Seminar on Fixed Point Theory **Cluj-Napoca 2002, 59-64**
http://www.math.ubbcluj.ro/~nodeacj/journal.htm

ON MULTIPLICATIVELY UNITARY PERFECT NUMBERS

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Abstract. In this paper we imtroduce the unitary multiplicative divisor function, and the notion of unitary multiplicative perfect number, unitary multiplicative superperfect number and multiplicatively unitary $k-l$ perfect number and characterize them.
Keywords: perfect number, unitary perfect number, divisor function, fixed points.
AMS Subject Classification: 11A25.

1 Introduction

Let $\sigma(n)$ be the sum of positive divisors of n,

$$\sigma(n) = \sum_{d|n} d,$$

$s(n)$ the sum of aliquet part of n, i. e. the positive divisors of n other than n itself, so that

$$s(n) = \sigma(n) - n.$$

It is well-known that a number n is called perfect if the sum of aliquot divisors of n is equal to n

$$s(n) = n,$$

or equivalently

$$\sigma(n) = 2n.$$

Perfect, amicable and sociable numbers are fixed points of the arithmetic function s and its iterates. (R. K. Guy [3], P. Erdős [1])
The Euclid-Euler theorem gives the form of even perfect numbers:

Lemma 1.1 *An even integer n is perfect iff there exist prime number p such that $n = 2^{p-1}q$, where $q = 2^p - 1$ is prime ("Mersenne prime").*

No odd perfect numbers are known.
The number n is called super-perfect if

$$\sigma(\sigma(n)) = 2n.$$

Suryanarayana and Kanold states [9], [5] the general form of even super-perfect numbers: $n = 2^{p-1}$, where $2^p - 1 = q$ is a prime (Mersenne prime).
No odd super-perfect numbers are known.
A divisor d of a natural number n is unitary divisor if $\left(d, \frac{n}{d}\right) = 1$, and n is unitary perfect if

$$\sigma^*(n) = 2n.$$

The notion of unitary perfect numbers introduced M. V. Subbarao and L. J. Waren in 1966 [8].
Five unitary perfect numbers are known, they are necessarilly even and its true that no unitary perfect numbers of the form $2^m s$ where s is a squarefree odd integer [2].
Sándor in [6] introduced the concept of multiplicatively divisor function $T(n)$ (see [4]) and multiplicatively perfect and superperfect number and characterize them. Let $T(n)$ denote the product of all divisors of n:

$$T(n) = \prod_{d|n} d.$$

We say that the number $n > 1$ multiplicatively perfect (or shortly m-perfect) if

$$T(n) = n^2,$$

and multiplicatively super-perfect (m-super-perfect), if

$$T(T(n)) = n^2.$$

In this paper we introduce the unitary muliplicative divisor function, and the notion of multiplicative unitary perfect, multiplicative unitary superperfect and k-l multiplicative unitary perfect numbers and characterize them.

2 Main results

Definition 2.1 *Let $T^*(n)$ denote the product of all unitary divisors of n:*

$$T^*(n) = \prod_{\substack{d \mid n \\ \left(d, \frac{n}{d}\right)}} d.$$

Let $T^{*k}(n)$ the kth iterate of $T^*(n)$:

$$T^{*k}(n) = T^*\left(T^{*(k-1)}(n)\right), \quad k \geq 1.$$

Definition 2.2 *The number $n > 1$ is* **multiplicatively unitary perfect** *(or shortly m unitary perfect) if*

$$T^*(n) = n^2,$$

multiplicatively unitary super-perfect *(m unitary super-perfect), if*

$$T^*(T^*(n)) = n^2,$$

and **multiplicatively unitary k-l perfect** *(m unitary $k-l$ perfect), if*

$$T^{*k}(n) = n^l.$$

First we prove the following result:

Lemma 2.1 *For $n \geq 1$*

$$T^*(n) = n^{\frac{\tau^*(n)}{2}}$$

where $\tau^*(n)$ denotes the number of unitary divisors of n.

Proof. If $d_1, d_2, \ldots, d_k$ are all unitary divisors of n, then

$$\{d_1, d_2, \ldots, d_k\} = \left\{\frac{n}{d_1}, \frac{n}{d_2}, \ldots, \frac{n}{d_k}\right\},$$

implying that

$$d_1 d_2 \ldots d_k = \frac{n}{d_1} \cdot \frac{n}{d_2} \cdots \frac{n}{d_k},$$

$$T(n) = n^{k/2}$$

where $k = \tau^*(n)$ denotes the number of unitary divisors of n.

□

Remark
The $T^*(n)$ function not multiplicative and not additive function.

Theorem 2.1 *All m-perfect numbers n have the following form: $n = p_1^{\alpha_1} p_2^{\alpha_2}$ where p_1, p_2 are distinct primes and $\alpha_1,\ \alpha_2 > 0$ natural numbers or $\alpha_1 = \alpha_2 = 0$.*
There are no $n > 1$ m-super-perfect numbers.

Proof. The $n = 1$ solution of the equation. If we assume that $n > 1$ we have $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ be the prime factorisation of $n > 1$. It is well-known that

$$\tau^*(n) = 2^{\omega(n)} = 2^k,$$

where $\omega(n)$ is the number of distinct prime divisors of n (see [7]).
We have by Lemma 2.1 that n multiplicatively unitary perfect iff

$$n^{\frac{\tau^*(n)}{2}} = n^2$$

or

$$\tau^*(n) = 4, \quad \omega(n) = 2,$$

which equivalent to the form $n = p_1^{\alpha_1} p_2^{\alpha_2}$.
Let $n > 1$ a natural number. We have

$$T^*(T^*(n)) = (T^*(n))^{\frac{\tau^*(T^*(n))}{2}} = n^{\frac{\tau^*(n)}{2} \frac{\tau^*(T^*(n))}{2}} \tag{2.1}$$

because $n > 1$

$$\tau^*(T^*(n)) = \tau^*\left(n^{\frac{\tau^*(n)}{2}}\right) = \tau^*(n). \tag{2.2}$$

From (2.1) and (2.2) we have that n unitary m superperfect number iff

$$n^{\left(\frac{\tau^*(n)}{2}\right)^2} = n^2,$$

or

$$(\tau^*(n))^2 = 8$$

which is impossible.

□

We now investigate the problem of the existence of $k - l$ unitary m perfect numbers. First we prove that:

Lemma 2.2 *If n a natural number such that $\omega(n) \geq l + 1$ we have:*

$$T^{*k}(n) \geq n^{2^{lk}}.$$

Proof. Let m be a smallest natural number with $\omega(m) = l + 1$ Because for $n \geq m$ with $\omega(n) \geq l + 1$

$$T^*(n) = n^{2^{\omega(n)}} \geq m^{2^{\omega(m)}} = T^*(m)$$

we have

$$T^*(n) = n^{2^{\omega(n)}}$$

the function $T^*(n)$ increasing function and therefore $T^{*k}(n)$ increasing too.
Let m be a smallest natural number with $\omega(n) = l+1$. We have $n \geq m$ and therefore

$$T^{*k}(n) \geq T^{*k}(m). \tag{2.3}$$

By induction and (2.1), (2.2) we have

$$T^{*k}(n) = n^{\left(\frac{\tau^*(n)}{2}\right)^k}. \tag{2.4}$$

But $\omega(n) = l + 1$ implies that

$$\tau^*(m) = 2^{l+1}. \tag{2.5}$$

From (2.3), (2.4), (2.5) we have

$$T^{*k}(n) \geq T^{*k}(m) \geq n^{2^{lk}}.$$

□

This lemma implies the following result:

Theorem 2.2 *If $l \neq 2^{mk}$ there are no $k-l$ m unitary perfect numbers.*
If $l = 2^{mk}$ every n natural number with $\omega(n) = m$ $k-l$ unitary m perfect number.

Theorem 2.3 *For every even perfect number and every k exist l suvh that n unitary $k-l$ perfect number.*

Proof. By Lemma 1.1, n even perfect iff $n = 2^{p-1}q$, where $q = 2^p - 1$ is prime which imlies that $\omega(n) = 2$, $\tau^*(n) = 4$ and

$$T^{*k}(n) = n^{\left(\frac{\tau^*(n)}{2}\right)^k} = n^{2^k} = n^l$$

with $l = 2^k$.

□

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