Seminar on Fixed Point Theory Cluj-Napoca, **Volume 2, 2001, 71-80**
http://www.math.ubbcluj.ro/~nodeacj/journal.htm

SOME FIXED POINT THEOREMS FOR COMMUTING MULTIVALUED OPERATORS

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Abstract. Existence results for the fixed points of some multivalued operators which commute are proved.

Keywords: multivalued operator, fixed point, fixed point structures, generalized contractions
AMS Subject Classification: 47H10, 47H20.

1. Introduction

In this paper we give some results concerning the existence of fixed points of some multivalued operators which commute.

In connection with such a study there have been published, in the latest fifteen years, some open problems, which inspired our investigations too (see, Rus [1], [3], [5] and [6]).

The aim of this paper is to include in a unique background also the proofs of some theorems containing conditions which imply $F_T = (SF)_T = \{x^*\}$. For other similar results, see also Rus [6] and Sîntămărian [7].

In the last section, the general framework of our study is of the fixed point structures. From this perspective, we proposed three open questions, which prove that the topic is far from being exhausted.

2. Definitions and notations

Let (X,d) be a metric space. Throughout this paper we use the following notations:

$$P(X) := \{A \subset X \mid A \neq \varnothing\};$$

$$P_{cl}(X) := \{A \in P(X) \mid A = \bar{A}\};$$

$$P_{cp}(X) := \{A \in P(X) \mid A \quad \text{is a compact set}\}.$$

Now, let (X,d) be a metric space and $A \in P(X)$. Then we denote:

$$\delta(A) := \sup\{d(a,b) \mid a,b \in A\};$$

$$P_b(X) := \{A \in P(X) \mid \delta(A) < +\infty\};$$

$$P_{b,cl}(X) := P_b(X) \cap P_{cl}(X).$$

For a multivalued operator $T : X \multimap X$, we denote:

$$I(T) := \{A \in P(X) \mid T(A) \subset A\};$$

$$I_b(T) := \{A \in I(T) \mid \delta(A) < +\infty\};$$

$$I_{b,cl}(T) := \{A \in I_b(T) \mid A = \bar{A}\}.$$

Definition 2.1. Let $T : X \multimap X$ be a multivalued operator. Then, by definition, an element $x \in X$ is:

i) a ***fixed point of T*** iff $x \in T(x)$;

ii) a ***strict fixed point of T*** iff $T(x) = \{x\}$.

We denote by:

$i')$ $F_T := \{x \in X \mid x \in T(x)\}$ ***the fixed points set of T***;

$ii'')$ $(SF)_T := \{x \in X \mid T(x) = \{x\}\}$ ***the strict fixed points set of T***.

We denote by $(X, S(X), M^{\circ})$ a ***fixed point structure*** on X (briefly, ***f.p.s.***). More details for this notion and its applications can be found in Rus [4].

Definition 2.2. Let $T : X \multimap X$. By definition, a subset $A \subset X$ is a ***fixed subset of T*** if $T(A) = A$.

Definition 2.3. **(Tarafdar-Vyborny [8]).** A multivalued operator $T : X \to P_{cl}(X)$ is said to be a ***multivalued topological contraction*** if

i) T is upper semicontinuous on X;

ii) for each $A \in P_{cl}(X)$ with $T(A) = A$ implies that A is a single point, i.e. $A = \{x^*\}$ for some point $x^* \in X$.

Definition 2.4. Let (X, d) be a metric space. Then, a multivalued operator $T : X \to P_{b,cl}(X)$ is called:

i) ***a-contraction***, if $a \in]0, 1[$ and $H(T(x), T(y)) \leqslant a \cdot d(x, y)$, for all $x, y \in X$;

ii) ***contractive***, if $H(T(x), T(y)) < d(x, y)$, for all $x, y \in X$, $x \neq y$;

iii) (δ, a)***-contraction***, if $a \in]0, 1[$ and $\delta(T(A)) \leqslant a \cdot \delta(A)$, for all $A \in I_b(T)$.

iv) (δ, φ)***-contraction***, φ is a comparison function and $\delta(T(A)) \leqslant \varphi(\delta(A))$, for all $A \in I_b(T)$.

Definition 2.5. Let X be a nonempty set and $T, S : X \multimap X$ multivalued operators. Then ***the composed of T and S*** is a multivalued operator denoted by $T \circ S : X \multimap X$ given by $(T \circ S)(x) := \bigcup_{y \in S(x)} T(y)$.

Definition 2.6. If X is a nonempty set and $T, S : X \multimap X$ are multi-valued operators, then we say that T commutes with S if $T \circ S = S \circ T$.

3. Known results

Theorem 3.1. (Rus [1]). *Let X be a set and $f, g : X \to X$ comutative singlevalued mappings.*

If f has a unique fixed point, then g will have, at least, one fixed point.

Theorem 3.2. (Rus [3]). *Let $T : X \to P(X)$. If there exists $n \in \mathbb{N}$ such that $\{x^*\}$ is the unique fixed set for T^n, then $(SF)_T = \{x^*\}$.*

Theorem 3.3. (Rus [6]). *Let (X, d) be a metric space and $T : X \to P_{b,cl}(X)$ a multivalued contraction.*

If $(SF)_T \neq \varnothing$, then $F_T = (SF)_T = \{x^*\}$.

(The proof of this result can be found in [7]).

Theorem 3.4. (Tarafdar-Vyborny [8]). *Let X be a Hausdorff compact topological space and $T : X \to P(X)$, a multivalued topological contraction.*

Then, $(SF)_T = \{x^*\}$.

Theorem 3.5. (Rus [4]). *Let (X, d) be a bounded complete metric space and $T : X \to P(X)$ a (δ, φ)-contraction.*

Then, $F_T = (SF)_T = \{x^*\}$.

Theorem 3.6. (Rus [3]). *Let X be a nonempty set and $T : X \to P(X)$ a multivalued operator. Then $(SF)_T$ is a fixed set for T.*

Theorem 3.7. (Rus [2]). *Let (X, d) be a complete metric space, $T : X \to P_b(X)$ and*
$\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$. We suppose that:

i) $(r \leqslant s, \quad r, s \in \mathbb{R}_+^5) \quad \Rightarrow \quad \varphi(r) \leqslant \varphi(s)$;

ii) there exists $p > 1$ such that $\varphi(r, pr, pr, r, r) < r$, for all $r > 0$;

iii) $r - \varphi(r, pr, pr, r, r) \to +\infty$, *when* $r \to +\infty$;

iv) φ is continuous;

v) $\delta(T(x), T(y)) \leqslant \varphi(d(x, y), \delta(x, T(x)), \delta(y, T(y)), D(x, T(y)), D(y, T(x)))$, *for all* $x, y \in X$.

Then, T has a unique fixed point x^* and $T(x^*) = \{x^*\}$.

4. Main results

We shall first prove the following result, which extends the Theorem 3.1 in the multivoque case and which will be frequently used in proofs.

Theorem 4.1. *Let* $X \neq \varnothing$ *and* $T : X \to P(X)$.

The following statements are equivalent:

a) $(SF)_T \neq \varnothing$;

b) There exists a multivalued operator $U : X \to P(X)$, *having the following properties:* (b_1) $\{x^*\}$ *is the unique fixed set for* U;

(b_2) U *commutes with* T.

Proof. **Necessity:** $a) \Longrightarrow b)$. Let $x^* \in (SF)_T$.

The constant operator $U : X \to P(X)$, defined by $U(x) = \{x^*\}$, $(\forall)\, x \in X$, has a unique fixed set, namely $\{x^*\}$. Moreover, U commutes whith T. Indeed, for all $x \in X$, we have:

$$(T \circ U)(x) = T(U(x)) = T(\{x^*\}) = \{x^*\}$$

and

$$(U \circ T)(x) = U(T(x)) = \bigcup_{y \in T(x)} U(y) = \{x^*\}.$$

We therefore have $T \circ U = U \circ T$.

Sufficiency: $b) \Longrightarrow a)$. Since $\{x^*\}$ is a fixed set for U, we shall have $U(x^*) = U(\{x^*\}) = \{x^*\}$. Then, by commutativity, it follows that $U(T(x^*)) = (U \circ T)(x^*) = (T \circ U)(x^*) = T(x^*)$, which shows that $T(x^*)$ is a fixed set of U. As, by hypothesis (b_1), the unique fixed set of U is $\{x^*\}$, we obtain that $T(x^*) = \{x^*\}$. Therefore, $(SF)_T \neq \varnothing$. $\square$

Remark 4.2. As an immediate consequence of Theorem 4.1, we obtain an interesting multivalued fixed point theorem due to I.A.Rus [3]. Indeed, taking $U = T^n$, the assumption (b_2) is satisfied. Hence, Theorem 4.1 generalizes Theorem 3.2 (which is Lemma 4.1 of Rus in [3]).

The next results furnish various answers to the Open Problem 3.1 proposed by I.A.Rus in [3], having the following enunciation:

Being given the commuting multivalued operators $T_1, T_2 : X \multimap X$, which are the conditions on T_1, such that $(SF)_{T_2} \neq \varnothing$?

Theorem 4.3. *Let* (X, d) *be a metric space and* $T_1, T_2 : X \to P_{b,cl}(X)$ *be two multivalued operators such that* $T_1 \circ T_2 = T_2 \circ T_1$.

In these conditions, if T_1 *is a multivalued a-contraction and* $(SF)_{T_1} \neq \varnothing$, *then* $F_{T_2} \neq \varnothing$.

Proof. If we assume that $x^* \in (SF)_{T_1}$, then we have:

$$T_1(T_2(x^*)) = (T_1 \circ T_2)(x^*) = (T_2 \circ T_1)(x^*) = T_2(T_1(x^*)) = T_2(x^*).$$

Therefore, $T_2(x^*)$ is a fixed set for T_1 and hence there exists $y^* \in T_2(x^*)$, such that $y^* \in F_{T_1}$. We note that all hypotheses of Theorem 3.3 are satisfied for T_1. By Theorem 3.3 it follows that $F_{T_1} = (SF)_{T_1} = \{x^*\}$. Finally, from $y^* \in F_{T_1}$ we obtain $y^* = x^*$. Thus we can write $x^* \in T_2(x^*)$, i.e. $F_{T_2} \neq \varnothing$. □

Remark 4.4. From the proof of Theorem 4.3, we have obtained, in fact, even more, namely that:

If one of the operators T_1 or T_2, which commute each other, has strict fixed points and satisfies a contraction condition, then the two operators have a common fixed point, i.e. $F_{T_1} \cap F_{T_2} \neq \varnothing$.

Remark 4.5. The problem which appears is if the Theorem 4.3 remains true if we replace the property of T_1, respectively T_2, of being multivalued contraction, by other metric conditions. To this question we will give an affirmative answer by the Theorem 5.3, in this paper.

Corollary 4.6. *Let $(X,\ d)$ be a metric space and $T_1, T_2 : X \to P_{b,cl}(X)$ be multivalued operators such that $T_1 \circ T_2 = T_2 \circ T_1$.*

In these conditions, if T_1 is a contractive operator and $(SF)_{T_1} \neq \varnothing$, then $F_{T_2} \neq \varnothing$.

Proof. The demonstration can be made analogously with the proof of Theorem 4.3, by using the definition of the contractive operator. □

Theorem 4.7. *Let X be a Hausdorff compact topological space and $T_1, T_2 : X \to P(X)$ be multivalued operators such that:*

i) T_1 *is a topological contraction;*

ii) $T_1 \circ T_2 = T_2 \circ T_1$.

Then, $(SF)_{T_2} \neq \varnothing$.

Proof. Using Theorem 3.4, we obtain $(SF)_{T_1} = \{x^*\}$. Note that, by Theorem 3.6, $\{x^*\}$ is the fixed set for T_1 and it is unique, since T_1 is a multivalued topological contraction. Hence, applying Theorem 4.1, we conclude that $(SF)_{T_2} \neq \varnothing$. □

Theorem 4.8. *Let $(X,\ d)$ be a bounded complete metric space and $T_1, T_2 : X \to P(X)$ such that:*

i) T_1 *is a* (δ, a)*- contraction and* $T_1(x)$ *is closed set for any* $x \in X$*;*

ii) $T_1 \circ T_2 = T_2 \circ T_1$.

Then, $(SF)_{T_2} \neq \varnothing$.

Proof. Since T_1 is a (δ, a) - contraction, based on Theorem 3.5, we have $F_{T_1} = (SF)_{T_1} = \{x^*\}$.

Moreover, because T_1 is a (δ, a)- contraction, then T_1 is clearly a topological contraction.

Consequently, $\{x^*\}$ is the unique fixed set for T_1. Finally, applying Theorem 4.1, it results that $(SF)_{T_2} \neq \varnothing$. $\square$

We now establish a more general result.

Theorem 4.9. *Let (X, d) be a complete metric space, $T_1, T_2 : X \to P_b(X)$ and $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$. We suppose that:*

i) $(r \leqslant s, \quad r, s \in \mathbb{R}_+^5) \quad \Rightarrow \quad \varphi(r) \leqslant \varphi(s)$;

ii) there exists $p > 1$ such that $\varphi(r, pr, pr, r, r) < r$, for all $r > 0$;

iii) $r - \varphi(r, pr, pr, r, r) \to +\infty$, *when* $r \to +\infty$;

iv) φ is continuous;

v) $\delta(T_1(x), T_1(y)) \leqslant \varphi(d(x,y)\,,\, \delta(x, T_1(x))\,,\, \delta(y, T_1(y))\,,\, D(x, T_1(y))\,,\, D(y, T_1(x)))$, *for all*
$x, y \in X$;

vi) $T_1 \circ T_2 = T_2 \circ T_1$;

vii) for each $A \in P(X)$ with $T_1(A) = A$ implies that A contains only one element.

In these conditions, $(SF)_{T_2} \neq \varnothing$.

Proof. We remark that all hypotheses of Theorem 3.7 are satisfied for T_1. So, we deduce $F_{T_1} = (SF)_{T_1} = \{x^*\}$. From the Theorem 3.6, we have that $(SF)_{T_1}$ is a fixed set for T_1 and taking into account the hypothesis vii), it is unique. Now, applying Theorem 4.1, we conclude that $(SF)_{T_2} \neq \varnothing$. $\square$

In 1999, I.A.Rus proposed the following open problem (Problem 21 in Rus [6]):

Which are the f.p.s. $(X, S(X), M^\circ)$ with the following property
$Y \in S(X), \quad T \in M^\circ(Y), \quad (SF)_T \neq \varnothing \;\Rightarrow F_T = (SF)_T = \{x^*\}$?

As a partial solution to the above-mentioned open problem in the case of f.p.s. of the contractions we shall prove:

Theorem 4.10. *Let (X, d) be a complete metric space and $T : X \to P_{b,cl}(X)$ a multivalued operator. We suppose that:*

i) $H(T^n(x), T^n(y)) \leqslant \alpha_n \cdot d(x,y), \quad (\forall) x, y \in X$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n < \infty$;

ii) there exists $n \in \mathbb{N}^*$ such that $\{x^*\}$ is the unique fixed set for T^n.

In these conditions, $F_T = (SF)_T = \{x^*\}$.

Proof. From the convergence of series $\sum\limits_{n=1}^{\infty} \alpha_n$ results that $(\exists) n \in \mathbb{N}^*$ such that $\alpha_n < 1$. In other words, for this n, T^n is a multivalued contraction. In addition, from ii) it follows that $(SF)_{T^n} \neq \varnothing$.

Now, we may invoke Theorem 3.3 to conclude that $F_{T^n} = (SF)_{T^n} = \{x^*\}$.

On the other hand, in virtue of Theorem 3.2, we have $(SF)_T \neq \varnothing$. Hence, $\varnothing \neq (SF)_T \subset (SF)_{T^n} = \{x^*\}$ implies $(SF)_T = \{x^*\}$.

Finally, from $x^* \in F_T \subset F_{T^n} = \{x^*\}$ we obtain $F_T = \{x^*\}$. □

Theorem 4.11. *Let X be a Hausdorff compact topological space and $T : X \to P(X)$.*

We suppose that there exists $n \in \mathbb{N}^*$ such that:

i) T^n *is a topological contraction;*

ii) F_{T^n} *is a singleton.*

In these conditions, $F_T = (SF)_T = \{x^*\}$.

Proof. Applying Theorem 3.4, we obtain $(SF)_{T^n} = \{x^*\}$. Now, taking into account the hypothesis ii), it follows that $\{x^*\} = (SF)_{T^n} \subset F_{T^n} = \{z\}$, hence $x^* = z$.

Consequently, $(SF)_{T^n} = F_{T^n} = \{x^*\}$. Finally, by the some arguments as in the proof of Theorem 4.10, we obtain the conclusion. □

Theorem 4.12. *Let (X, d) be a bounded complete metric space and $T : X \to P(X)$ a multivalued operator.*

We suppose that there exists $n \in \mathbb{N}^*$ such that:

i) T^n *is a* (δ, φ) *- contraction;*

ii) $\{x^*\}$ *is the unique fixed set for* T^n.

In these conditions, $F_T = (SF)_T = \{x^*\}$.

Proof. Using Theorem 3.5, we obtain $F_{T^n} = (SF)_{T^n} = \{x^*\}$.

Further on, the reasoning is similar to the proof of Theorem 4.10. □

Theorem 4.13. *Let $(X, S(X), M^\circ)$ be a s.f.p.s. and (θ, η) a compatible pair with $(X, S(X), M^\circ)$. Let $Y \in \eta(Z)$ and $T \in M^\circ(Y)$.*

We suppose that:

i) $\theta \mid \eta(Z)$ *has the intersection property;*

ii) there exists $n \in \mathbb{N}^*$ *such that*

ii_1) T^n *is a* (θ, φ) *- contraction;*

ii_2) for each $A \subset X$ with $T^n(A) = A$ implies that A consists of single point;

ii_3) F_{T^n} *is a singleton.*

In these conditions, $F_T = (SF)_T = \{x^*\}$.

Proof. From the first general strict fixed point principle (see Rus [4], Theorem 4.2) we have that $(SF)_{T^n} \neq \varnothing$.

From ii_3) it follows that $\varnothing \neq (SF)_{T^n} \subset F_{T^n} = \{x^*\}$ and so, $F_{T^n} = (SF)_{T^n} = \{x^*\}$.

On the other hand, on the basis of Theorem 3.2 it results $(SF)_T \neq \varnothing$. Further on, the demonstration is made analogously with the proof of Theorem 4.10. □

5. Fixed point structure with the common fixed property

The purpose of this section is to extend in the multivoque case, the so-called ***fixed point structure with the common fixed point property***, introduced by I.A.Rus for singlevalued operators (see [5]). We begin with a new problem in connection with our topic, which can be formulated in terms of the fixed point structures.

Open problem 5.1. Which are the fixed point structures $(X, S(X), M^\circ)$, with the following property:
(c) $Y \in S(X)$ and $T_1, T_2 \in M^\circ(Y)$ such that $T_1 \circ T_2 = T_2 \circ T_1$ implies that $F_{T_1} \cap F_{T_2} \neq \varnothing$?

Definition 5.2. By definition, a fixed point structure which satisfies the condition (c) is called ***fixed point structure with the common fixed point property***.

To exemplify this concept, we will establish the following result, using Reich's strict fixed point structure.

Theorem 5.3. *Let (X, d) be a complete metric space. For* $Y \in P(X)$, *let* $M^\circ(Y) = \{T : Y \to P_{cl}(Y) \mid (\exists)\, a, b, c \in \mathbb{R}_+,\ a + b + c < 1,$ *such that* $\delta(T(x), T(y)) \leqslant a \cdot d(x, y) + b \cdot \delta(x, T(x)) + c \cdot \delta(y, T(y)),$ *for all* $x, y \in X\}$.

Let $T_1, T_2 \in M^\circ(Y)$ *such that* $T_1 \circ T_2 = T_2 \circ T_1$.

Then, $F_{T_1} = (SF)_{T_1} = F_{T_2} = (SF)_{T_2} = \{x^*\}$.

Proof. By Reich's strict fixed point theorem we have $F_{T_1} = (SF)_{T_1} = \{x^*\}$ and $F_{T_2} = (SF)_{T_2} = \{y^*\}$. We shall show that $x^* = y^*$. We suppose, on the contrary, that $x^* \neq y^*$. Then, we have $T_1(T_2(x^*)) = T_2(T_1(x^*)) = T_2(x^*)$ and hence there exists $z \in T_2(x^*)$, such that $z \in F_{T_1}$. So, $z = x^*$. Thus we can write $x^* \in T_2(x^*)$, i.e. $x^* \in F_{T_2}$. Finally, we obtain the contradiction $x^* = y^*$. Therefore , in these conditions, the multivalued operators T_1 and T_2 have a unique common strict fixed point. □

At the end of this paper we formulate other two open questions.

Open problem 5.4. Let $(X, S(X), M^\circ)$, be a f.p.s., $Y \in S(X)$ and $T_1, T_2 \in M^\circ(Y)$ such that $T_1 \circ T_2 = T_2 \circ T_1$.

Establish conditions on T_1 and T_2, which imply that $F_{T_1} \cap F_{T_2} \neq \varnothing$.

For this problem, we give a partial answer by the Remark 4.4, in the present paper.

We now suggest another direction of investigation, changing the previous problem into a problem of existence of the coincidence points.

Open problem 5.5. Let $(X, S(X), M^{\circ})$ be a f.p.s., $Y \in S(X)$ and $T_1, T_2 \in M^{\circ}(Y)$ such that $T_1 \circ T_2 = T_2 \circ T_1$.

Establish conditions on T_1 and T_2, which imply that there exists $x^* \in Y$ such that $T_1(x^*) \cap T_2(x^*) \neq \emptyset$.

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