_1, f_2) : X^{p+1} \to X,$$

$$f(z_0, \ldots, z_p) = \left(A + \frac{y_p}{x_0}, A + \frac{x_p}{y_{p-q}}\right)$$

and for all $z_0, \ldots, z_p \in X$ we have:

$$(A, A) \le f(z_0, \ldots, z_p) \le \left(A + \frac{B}{A}, A + \frac{B}{A}\right),$$

but

$$A + \frac{B}{A} \leq B \Longleftrightarrow B \geq \frac{A^2}{A-1}$$

and therefore $f(X, \ldots, X) \subseteq X$.

With this notation the system (6) can be written

$$z_{n+p+1} = f(z_n, \ldots, z_{n+p-q}, \ldots, z_{n+p}).$$

For all $z_0, \ldots, z_p, \tilde{z}_0, \ldots, \tilde{z}_p \in X$ we have:

$$|f_1(z_0, \ldots, z_p) - f_1(\tilde{z}_0, \ldots, \tilde{z}_p)| \leq \frac{y_p}{x_0 \tilde{x}_0} \cdot |x_0 - \tilde{x}_0| + \frac{1}{\tilde{x}_0} \cdot |y_p - \tilde{y}_p| \leq$$

$$\leq \frac{B}{A^2} \cdot |x_0 - \tilde{x}_0| + \frac{1}{A} \cdot |y_p - \tilde{y}_p|,$$

$$|f_2(z_0, \ldots, z_p) - f_2(\tilde{z}_0, \ldots, \tilde{z}_p)| \leq \frac{1}{y_{p-q}} \cdot |x_p - \tilde{x}_p| + \frac{\tilde{x}_p}{y_{p-q} \tilde{y}_{p-q}} \cdot |y_{p-q} - \tilde{y}_{p-q}| \leq$$

$$\leq \frac{1}{A} \cdot |x_p - \tilde{x}_p| + \frac{B}{A^2} \cdot |y_{p-q} - \tilde{y}_{p-q}|,$$

and we obtain

$$\delta(f(z_0, \ldots, z_p), f(\tilde{z}_0, \ldots, \tilde{z}_p)) \leq \frac{B}{A^2} \cdot \max\{\delta(z_0, \tilde{z}_0), \delta(z_{p-q}, \tilde{z}_{p-q})\} + \frac{1}{A} \cdot \delta(z_p, \tilde{z}_p)$$

For this case we consider the function

$$\varphi : \mathbb{R}_+^{p+1} \to \mathbb{R}_+$$

$$\varphi(r_0, \ldots, r_p) = \frac{B}{A^2} \cdot \max\{r_0, r_{p-q}\} + \frac{1}{A} \cdot r_p$$

which is a (c)-comparison function p+1 dimensional. From Theorem 3.1 we obtain that operator $A_f : X^{p+1} \to X^{p+1}$, defined by (2), is a Picard operator and therefore from the Theorem 2.1 we deduce the global asymptotic stability for the unique equilibrium solution.

Case 2. $q > p$

In this case we define

$$f = (f_1, f_2) : X^{q+1} \to X,$$

$$f(z_0, \ldots, z_q) = \left(A + \frac{y_q}{x_{q-p}}, A + \frac{x_q}{y_0}\right).$$

The invariance property of the set X to the operator f is also true in this case. Thus, the system (6) can be written in following form:

$$z_{n+p+1} = f(z_{n+p-q}, \ldots, z_n, \ldots, z_{n+p}).$$

For all $z_0, \ldots, z_p, \tilde{z}_0, \ldots, \tilde{z}_p \in X$ we have:

$$|f_1(z_0, \ldots, z_q) - f_1(\tilde{z}_0, \ldots, \tilde{z}_q)| \le \frac{\tilde{y}_q}{x_{q-p}\tilde{x}_{q-p}} \cdot |x_{q-p} - \tilde{x}_{q-p}| + \frac{1}{x_{q-p}} \cdot |y_q - \tilde{y}_q| \le$$

$$\le \frac{B}{A^2} \cdot |x_{q-p} - \tilde{x}_{q-p}| + \frac{1}{A} \cdot |y_q - \tilde{y}_q|,$$

$$|f_2(z_0, \ldots, z_q) - f_2(\tilde{z}_0, \ldots, \tilde{z}_q)| \le \frac{1}{y_0} \cdot |x_q - \tilde{x}_q| + \frac{\tilde{x}_q}{y_0\tilde{y}_0} \cdot |y_0 - \tilde{y}_0| \le$$

$$\le \frac{1}{A} \cdot |x_q - \tilde{x}_q| + \frac{B}{A^2} \cdot |y_0 - \tilde{y}_0|,$$

and we deduce:

$$\delta(f(z_0, \ldots, z_p), f(\tilde{z}_0, \ldots, \tilde{z}_p)) \le \frac{B}{A^2} \max\{\delta(z_0, \tilde{z}_0), \delta(z_{q-p}, \tilde{z}_{q-p})\} + \frac{1}{A}\delta(z_q, \tilde{z}_q).$$

In this case we consider the function

$$\varphi : \mathbb{R}_+^{q+1} \to \mathbb{R}_+$$

$$\varphi(r_0, \ldots, r_q) = \frac{B}{A^2} \cdot \max\{r_0, r_{q-p}\} + \frac{1}{A} \cdot r_q$$

which is a (c)-comparison function $q+1$ dimensional. From Theorem 3.1 we obtain that operator $A_f : X^{q+1} \to X^{q+1}$, defined by (2), is a Picard operator and therefore from the Theorem 2.1 we deduce the global asymptotic stability for the unique equilibrium solution.□

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