Seminar on Fixed Point Theory Cluj-Napoca, **Volume 2, 2001, 3-22**
http://www.math.ubbcluj.ro/~nodeacj/journal.htm

FIXED POINTS AND SELECTIONS FOR MULTI-VALUED OPERATORS

Adrian Petruşel

Department of Applied Mathematics
Babeş-Bolyai University, Cluj-Napoca, Romania
E-mail: petrusel@math.ubbcluj.ro

Abstract. The purpose of this paper is to report several results in the fixed point theory and selections theory for multi-valued operators.

Keywords: multi-valued operator, fixed point, decomposable set.
AMS Subject Classification: 47H10, 54H25.

1 Introduction

Fixed point theory is one of the most powerful tool for the study of a lot of problems arising in pure and applied mathematics. The main purpose of this paper is to report several fixed point results and selection theorems for multi-valued operators. The structure of this paper is the following: first of all, in the second section, we consider some notations, notions and preliminary results that we need throughout the paper. Then in the third paragraph, we present some Krasnoselskii-type fixed points theorems for multi-valued operators. In section 4, we concentrate on some properties of the fixed points set of a multi-valued operator. Finally, in the last paragraph we consider some selection theorems for multi-valued operators with decomposable values. For more details and further results, we refer to [49], [51] and [64].

2 Preliminaries

Let (X, d) be a metric space, $x_0 \in X$ and $r > 0$. We denote:

$B(x_0; r) = \{x \in X : \ d(x_0, x) < r\}$, $\widetilde{B}(x_0; r) = \{x \in X : \ d(x_0, x) \leq r\}$,
$\mathcal{P}(X) = \{A : \ A \text{ is a subset of } X\}$, $P(X) = \{A \in \mathcal{P}(X) : \ A \text{ is nonempty}\}$,
$P_p(X) = \{A \in P(X) : \ A \text{ has the property } "p"\}$, where "$p$" could be: cl = closed, b = bounded, cp = compact, cv = convex (for normed spaces X), etc.

If $A, B \in P(X)$, we define the functional:

$D : P(X) \times P(X) \to \mathbb{R}_+$, $D(A, B) = \inf\{d(a, b)| \ a \in A, \ b \in B\}$

and the following generalized functionals:

$\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, $\delta(A, B) = \sup\{d(a, b)| \ a \in A, \ b \in B\}$

$\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, $\rho(A, B) = \sup(D(a, B)| \ a \in A\}$

$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, $H(A,B) = \max\{\rho(A,B), \rho(B,A)\}$.

Throughout the paper, the symbol $\mathcal{M}$ indicates the family of all metric spaces. Let $X \in \mathcal{M}$. The space X is called an absolute retract for metric spaces (briefly $X \in AR(\mathcal{M})$) if, for any $Y \in \mathcal{M}$ and any $Y_0 \in P_{cl}(X)$, every continuous function $f_0 : Y_0 \to X$ has a continuous extension over Y, that is $f : Y \to X$. Obviously, any absolute retract is arcwise connected. Let $X, Y \in \mathcal{M}$ and $T : X \to P(Y)$ be a multifunction. We will denote by the symbol $G(T) = \{(x,y) \in X \times Y |\ y \in T(x)\}$ the graph of T. If $T : X \to P(X)$ is a multi-valued operator then we denote by $FixT$ the fixed points set of T , i.e. $FixT = \{x \in X |\ x \in T(x)\}$. A multi-function $T : X \to P_{cl}(Y)$ is, by definition, lower semi-continuous (briefly, l.s.c.) if, for any open set A of Y, the set $T^{-1}(A) := \{x \in X : T(x) \cap A \neq \emptyset\}$ is open in X. When for any open set A of Y , the set $\{x \in X : T(x) \subset A\}$ is open in X, we say that T is upper semi- continuous (briefly u.s.c.). A continuous multi-function $T : X \to P_{cl}(Y)$ is, by definition, both l.s.c. and u.s.c. Also, T is said to be closed if and only if the set $G(T)$ is closed in $X \times Y$.

Definition 2.1. (see Rus-Petruşel-Sîntămărian [73]) Let (X, d) be a metric space and $T : X \to P(X)$ a multi-valued operator. By definition, T is a multi-valued weakly Picard operator (briefly m.w.P.o.) if and only if for all $x \in X$ and all $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

i) $x_0 = x$, $x_1 = y$

ii) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$

iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

Let us remark that a sequence $(x_n)_{n\in\mathbb{N}}$ satisfying the conditions (i) and (ii) in the previous definition is, by definition, a sequence of successive approximations of T, starting from (x, y).

Definition 2.2. (see Covitz-Nadler [23]) Let (X, d) be a metric space. Then $T : X \to P_{cl}(X)$ is a multi-valued L-contraction, if there exists $L \in]0, 1[$ such that: $H(T(x), T(y)) \leq Ld(x, y),$ for every $x, y \in X$.

Definition 2.3. (see Reich [66]) Let (X, d) be a metric space. Then $T : X \to P_{cl}(X)$ is said to be a multi-valued Reich-type operator if there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$, $\alpha + \beta + \gamma < 1$ such that: $H(T(x), T(y)) \leq \alpha d(x, y) + \beta D(x, T(x)) + \gamma D(y, T(y)),$ for all $x, y \in X$.

Definition 2.4. (see Rus [70] and Petruşel [49]) Let (X, d) be a complete metric space. A multi-valued operator $T : X \to P_{cl}(X)$ is said to be a multi-valued Rus-type graphic-contraction if G(T) is closed and the following condition is satisfied: there exist $\alpha, \beta \in \mathbb{R}_+$, $\alpha + \beta < 1$ such that: $H(T(x), T(y)) \leq \alpha d(x, y) + \beta D(y, T(y)),$ for every $x \in X$ and every $y \in T(x)$. Moreover, the operator T is a multi-valued graphic-contraction if T is a multi-valued Rus-type contraction with $\beta = 0$.

Definition 2.5. (see Frigon-Granas [29] and Petruşel [59]) Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. The multi-valued operator T is called a Frigon-Granas-type operator if $T : \widetilde{B}(x_0; r) \to P_{cl}(X)$ and satisfies the following assertion:

i) there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$, $\alpha + \beta + \gamma < 1$ such that:

$$H(T(x), T(y)) \leq \alpha d(x, y) + \beta D(x, T(x)) + \gamma D(y, T(y)), \text{ for all } x, y \in \widetilde{B}(x_0; r)$$

We also recall the definition of Kuratowski's measure of non-compactness :

Definition 2.6. Let X be a metric space and S a bounded subset of X. We set $\alpha(S) = \inf\{\varepsilon > 0 |$there exists m and exists $S_1, ..., S_m$ such that $S = \bigcup_{i \leq m} S_i$ and $diam(S_i) \leq \varepsilon\}$.

Definition 2.7. Let X a Banach space and $T : X \to P_{b,cl}(X)$ be a multivalued operator. Then T is said to be an (α, L)-contraction if and only if there exists $L \in]0, 1[$ such that $\alpha(T(A)) \leq L\alpha(A)$, for each $A \in P_b(X)$. (where α is the Kuratowski measure of non-compactness on X)

A multivalued operator $T : X \to P(X)$ is called compact if and only if $T(A)$ is relatively compact, for each $A \in P_b(X)$.

Definition 2.8. Let $F : [a, b] \times \mathbb{R}^n \to P_{cl}(\mathbb{R}^n)$ a multivalued operator. Then F is called integrably bounded if and only if there exists a function $r \in L^1[a, b]$ such that for every $v \in F(t, x)$, $|v| \leq r(t)$ a.e.

Definition 2.9. Let (X, d) be a metric space and E be a Banach space. Then $F : X \to P(E)$ is called:

a) locally selectionable at $x_0 \in X$ if and only if for all $y_0 \in F(x_0)$ there exist a neighborhood $N(x_0)$ and a continuous function $f : N(x_0) \to E$ such that $f(x_0) = y_0$ and $f(x) \in F(x)$, for each $x \in N(x_0)$.

b) measurable if and only if $F^{-1}(C)$ belongs to the σ-algebra $\mathcal{B}$ of Borel subsets of X, for each closed subset C of E.

Definition 2.10. Let X be a nonempty set and E be a Banach space and $F : X \to P(E)$ be a multivalued operator.

i) The set defined by $F^{-1}(y) = \{x \in X | y \in F(x)\}$ is said to be the fibre of F at the point $y \in E$.

ii) The singlevalued operator $f : X \to E$ is a selection for F if and only if $f(x) \in F(x)$ for every $x \in X$.

Definition 2.11. (see Deguire-Lassonde [24] and [25]) Let X be a topological space and $\{Y_i | i \in I\}$ an arbitrary family of topological spaces.

i) We say that $\{f_i : X \to Y_i | i \in I\}$ is a selecting family for the family of multivalued operators $\{F_i : X \to \mathcal{P}(Y_i) | i \in I\}$ if and only if for each $x \in X$ there exists $i \in I$ such that $f_i(x) \in F_i(x)$.

ii) If $\{Y_i |\ i \in I\}$ is an arbitrary family of convex subsets of a Hausdorff topological vector space then the family $\{F_i : X \to \mathcal{P}(Y_i) | i \in I\}$ is said to be of Ky Fan-type if and only if each F_i has convex values and open fibres and for every $x \in X$ there is $i \in I$ such that $F_i(x) \neq \emptyset$.

Definition 2.12. The topological space X has the compact fixed point property if and only if every continuous mapping $f : X \to X$ with relatively compact image has a fixed point.

We consider now some known results that will be used in the following sections.

Theorem 2.13. (see [75]) *Let X be a metric space and Y be a closed subset of a Banach space Z. Assume that the multivalued operator $F : X \times Y \to P_{cl,cv}(Y)$ satisfies the following conditions:*

i) $H(F(x, y_1), F(x, y_2)) \leq L\|y_1 - y_2\|$, for each $(x, y_1), (x, y_2) \in X \times Y$;

ii) for every $y \in Y$, $F(\cdot, y)$ is l.s.c. on the space X.

Then there exists a continuous mapping $f : X \times Y \to Y$ such that :

$$f(x, y) \in F(x, f(x, y)), \text{ for each } (x, y) \in X \times Y$$

.

Theorem 2.14. (see [51]) *Let X be a Banach space and $F_1, F_2 : X \to P_{b,cl}(X)$ be two multivalued operators, such that F_1 is a L-contraction and F_2 is compact. Then $F_1 + F_2$ is (α, L)-contraction.*

3 Fixed point theorems for the sum of two multivalued operators

A first multivalued version of the Krasnoselskii's fixed point principle is:

Theorem 3.1. *Let X be a Banach space, $Y \in P_{cl,cv}(X)$ and $A : Y \to P_{b,cl,cv}(X)$, $B : Y \to P_{cp,cv}(X)$ two multivalued operators. If the following conditions are satisfied:*

(i) $A(y_1) + B(y_2) \subset Y$, for each $y_1, y_2 \in Y$;

(ii) A is L-contraction;

(iii) B is l.s.c. and $B(Y)$ is relatively compact;

then $Fix(A + B) \neq \emptyset$.

Proof. Let $C : Y \to \mathcal{P}(Y)$ be a multivalued operator defined as follows:

a) for each $x \in Y$ consider the multivalued operator $T_x : Y \to P_{cp,cv}(Y)$, $T_x(y) = A(y) + B(x)$. Since T_x is multivalued L-contraction (indeed, on have: $H(T_x(y_1), T_x(y_2)) = H(A(y_1)+B(x), A(y_2)+B(x)) \leq H(A(y_1), A(y_2)) \leq L\|y_1-y_2\|$, for each $y_1, y_2 \in Y$), from Covitz-Nadler fixed point theorem (see [23]) it follows that for every $x \in Y$ the fixed point set for the multifunction T_x, $Fix\, T_x = \{y \in Y | \ y \in A(y) + B(x)\}$ is nonempty and closed.

b) From Theorem 2.13. it follows that there exists a continuous mapping $f : Y \times Y \to Y$ such that $f(x, y) \in A(f(x, y)) + B(x)$. (Let us observe that the multivalued operator $F : Y \times Y \to P_{cp,cv}(Y)$ defined by $F(x, y) = A(y) + B(x)$, for each $(x, y) \in Y \times Y$ satisfies the hypothesis of Theorem 2.13.) Let us define $C(x) = Fix\, T_x$, $C : Y \to P_{cl}(Y)$. Let us consider the singlevalued operator $c : Y \to Y$ defined by $c(x) = f(x, x)$, for each $x \in Y$.

Then c is a continuous mapping having the property that $c(x) = f(x, x) \in A(f(x, x)) + B(x) = A(c(x)) + B(x)$, for each $x \in Y$.

Now, we will prove that $c(Y)$ is relatively compact. For this purpose it is sufficient to show that $C(Y)$ is relatively compact. Let us observe that $C(Y)$ is totally bounded:

Indeed $B(Y)$ being relatively compact it is also totally bounded. So, there exists $Z = \{x_1, \ldots, x_n\} \subset Y$ such that $B(Y) \subset \{z_1, \ldots, z_n\} + B(0, (1 - L)\varepsilon) \subset \bigcup_{i=1}^{n} B(x_i) +$

$B(0,(1-L)\varepsilon)$ (where $z_i \in B(x_i)$, for each $i = 1,2,\dots,n$). It follows that, for each $x \in Y$, $B(x) \subset \bigcup_{i=1}^{n} B(x_i) + B(0,(1-L)\varepsilon)$ and hence there exists an element $x_k \in Z$ such that $\rho(B(x), B(x_k)) < (1-L)\varepsilon$. Then:

$$\rho(C(x), C(x_k)) = \rho(Fix\, T_x, Fix\, T_{x_k}) \le \frac{1}{1-L} \sup_{y\in Y} \rho(T_x(y), T_{x_k}(y)) =$$

$$\frac{1}{1-L} \sup_{y\in Y} \rho(A(y)+B(x), A(y)+B(x_k)) \le \frac{1}{1-L} \sup_{y\in Y} \rho(B(x), B(x_k)) <$$

$$< \frac{1}{1-L}(1-L)\varepsilon = \varepsilon$$

It follows that for each $u \in C(x)$ there is $v_k \in C(x_k)$ such that $\|u - v_k\| < \varepsilon$. Hence, for each $x \in Y$, $C(x) \subset Q + B(0,\varepsilon)$, where $Q = \{v_1,\dots,v_k,\dots,v_n\}$, $v_i \in C(x_i)$, $i = 1,2,\dots,n$.

Since in a Banach space a totally bounded set is relatively compact the conclusion follows.

Finally, let us observe that the mapping $c : Y \to Y$ satisfies the assumptions of Schauder's fixed point theorem. Let $x^* \in Y$ be a fixed point for c. On have that $x^* = c(x^*) \in A(c(x^*)) + B(x^*) = A(x^*) + B(x^*)$.

Using the measures of non-compactness technique (see [51]) another fixed point result for the sum of two multi-valued operators is the following:

Theorem 3.2. *Let X be a Banach space, $Y \in P_{b,cl,cv}(X)$ and $A, B : Y \to P_{cp,cv}(X)$ two multivalued operators. If the following conditions are satisfied:*

(i) $A(y) + B(y) \subset Y$, for each $y \in Y$;

(ii) A is L-contraction;

(iii) B is u.s.c. and compact;

then $Fix(A+B) \neq \emptyset$.

As application, from Theorem 3.2., we get the following existence result for an integral inclusion (see [51] for more details).

Theorem 3.3. *Consider the following Fredholm-Volterra integral inclusion:*

$$x(t) \in \lambda_1 \int_a^b K_1(t,s,x(s)) + \lambda_2 \int_a^t K_2(t,s,x(s))ds, \quad t \in [a,b]. \tag{3.1}$$

We assume that:

i) $K_1 : [a,b]\times[a,b]\times\mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ is a lower semicontinuous, measurable and integrably bounded multivalued operator

ii) $K_2 : [a,b]\times[a,b]\times\mathbb{R}^n \to P_{cp,cv}(\mathbb{R}^n)$ is an upper semicontinuous, measurable and integrably bounded (by an integrable function m_{K_2}) multivalued operator

iii) there exists $L > 0$ such that

$H(K_1(t,s,u_1), K_1(t,s,u_2)) \le L\|u_1 - u_2\|$, *for each* $(t,s,u_1),(t,s,u_2) \in [a,b]\times[a,b]\times\mathbb{R}^n$

iv) λ_2 *satisfy the following relation:*

$$|\lambda_2| \leq \frac{R}{2M_{K_2}(b-a)}, \text{ where } R \geq \frac{\delta}{1 - \dfrac{|\lambda_1|L}{2\tau}}$$

(with $M_{K_2} = \max\limits_{t\in[a,b]} m_{K_2}(t)$, $\tau > |\lambda_1|L$ *and* δ *is an upper bound for the set of continuous selections for the multivalued operator* $t \mapsto \lambda_2 \int_a^b K_2(t,s,y(s))ds$, $y \in C[a,b]$*).*

Then, there exists $y_0 \in C[a,b]$, *such that the integral inclusion (3.1.) has a solution* $y^* \in \widetilde{B}(y_0;R) \subset C[a,b]$.

Sketch of the proof. Let $A, B : C[a,b] \to \mathcal{P}(C[a,b])$ be two multivalued operators given by

$$A(y) = \left\{ u \in C[a,b] |\ u(t) \in \lambda_1 \int_a^t K_1(t,s,y(s))ds, \quad t \in [a,b] \right\}$$

$$B(y) = \left\{ v \in C[a,b] |\ v(t) \in \lambda_2 \int_a^t K_2(t,s,y(s))ds, \quad t \in [a,b] \right\}$$

Obviously $y^* \in Fix(A+B)$ if and only if y^* is a solution for (3.1.).

Then on can prove that the multifunctions A and B satisfies the assumptions of Theorem 3.2. □

4 Properties of the fixed points set for multifunctions

Contrary to the single-valued case, the fixed points set for a multivalued contraction is not necessarily a singleton and hence it is of interest to study some properties (compactness, absolute retract property , data dependence etc.) of it.

In this framework, J. Saint-Raymond established the following theorem.

Theorem 4.1. ([65]) *Let* T *be a multi-valued contraction from the complete metric space* X *to itself. If* T *takes compact values, the fixed points set* $FixT$ *is compact too.*

As regard to the same problem, B. Ricceri stated the following very important theorem:

Theorem 4.2. ([69]) *Let* E *be a Banach space and let* X *be a nonempty, closed, convex subset of* E. *Suppose* $T : X \to P_{cl,cv}(X)$ *is a multi-valued contraction. Then* $FixT$ *is an absolute retract for metric spaces.*

Using a similar idea as in [29] we shall prove first the following existence result:

Theorem 4.3. *Let* (X,d) *be a complete metric space* $x_0 \in X$, $r > 0$ *and* $T : \widetilde{B}(x_0,r) \to P_{cl}(X)$ *a multivalued operator satisfying the following assertions:*

i) T *is a Reich-type multivalued operator*

ii) $D(x_0, T(x_0)) < \dfrac{1-(\alpha+\beta+\gamma)}{1-\gamma} r$.

Then $FixT \neq \emptyset$.

Proof. We shall prove, by induction, the existence of a sequence $(x_n)_{n\in\mathbb{N}}$ in $\widetilde{B}(x_0, r)$ such that

a) $x_n \in T(x_{n-1})$, for each $n \in \mathbb{N}^*$

b) $d(x_{n-1}, x_n) < (\alpha+\beta+\gamma)^{n-1}\dfrac{1-(\alpha+\beta+\gamma)}{1-\gamma}r$, for each $n \in \mathbb{N}^*$.

For $n = 1$ the assertions a) and b) are true.

If the relations hold for $i \in \{1, n\}$ let us show the existence of x_{n+1}.

We have:

$$H(T(x_{n-1}), T(x_n)) \leq \alpha d(x_{n-1}, x_n) + \beta D(x_{n-1}, T(x_{n-1}) + \gamma D(x_n, T(x_n)) \leq$$

$$\leq \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, x_n) + \gamma H(T(x_{n-1}), T(x_n)).$$

Hence

$$H(T(x_{n-1}), T(x_n)) \leq \frac{\alpha+\beta}{1-\gamma} d(x_{n-1}, x_n) < \frac{\alpha+\beta}{1-\gamma}(\alpha+\beta+\gamma)^{n-1}\frac{1-(\alpha+\beta+\gamma)}{1-\gamma} r <$$

$$< (\alpha+\beta+\gamma)^n \frac{1-(\alpha+\beta+\gamma)}{1-\gamma} r.$$

So we get that there exists $x_{n+1} \in T(x_n)$ such that

$$d(x_n, x_{n+1}) < (\alpha+\beta+\gamma)^n \frac{1-(\alpha+\beta+\gamma)}{1-\gamma} r,$$

proving the existence of the point x_{n+1} satisfying a) and b). Let us denote by $l = \alpha+\beta+\gamma$. Further on, we shall prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Indeed

$$d(x_n, x_{n+p}) < (l^n + \ldots + l^{n+p-1})\frac{1-l}{1-\gamma} r \leq \frac{l^n}{1-\gamma} r \to 0, \quad \text{as} \quad n \to \infty.$$

This implies that $x_n \to x^*$ as $n \to \infty$ and because T has closed values we get by standard arguments that $x^* \in T(x^*)$. □

Remark 4.4. When $\beta = \gamma = 0$ we get Theorem 3.1 from [29].

Remark 4.5. When $\alpha = 0$ and $\beta = \gamma = h$ (with $h \in \mathbb{R}$, $0 < h < 1/2$) we obtain a fixed point result for a Kannan-type multivalued operator.

Another result of this type is:

Theorem 4.6. *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T : \widetilde{B}(x_0; r) \to P_{cl}(X)$ satisfying:*

i) T is a graphic-contraction

ii) T is a closed multifunction

iii) $D(x_0, T(x_0)) < (1-\alpha)r$.

Then $FixT \neq \emptyset$.

For the fixed points set of a Frigon-Granas type multifunction we have:

Theorem 4.7. *Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let us suppose that $T : \widetilde{B}(x_0; r) \to P_{cp}(X)$ satisfies the following two conditions:*

i) there exist $\alpha, \beta \in \mathbb{R}_+$, $\alpha + 2\beta < 1$ *such that*

$$H(T(x), T(y)) \leq \alpha d(x, y) + \beta[D(x, T(x)) + D(y, T(y))], \text{ for each } x, y \in \widetilde{B}(x_0; r)$$

ii) $D(x_0, T(x_0)) < [1 - (\alpha + 2\beta)](1 - \beta)^{-1} r$.

Then the fixed points set $FixT$ *is compact.*

Proof. It is not difficult to see that $FixT \in P_{cl}(\widetilde{B}(x_0; r))$. Assume that $FixT$ is not compact. Because $FixT$ is complete, it cannot be precompact, so there exist $\delta > 0$ and a sequence $(x_i)_{i \in \mathbb{N}} \subset FixT$ such that $d(x_i, x_j) \geq \delta$, for each $i \neq j$. Put $\rho = \inf\{R|$ there exists $a \in \widetilde{B}(x_0; r)$ such that $B(a, R)$ contains infinitely many x_i;s$\}$. Obviously $\rho \geq \frac{\delta}{2} > 0$. Let $\varepsilon > 0$ such that $\varepsilon < \frac{1 - \alpha - 2\beta}{1 + \alpha} \rho$ and choose $a \in \widetilde{B}(x_0; r)$ such that the set $J = \{i : \ x_i \in B(a, \rho + \varepsilon)\}$ is infinite.

For each $i \in J$, we have

$$D(x_i, T(a)) \leq H(T(x_i), T(a)) \leq \alpha d(x_i, a) + \beta_i[D(x_i, T(x_i)) + D(a, T(a))] =$$

$$= \alpha d(x_i, a) + \beta D(a, T(a)) < \alpha(\rho + \varepsilon) + \beta d(a, y), \text{ for every } y \in T(a).$$

Then

$$D(x_i, T(a)) < \alpha(\rho + \varepsilon) + \beta[d(a, x_i) + d(x_i, y)] < \alpha(\rho + \varepsilon) + \beta(\rho + \varepsilon) + \beta d(x_i, y),$$

for every $y \in T(a)$. Taking $\inf\limits_{y \in T(a)}$ we get : $D(x_i, T(a)) \leq (\alpha + \beta)(\rho + \varepsilon)(1 - \beta)^{-1}$, for each $i \in J$. So, we can choose some $y_i \in T(a)$ such that $d(x_i, y_i) \leq (\alpha + \beta)(\rho + \varepsilon)(1 - \beta)^{-1}$, for each $i \in J$. By the compactness of $T(a)$ there exists $b \in T(a)$ such that the following set: $J' = \{i \in J | \ d(y_i, b) < \varepsilon\}$ is infinite. Then, for each $i \in J'$ we get $d(x_i, b) \leq d(x_i, y_i) + d(y_i, b) < (\alpha + \beta)(\rho + \varepsilon)(1 - \beta)^{-1} + \varepsilon = (\alpha + \beta)(1 - \beta)^{-1}\rho + \varepsilon\left(1 + (\alpha + \beta)(1 - \beta)^{-1}\right) < \rho$. This contradicts the definition of ρ, because the set $B(b, R)$ contains infinitely many x_i's. $\left(\text{where } R = (\alpha + \beta)\rho(1 - \beta)^{-1} + \varepsilon\left(1 + (\alpha + \beta)(1 - \beta)^{-1}\right)\right)$. □

Corollary 4.8. *Let* (X, d) *be a complete metric space and* $T : X \to P_{cp}(X)$ *be a multivalued operator. Let us suppose that there exist* $\alpha, \beta \in \mathbb{R}_+$ *with* $\alpha + 2\beta < 1$ *such that*

$$H(T(x), T(y)) \leq \alpha d(x, y) + \beta[D(x, T(x)) + D(y, T(y))],$$

for each $x, y \in X$.

Then the fixed points set $FixT$ *is compact.*

Only minor modifications of the above technique are needed in order to obtain the following result:

Theorem 4.9. *Let* (X, d) *be a complete metric space and* $T : X \to P_{cp}(X)$ *be a multivalued operator. If the following assertions are true:*

i) there exist $\alpha, \beta \in \mathbb{R}_+$, $\alpha + \beta < 1$ *such that*

$$H(T(x), T(y)) \leq \alpha d(x, y) + \beta D(x, T(x)), \quad \text{for each} \quad x, y \in X$$

ii) $D(x_0, T(x_0)) < [1 - (\alpha + \beta)](1 - \beta)^{-1} r$.

then the fixed points set $FixT$ is compact.

We consider now another result regarding the structure of the set of all fixed points for a multi-valued Reich-type operator with convex values.

Theorem 4.10. *Let E be a Banach space, $X \in P_{clc,cv}(E)$ and $T : X \to P_{cl,cv}(X)$ be a l.s.c. multi-valued Reich-type operator. Then $FixT \in AR(\mathcal{M})$.*

Proof. Let us remark first that $FixT \in P_{cl}(X)$ (see for example [73]). Let K be a paracompact topological space, $A \in P_{cl}(K)$ and $\psi : A \to FixT$ a continuous mapping. Using Theorem 2 from B. Ricceri [69] (taking $G(t) = X$, for each $t \in K$) it follows the existence of a continuous function $\varphi_0 : K \to X$ such that $\varphi_0|_A = \psi$. We next consider $q \in]1, (\alpha+\beta+\gamma)^{-1}[$. We claim that there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of continuous functions from K to X with the following properties:

(i) $\varphi_n|_A = \psi$

(ii) $\varphi_n(t) \in T(\varphi_{n-1}(t))$, for all $t \in K$

(iii) $\|\varphi_n(t) - \varphi_{n-1}(t)\| \leq [(\alpha+\beta+\gamma)q]^{n-1}\|\varphi_1(t) - \varphi_0(t)\|$, for all $t \in K$.

To see this, we proceed by induction on n. Clearly, for each $t \in A$ we have that $\psi(t) \in T(\varphi_0(t))$. On the other hand, the multi-function $t \mapsto T(\varphi_0(t))$ is l.s.c. on K with closed, convex values and hence using again Theorem 2 in [69] it follows that there is a continuous function $\varphi_1 : K \to X$ such that $\varphi_1|_A = \psi$ and $\varphi_1(t) \in T(\varphi_0(t))$, for all $t \in K$. Hence, the conditions (i), (ii), (iii) are true for φ_1. Suppose now we have constructed p continuous functions $\varphi_1, \varphi_2, \ldots, \varphi_p$ from K to X in such a way that (i), (ii), (iii) are true for $n \in \{1, 2, \ldots, p\}$. Using the Reich-type contraction condition for T, we have

$$D(\varphi_p(A), T(\varphi_p(t))) \leq H(T(\varphi_{p-1}(t)), T(\varphi_p(t))) \leq$$

$$\leq \alpha\|\varphi_{p-1}(t) - \varphi_p(t)\| + \beta D(\varphi_{p-1}(t), T(\varphi_{p-1}(t))) + \gamma D(\varphi_p(t), T(\varphi_p(t))) \leq$$

$$\leq \alpha\|\varphi_{p-1}(t) - \varphi_p(t)\| + \beta\|\varphi_{p-1}(t) - \varphi_p(t)\| + \gamma D(\varphi_p(t), T(\varphi_p(t)))$$

so that

$$D(\varphi_p(t), T(\varphi_p(t))) \leq (\alpha+\beta)(1-\gamma)^{-1}\|\varphi_p(t) - \varphi_{p-1}(t)\| \leq$$

$$\leq (\alpha+\beta)(1-\gamma)^{-1}[(\alpha+\beta+\gamma)q]^{p-1}\|\varphi_1(t) - \varphi_0(t)\| < (\alpha+\beta+\gamma)^p q^{p-1}\|\varphi_1(t) - \varphi_0(t)\| <$$

$$< [(\alpha+\beta+\gamma)q]^p\|\varphi_1(t) - \varphi_0(t)\|.$$

We next define

$$Q_p(t) = \begin{cases} B(\varphi_p(t), [(\alpha+\beta+\gamma)q]^p\|\varphi_1(t) - \varphi_0(t)\|), & \text{if} \quad t \in K \text{ and } \varphi_1(t) \neq \varphi_0(t) \\ \{\varphi_p(t)\}, & \text{if} \quad \varphi_1(t) = \varphi_0(t) \end{cases}$$

Obviously $T(\varphi_p(t)) \cap Q_p(t) \neq \emptyset$, for all $t \in K$. We can apply now (taking $G(t) = F(\varphi_p(t))$, $f(t) = \varphi_p(t)$ and the mapping $g(t) = [(\alpha+\beta+\gamma)q]^p\|\varphi_1(t) - \varphi_0(t)\|$, for all $t \in K$). Proposition 3 from [69], we obtain that the multi-function $t \mapsto \overline{T(\varphi_p(t)) \cap Q_p(t)}$ is l.s.c. on K with nonempty, closed, convex values. Because of Theorem 2 in [69], this produces a continuous function $\varphi_{p+1} : K \to X$ such that $\varphi_{p+1}|_t = \psi$ and $\varphi_{p+1}(t) \in \overline{T(\varphi_p(H)) \cap Q_p(t)}$, for all $t \in T$. Thus the existence of the sequence $\{\varphi_n\}$ is established. Consider now the open covering of K defined by the formula: $(\{t \in K|\ \|\varphi_1(t) - \varphi_0(t)\| < \lambda\})_{\lambda>0}$. Moreover, because of (iii) and the fact that X is complete, the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ converges uniformly on each of the following set $K_\lambda = \{t \in K|\ \|\varphi_1(t) - \varphi_0(t)\| < \lambda\}$ $(\lambda > 0)$. Let $\varphi : K \to X$ be the pointwise limit of $(\varphi_n)_{n\in\mathbb{N}}$. Obviously φ is continuous and $\varphi|_A = \psi$. Moreover, a simple computation ensures that : $\varphi(t) \in T(\varphi(t))$ for all $t \in K$ and this completes the proof. □

Remark 4.11. *If $\beta = \gamma = 0$ then the previous theorem coincides with B. Ricceri's result (Theorem 4.2 below).*

Remark 4.12. *Of course, it is also possible to formulate version of Theorem 4.10., where T is a multi-valued Rus-type graphic contraction. It is an open question if such a result holds for a Frigon-Granas type multi-function.*

The second aim of this section is to study the data dependence problem of the fixed points set for some Frigon-Granas-type multifunctions.

Theorem 4.13. *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T_1, T_2 : \widetilde{B}(x_0; r) \to P_{cl}(X)$ be multifunctions satisfying the following assertions:*

i) T_i is a multivalued α_i-contraction, for $i \in \{1, 2\}$

ii) $\delta(x_0, T_i(x_0)) < (1 - \alpha_i)r$, for $i \in \{1, 2\}$

iii) there exists $\eta > 0$ such that $H(T_1(x), T_2(x)) \leq \eta$, for each $x \in \widetilde{B}(x_0; r)$.

Then:

a) $FixT_i \in P_{cl}(\widetilde{B}(x_0; r))$ for $i \in \{1, 2\}$

b) $H(FixT_1, FixT_2) \leq \dfrac{\eta}{1 - \max\{\alpha_1, \alpha_2\}}$.

Proof. a) We denote by T each of the two multifunctions T_1 and T_2. For each $x \in \widetilde{B}(x_0; r)$ we have that $T(x) \subset \widetilde{B}(x_0; r)$. Indeed, let $u \in T(x)$ be arbitrary. Then

$$d(x_0, u) \leq d(x_0, y) + d(y, u) \leq (1 - \alpha)r + d(y, u) \quad \text{for each} \quad y \in T(x_0).$$

Taking the infimum after $y \in T(x_0)$ we get:

$$d(x_0, u) \leq (1 - \alpha)r + D(u, T(x_0)) \leq (1 - \alpha)r + H(T(x), T(x_0)) \leq$$

$$\leq (1 - \alpha)r + \alpha d(x, x_0) \leq r,$$

proving the fact that $T : \widetilde{B}(x_0; r) \to P_{cl}(\widetilde{B}(x_0; r))$. The conclusion a) follows from Covitz-Nadler theorem (see [23]) and the part b) from Rus-Petruşel-Sîntămărian result in [73]. □

Remark 4.14. *It is an open problem, the data dependence of the fixed points set for the Frigon-Granas-type multifunctions satisfying the weaker condition*

$$D(x_0, T(x_0)) < (1 - \alpha)r.$$

Similarily, we can prove:

Theorem 4.15. *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T_1, T_2 : \widetilde{B}(x_0; r) \to P_{cl}(X)$ be multifunctions such that:*

i) T_i is a multivalued Reich-type operator with $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}_+$, $\alpha_i + \beta_i + \gamma_i < 1$, for each $i \in \{1, 2\}$.

ii) $\delta(x_0, T_i(x_0)) < \frac{1-(\alpha_i+\beta_i+\gamma_i)}{1-\gamma_i} r$ *for* $i \in \{1,2\}$.

iii) there exists $\eta > 0$ *such that* $H(T_1(x), T_2(x)) \leq \eta$, *for each* $x \in \widetilde{B}(x_0; r)$.

Then:

a) $FixT_i \in P_{cl}(\widetilde{B}(x_0, r))$, *for* $i \in \{1,2\}$

b) $H(FixT_1, FixT_2) \leq \frac{1-\min\{\gamma_1,\gamma_2\}}{1-\max\{\alpha_1+\beta_1+\gamma_1, \alpha_2+\beta_2+\gamma_2\}}$.

Let us remark now that, if $T : \widetilde{B}(x_0; r) \to P_{cl}(X)$ is a Frigon-Granas-type multifunction, such that

$$D(x_0, T(x_0)) < (1-\alpha)r,$$

then T is not necessarily a m.w.P.o.

This situation suggests the following concept:

Definition 4.16. Let (X, d) be a metric space, $Y \in P(X)$ and $T : Y \to P(X)$ be a multivalued operator. By definition, T is a multivalued weakly pseudo-Picard operator (briefly m.w.p.P.o.) if and only if there exist $x_0 \in Y$, $y_0 \in T(x_0)$ and a sequence $(u_n)_{n\in\mathbb{N}} \subset Y$ such that

i) $u_0 = x_0$, $u_1 = y_0$

ii) $u_{n+1} \in T(u_n)$ for all $n \in \mathbb{N}$

iii) the sequence $(u_n)_{n\in\mathbb{N}}$ converges and its limit is a fixed point of T.

Example 4.17. *Let* (X, d) *be a complete metric space,* $x_0 \in X$, $r > 0$ *and* $T : \widetilde{B}(x_0; r) \to P_{cl}(X)$ *a multifunction such that* $D(x_0, T(x_0)) < (1-\alpha)r$. *Then* T *is a m.w.p.P.o.*

Proof. The conclusion follows from Frigon-Granas theorem (see [29] Theorem 3.1).□

Example 4.18. *Let* (X, d) *be a complete metric space,* $x_0 \in X$, $r > 0$ *and* $T : \widetilde{B}(x_0; r) \to P_{cl}(X)$ *a Reich-type multifunction, such that*

$$D(x_0, T(x_0)) < \frac{1-(\alpha+\beta+\gamma)}{1-\gamma} r.$$

Then T *is a m.w.p.P.o.*

Proof. The conclusion follows from Theorem 4.3. □

Example 4.19. *Let* (X, d) *be a complete metric space,* $\varepsilon > 0$ *and* $T : X \to P_{cl}(X)$ *be a multi-valued operator satisfying the following assertions:*

i) there exists $\alpha \in [0, 1[$ *such that for each* $x, y \in X$ *with* $d(x, y) < \varepsilon$

$$H(T(x), T(y)) \leq \alpha d(x, y).$$

ii) there exists $x_0 \in X$ *such that* $D(x_0, T(x_0)) < \varepsilon$.

Then T *is a m.w.p.P.o.*

Proof. The conclusion is an easy consequence of Theorem 3.1 in [10]. □

Example 4.20. *Let* (X, d) *be a complete metric space and let* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a map which satisfies:*

a) φ *is nondecreasing and continuous from the right*

b) for each $u > 0$, $\sum_{n\geq 0} \varphi^n(u) < \infty$

c) $\varphi(u) = 0$ if and only if $u = 0$.

Assume that $T : X \to P_{b,cl}(X)$ is a multifunction for which there exists $\varepsilon > 0$ such that the following two conditions are satisfied:

i) for each $x, y \in X$ with $d(x, y) < \varepsilon$ we have that

$$H(T(x), T(y)) \leq \varphi(d(x, y))$$

ii) there exists $x_0 \in X$ such that $D(x_0, T(x_0)) < \varepsilon$.
Then T is a m.w.p.P.o.

Proof. The conclusion follows from Theorem 3.1 in [10]. □

In [74], we proved the data dependence of the fixed points set for so-called c-m.w.P.o. It is an open question, if some similar results can be obtained for some multivalued weakly pseudo-Picard operators.

5 Fixed points and selections for multifunctions with decomposable values

Throughout this section $(\Omega, \mathcal{A}, \mu)$ is a complete σ-finite nonatomic measure space and E is a Banach space. Let $L^1(\Omega, E)$ be the Banach space of all measurable functions $u : \Omega \to E$ which are Bochner μ-integrable. We call a set $K \subset L^1(\Omega, E)$ decomposable if for all $u, v \in K$ and each $A \in \mathcal{A}$:

$$u\chi_A + v\chi_{\Omega \setminus A} \in K, \tag{5.2}$$

where χ_A stands for the characteristic function of the set A.

This notion is, somehow, similar to convexity, but there exist also major differences. For example, the following theorem is a "decomposable" version of the well-known Michael's selection theorem for l.s.c. multifunctions.

Theorem 5.1. (see [15]) *Let (X, d) be a separable metric space, E a separable Banach space and let $F : X \to P_{cl,dec}(L^1(\Omega, E))$ be a l.s.c. multivalued operator. Then F has a continuous selection.*

Our first result, concerning the existence of continuous selections for a locally selectionable multifunction, is as follows:

Lemma 5.2. *Let (X, d) be a separable metric space, $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite and nonatomic measure space and E be a Banach space. Let $F : X \to P_{dec}(L^1(\Omega, E))$ be a locally selectionable multivalued operator. Then F has a continuous selection.*

Proof We associate to any $y \in X$ and $z \in F(y)$ an open neighborhood $N(y)$ and a local continuous selection $f_y : N(y) \to L^1(\Omega, E)$, satisfying $f_y(y) = z$ and $f_y(x) \in F(x)$ when $x \in N(y)$. We denote by $\{V_n\}_{n\in\mathbb{N}^*}$ a countable locally finite open refinement of the open covering $\{N(y)|\ y \in X\}$ and by $\{\psi_n\}_{n\in\mathbb{N}^*}$ a continuous partition of unity associated to $\{V_n\}_{n\in\mathbb{N}^*}$.

Then, for each $n \in \mathbb{N}^*$ there exist $y_n \in X$ such that $V_n \subset N(y_n)$ and a continuous function $f_{y_n} : N(y_n) \to L^1(\Omega, E)$ with $f_{y_n}(y_n) = z_n$, $f_{y_n}(x) \in F(x)$, for all $x \in N(y_n)$.

We define $\lambda_0(x) = 0$ and $\lambda_n(x) = \sum_{m \le n} \psi_m(x)$, $n \in \mathbb{N}\mathbb{N}^*$. Let $g_{m,n} \in L^1(\Omega, \mathbb{R}_+)$ be the function defined by $g_{m,n}(t) = \|z_n(t) - z_m(t)\|$, for each $m, n \ge 1$.

We arrange these functions into a sequence $\{g_k\}_{k \in \mathbb{N}\mathbb{N}^*}$.

Consider the function $\tau(x) = \sum_{m,n \ge 1} \psi_m(x)\psi_n(x)$. From Lemma 1 in [15], there exists a family $\{\Omega(\tau, \lambda)\}$ of measurable subsets of Ω such that:
(a) $\Omega(\tau, \lambda_1) \subseteq \Omega(\tau, \lambda_2)$, if $\lambda_1 \le \lambda_2$
(b) $\mu(\Omega(\tau_1, \lambda_1)\Delta\Omega(\tau_2, \lambda_2)) \le |\lambda_1 - \lambda_2| + 2|\tau_1 - \tau_2|$
(c) $\int_{\Omega(\tau,\lambda)} g_n d\mu = \lambda \int_\Omega g_n d_\mu$, $\forall\ n \le \tau_0$ for all $\lambda, \lambda_1, \lambda_2 \in [0, 1]$, and all $\tau_0, \tau_1, \tau_2 \ge 0$.

Define $f_n(x) = f_{y_n}(x)$ and $\chi_n(x) = \chi_{\Omega(\tau(x),\lambda_n(x)) \setminus \Omega(\tau(x),\lambda_{n-1}(x))}$ for each $n \in \mathbb{N}\mathbb{N}^*$.

Let us consider the singlevalued operator $f : X \to L^1(\Omega, E)$, defined by $f(x) = \sum_{n \ge 1} f_n(x)\chi_n(x)$, $x \in X$.Then, f is continuous because the functions τ and λ_n are continuous, the characteristic function of the set $\Omega(\tau, \lambda)$ varies continuously in $L^1(\Omega, E)$ with respect to the parameters τ and λ and because the summation defining f is locally finite. On the other hand, from the properties of the sets $\Omega(\tau, \lambda)$ (see [64] for example) and because F has decomposable values, it follows that f is a selection of F. □

The following result is a selection theorem for the intersection of two multi-valued operators.

Theorem 5.3. *Let (X, d) be a separable metric space, E a separable Banach space, $F : X \to P_{cl,dec}(L^1(\Omega, E))$ be a l.s.c. multivalued operator and $G : X \to P_{dec}(L^1(\Omega, E))$ be with open graph. If $F(x) \cap G(x) \ne \emptyset$ for each $x \in X$ then there exists a continuous selection of $F \cap G$.*

Proof Let $x_0 \in X$ and for each $y_0 \in F(x_0)$ we define the multifunction

$$F_0(x) = \begin{cases} \{y_0\}, & \text{if } x = x_0 \\ F(x), & \text{if } x \ne x_0. \end{cases}$$

Obviously $F_0 : X \to \mathcal{P}_{cl,dec}(L^1(\Omega, E))$ is l.s.c. From Theorem 5.1. there exists a continuous selection f of F_0, i.e. $f_0(x_0) = y_0$ and $f_0(x) \in F(x)$, for each $x \in X$, $x \ne x_0$. Using Proposition 4, p.81 in [5] it follows that $F \cap G$ is locally selectionable at x_0 and has decomposable values. From Lemma 5.2. the conclusion follows. □

An important result is the following Browder-type selection theorem:

Theorem 5.4. *Let E be a Banach space such that $L^1(\Omega, E)$ is separable. Let K be a nonempty, paracompact, decomposable subset of $L^1(\Omega, E)$ and let $F : K \to P_{dec}(K)$ be a multivalued operator with open fibres. Then F has a continuous selection.*

Proof For each $y \in K$, $F^{-1}(y)$ is an open subset of K. Since K is paracompact it follows that the open covering $\{F^{-1}(y)\}_{y \in K}$ admits a locally finite open refinement, let say $K = \bigcup_{j \in J} F^{-1}(y_j)$, with $y_j \in K$. Let $\{\psi_j\}_{j \in J}$ be a continuous partition of unity subordinate to $\{F^{-1}(y_j)\}_{j \in J}$. Using the same construction as in the proof of Lemma 5.2., one can construct a continuous function $f : K \to K$, $f(x) = \sum_{j \in J} f_j(x)\chi_j(x)$, where $f_j(x) \in F(x)$ for each $x \in K$. This function is a continuous selection for F. □

Theorem 5.5. *Let E be a Banach space such that $L^1(\Omega, E)$ is separable. Let I be an arbitrary set of indices, $\{K_i | i \in I\}$ be a family of nonempty, decomposable subsets of $L^1(\Omega, E)$ and X a paracompact space. Let us suppose that the family $\{F_i : X \to \mathcal{P}_{dec}(K_i) | i \in I\}$ is of Ky Fan-type. Then there exists a selecting family for $\{F_i\}_{i\in I}$.*

Proof Let $\{U_i\}_{i\in I}$ be the open covering of the paracompact space X given by $U_i = \{x \in X |\ F_i(x) \neq \emptyset\}$ for each $i \in I$. It follows that there exists a locally finite open cover $\{W_i\}_{i\in I}$ such that $\overline{W_i} \subset U_i$ for $i \in I$. Let $V_i = \overline{W_i}$. For each $i \in I$ let us consider the multivalued operator $G_i : X \to \mathcal{P}(K_i)$, defined by the relation

$$G_i(x) = \begin{cases} F_i(x), & \text{if } x \in V_i \\ K_i, & \text{if } x \notin V_i. \end{cases}$$

Then G_i is a multifunction with nonempty and decomposable values having open fibres (indeed, $G_i^{-1}(y) = F_i^{-1}(y) \cup (X \backslash V_i))$, for each $i \in I$.

Using Theorem 4.4. we have that there exists $f_i : X \to K_i$ continuous selection for G_i $(i \in I)$, for each $i \in I$. It follows that for each $x \in X$ there exists $i \in I$ such that $x \in V_i$ and hence $f_i(x) \in G_i(x) = F_i(x)$, proving that $\{f_i\}_{i\in I}$ is a selecting family for $\{F_i\}_{i\in I}$. □

By a similar argument we have:

Theorem 5.6. *Let E be a separable Banach space and X a separable metric space. Let I be an arbitrary set of indices, $\{K_i | i \in I\}$ be a family of nonempty, closed, decomposable subsets of $L^1(\Omega, E)$. Let $\{F_i : X \to \mathcal{P}_{cl,dec}(K_i) | i \in I\}$ be a family of l.s.c. multivalued operators such that for each $x \in X$ there is $i \in I$ such that $F_i(x) \neq \emptyset$. Then $\{F_i\}_{i\in I}$ has a selecting family.*

References

[1] M.C. Anisiu, *Point-to-set mappings. Continuity*, Seminar on Fixed Point Theory, Babeş-Bolyai Univ. , 1981, 1-100.

[2] M.C. Anisiu, O. Mark, *Some properties of the fixed points set for multifunctions*, Studia Univ. Babeş-Bolyai, Mathematica, XXV(1980), 77-79.

[3] Q.H. Ansari, A. Idzik, J.-C. Yao, *Coincideance and fixed point theorems with applications*, Topol. Meth. Nonlinear Anal., 15(2000), 191-202.

[4] J.P. Aubin, *Viability Theory*, Birkhauser, Basel, 1991.

[5] J.P. Aubin, A. Cellina, *Dif and only iferential inclusions*, Springer, Berlin, 1984.

[6] J.P. Aubin, H. Frankowska, *Set-Valued Analysis*, Birkhauser, Basel, 1990.

[7] R.J. Aumann, *Integrals of set-valued functions*, J. Math. Anal. Appl., 12(1965), 1-12.

[8] C. Avramescu, *Teoreme de punct fix pentru aplicaţii multivoce contractante definite în spaţii uniforme*, An. Univ. Craiova, 1(1970), 63-67.

[9] Y.M. Ayerbe Toledano, T. Dominguez Benavides, L. López Acedo, *Measures of non-compactness in metric fixed point theory*, Birkhäuser Verlag, Basel, 1997.

[10] E. Barcz, *Some fixed point theorems for multivalued mappings*, Dem. Math., 16(1983), 735-744.

[11] T. Bânzaru, B. Rendi, *Topologies on spaces of subsets and multivalued mappings*, Ed. Univ. Timişoara, 1997.

[12] G. Beer, *Topologies on closed and closed convex sets*, Kluwer Acad. Publ., Dordrecht, 1993.

[13] C. Berge, *Espaces topologiques. Fonctions multivoques*, Dunod, Paris, 1959.

[14] K. Border, *Fixed point theorems with applications to economics and game theory*, Cambridge Univ. Press, Cambridge, 1985.

[15] A. Bressan, G. Colombo, *Extensions and selections of maps with decomposable values*, Studia Math., 90(1988), 69-86.

[16] A. Bressan, A. Cellina, A. Fryszkowski, *A class of absolute retracts in spaces of integrable functions*, Proc. A.M.S., 12(1991), 413-418.

[17] F.E. Browder, *The fixed point theory of multivalued mappings in topological spaces*, Math. Ann., 177(1968), 283-301.

[18] C. Castaing, M. Valadier, *Convex analysis and measurable multifunctions*, Springer, Berlin, 1977.

[19] A. Cellina, G. Colombo, *Some qualitative and quantitative results on dif and only iferential inclusions*, Rend. Sem. Mat. Univ. Polit. Torino,, 48(1990), 105-124.

[20] A. Cernea, *Incluziuni diferenţiale şi aplicaţii*, Editura Univ. Bucureşti, 2000.

[21] G. Colletti, G. Regoli, *Continuous selections and multivalued dif and only iferential equations*, Riv. Mat. Univ. Parma, 7(1981), 361-366.

[22] A. Constantin, *Stability of solution set of dif and only iferential equations with multivalued right-hand side*, J. Dif and only if. Eq., 114(1994), 243-252.

[23] H. Covitz, S.B. Nadler jr., *Multivalued contraction mappings in generalized metric spaces*, Israel J. Math., 8(1970), 5-11.

[24] P. Deguire, *Browder-Fan fixed point theorem and related results*, Discuss. Math. Differential Inclusions, 15(1995), 149-162.

[25] P.Deguire, M. Lassonde, *Familles selectantes*, Topological Methods in Nonlinear Anal., 5(1995), 261-269.

[26] K. Deimling, *Multivalued differential equations*, W. de Gruyter, 1992.

[27] J. Dugundji, A. Granas, *Fixed point theory*, P.W.N., Warszawa, 1982.

[28] Z. Dzedzej, B.D. Gelman, *Dimension of the solution set for differential inclusions*, Demonstratio Math., 26(1993), 149-158.

[29] M. Frigon, A. Granas, *Résultats du type de Leray-Schauder pour les contractions multivoques*, Topol. Math. Nonlinear Anal. 4(1994), 197-208.

[30] K. Goebel, W.A. Kirk, *Topics in metric fixed point theory*, Cambridge Univ. Press, 1990.

[31] L. Gorniewicz, S.A. Marano, *On the fixed point set of multivalued contractions*, Rend. Circolo Mat. Palermo, 40(1996), 139-145.

[32] L. Gorniewicz, S.A. Marano, M. Slosarki, *Fixed points of contractive multivalued maps*, Proc. AMS, 124(1996), 2675-2683.

[33] F. Hiai, H. Umegake, *Integrals, conditional expectations and martingales of multivalued functions*, J. Multiv. Anal., 7(1977), 149-182.

[34] C.J. Himmelberg, F.S. Van Vieck, *Lipschitzian generalized differential equations*, Rend. Sem. Mat. Univ. Padova, 48(1973), 159-169.

[35] C.J. Himmelberg, *Measurable relations*, Fund. Math., 87(1975), 53-72.

[36] S. Hu, N. Papageorgiou, *Handbook of multivalued analysis*, Kluwer Acad. Publ.,vol.1-2, 1997-1999.

[37] C.F.K. Jung, *On generalized complete metric spaces*, Bull. A.M.S., 75(1969), 113-116.

[38] M. Kisielewicz, *Differential inclusions and optimal control*, Kluwer Acad. Press, 1991.

[39] K. Kuratowski, C. Ryll-Nardzewski, *A general theorem on selectors*, Bull. Polon. Acad. Sci., 13(1965), 397-403.

[40] T.C. Lim, *On fixed point stability for set-valued contractive mappings with applications to generalized differential equations*, J. Math. Anal. Appl., 110(1985), 436-441.

[41] S.A. Marano, *Fixed points of multivalued contractions*, Rend. Circolo Mat. Palermo, 48(1997), 171-177.

[42] E. Michael, *Continuous selections* (I), Ann. Math., 63(1956), 361-382.

[43] M. Mureşan, *An introduction to set-valued analysis*, Cluj Univ. Press, 1999.

[44] S.B. Nadler jr., *Multivalued contraction mappings*, Pacific J. Math., 30(1969), 475-488.

[45] C. Olech, *Decomposability as substitute for convexity*, Lect. Notes in Math., 1091, Springer, Berlin, 1984.

[46] D. O'Regan, *Fixed point theory for compact upper semi-continuous or lower semi-continuoues set valued maps*, Proc. A.M.S., 125(1997), 875-881.

[47] D. O'Regan, R. Precup, *Fixed point theorems for set-valued maps and existence principles for integral inclusions*, J. Math. Anal. Appl., 245(2000), 594-612.

[48] D. O'Regan, R. Precup, *Theorems of Leray-Schauder type and applications*, Gordon and Breach Sci. Publ., 2001.

[49] A. Petruşel, *Generalized multivalued contractions*, NonLinear Anal. T.M.A., 2001, 47(2001), 649-659.

[50] A. Petruşel , *Integral Inclusions. Fixed Point Approaches*, Ann. Soc. Math. Polonae, Commentationes Math., XL(2000), 147-158

[51] A. Petruşel, *Multivalued operators and fixed points*, Pure Math.Appl., 11(2000), 361-368.

[52] A. Petruşel, *Multivalued operators and continuous selections*, Pure Math. Appl., 9(1998), 165-170.

[53] A. Petruşel, *Fixed points for multifunctions on generalized metric spaces with applications to a multivalued Cauchy problem*, Comment. Math. Univ. Carolinae, 38(1997), 657-663.

[54] A. Petruşel, E. Kirr, *Continuous dependence on parameters of the fixed points set for some set-valued operators*, Discussiones Math.-Differential Inclusions, 17(1997), 29-41.

[55] A. Petruşel, *On a theorem by Roman Wegrzyk*, Demonstratio Math., 29(1996), 637-641.

[56] A. Petruşel, *A topological property of the fixed points set for a class of multivalued operators*, Mathematica, 40(63), No.2(1998), 269-275.

[57] A. Petruşel, E. Kirr, *Continuous dependence and fixed points for some multivalued operators*, Revue D'Analse Num. Th.Approx., 26(1997), 99-101.

[58] A. Petruşel, *On the fixed points set for some contractive multivalued operators*, Mathematica, Cluj, 2(2000). (to appear)

[59] A. Petruşel, *On Frigon-Granas type operators*, (to appear)

[60] A. Petruşel, I.A.Rus, *Dynamics on* $(P_{cp}(X), H_d)$ *generated by a finite family of multi-valued operators on* (X, d). (to appear)

[61] A. Petruşel, *Singlevalued and multivalued Meir-Keeler type operators*, Revue D'Analse Num. Th.Approx., 2001. (to appear)

[62] A. Petruşel, (ε, φ)*-locally contractive multivalued mappings and applications*, Studia Univ. Babeş-Bolyai, Seria Mathematica, 36(1991), 24-36.

[63] A. Petruşel, *Fixed points of retractible multivalued operators*, Studia Univ. Babeş-Bolyai, Seria Mathematica, 38(1993), 57-63.

[64] A. Petruşel, G. Moţ, *Convexity and Decomposability in Multivalued Analysis*, Lecture Notes in Economics and Mathemaical Sciences, (Proc. of the Generalized Convexity/Monotonicity Conference, Samos, Greece, 1999, Editors: N. Hadjisavvas, J.-M. Martinez Legaz, J. P. Penot), Springer, Berlin, 2001, 333-341.

[65] J.S. Raymond, *Multivalued contractions*, Set-Valued Analysis, 2(1994), 559-571.

[66] S. Reich, *Kannan's fixed point theorem*, Boll. U.M.I., 4(1971), 1-11.

[67] S. Reich, *Fixed point of contractive functions*, Boll. U.M.I., 5(1972), 26-42.

[68] D. Repovš, P.V. Semenov, *Continuous selections of multivalued mappings*, Kluwer Academic Publ., Dordrecht, 1998.

[69] B. Ricceri, *Une propriété topologique de l'ensemble des points fixed d'une contraction multivoque à valeurs convexes*, Atti. Acc. Lincei, LXXXI(1987), 283-286.

[70] I.A. Rus, *Fixed point theorems for multivalued mappings in complete metric spaces*, Math. Japonica, 20(1975), 21-24.

[71] I.A. Rus, *Technique of the fixed point structures for multivalued mappings*, Math. Japonica, 2(1993), 289-296.

[72] I.A. Rus, *Generalized contractions and applications*, Cluj Univ. Press, 2001.

[73] I.A.Rus, A Petruşel şi A. Sîntămărian, *Data dependence for multi-valued weakly Picard opeartors* Studia Univ. Babeş-Bolyai, Seria Mathematica, Cluj, 46(2), 2001, 111-121.

[74] I.A.Rus, A. Petruşel şi A. Sîntămărian, *Data dependence for multi-valued c-weakly Picard opeartors.* (to appear)

[75] L. Rybinski, *An application of the continuous selection theorem to the study of the fixed points of multivalued mappings*, J. Math. Anal. Appl., 153(1990), 391-396.

[76] A. Sîntămărian, *Metrical strict fixed point theorms for multivalued mappings*, Seminar on Fixed Point Theory Cluj-Napoca, 1997, 27-31.

[77] H. Schirmer, *Properties of the fixed point set of contractive multifunctions*, Canad. Math. Bull., 13(1970), 169-173.

[78] R.E. Smithson, *Fixed points for contractive multifunctions*, Proc. A.M.S., 27(1971), 192-194.

[79] R. Wegrzyk, *Fixed point theorems for multifunctions and their applications to functional equations*, Diss. Math., 201(1982).

[80] H.K. Xu, *ε-chainability and fixed points of set-valued mappings in metric spaces*, Math. Japonica, 39(1994), 353-356.

[81] H.-K. Xu, I. Beg, *Measurability of fixed points sets of multivalued random operators*, J. Math. Anal. Appl., 225(1998), 62-72.

[82] M. Yamaguti, M. Hata, J. Kigani, *Mathematics of fractals*, Translations Math.Monograph, vol.167, AMS Providence, Rhode Island 1997.

[83] G.X.Z. Yuan, *KKM theory and applications in nonlinear analysis*, Marcel Dekker, New York, 1999.