ON MULTIPLICATIVELY UNITARY PERFECT NUMBERS

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Abstract. In this paper we introduce the unitary multiplicative divisor function, and the notion of unitary multiplicative perfect number, unitary multiplicative superperfect number and multiplicatively unitary $k - l$ perfect number and characterize them.

Keywords: perfect number, unitary perfect number, divisor function, fixed points.

AMS Subject Classification: 11A25.

1 Introduction

Let $\sigma(n)$ be the sum of positive divisors of $n$,

$$\sigma(n) = \sum_{d|n} d,$$

$s(n)$ the sum of aliquet part of $n$, i. e. the positive divisors of $n$ other than $n$ itself, so that

$$s(n) = \sigma(n) - n.$$

It is well-known that a number $n$ is called perfect if the sum of aliquot divisors of $n$ is equal to $n$

$$s(n) = n,$$

or equivalently

$$\sigma(n) = 2n.$$

Perfect, amicable and sociable numbers are fixed points of the arithmetic function $s$ and its iterates. (R. K. Guy [3], P. Erdős [1])

The Euclid-Euler theorem gives the form of even perfect numbers:

Lemma 1.1 An even integer $n$ is perfect iff there exist prime number $p$ such that $n = 2^{p-1}q$, where $q = 2^p - 1$ is prime (“Mersenne prime”).
No odd perfect numbers are known.
The number $n$ is called super-perfect if
$$\sigma(\sigma(n)) = 2n.$$  
Suryanarayana and Kanoldt states [9], [5] the general form of even super-perfect numbers: $n = 2^{p-1}$, where $2^p - 1 = q$ is a prime (Mersenne prime).
No odd super-perfect numbers are known.
A divisor $d$ of a natural number $n$ is unitary divisor if $(d, \frac{n}{d}) = 1$, and $n$ is unitary perfect if
$$\sigma^*(n) = 2n.$$  
The notion of unitary perfect numbers introduced M. V. Subbarao and L. J. Waren in 1966 [8].
Five unitary perfect numbers are known, they are necessarily even and its true that no unitary perfect numbers of the form $2^m s$ where $s$ is a squarefree odd integer [2].
Sándor in [6] introduced the concept of multiplicatively divisor function $T(n)$ (see [4]) and multiplicatively perfect and superperfect number and characterize them. Let $T(n)$ denote the product of all divisors of $n$:
$$T(n) = \prod_{d|n} d.$$  
We say that the number $n > 1$ multiplicatively perfect (or shortly $m$-perfect) if
$$T(n) = n^2,$$  
and multiplicatively super-perfect ($m$-super-perfect), if
$$T(T(n)) = n^2.$$  
In this paper we introduce the unitary multiplicative divisor function, and the notion of multiplicative unitary perfect, multiplicative unitary superperfect and $k$-$l$ multiplicative unitary perfect numbers and characterize them.

2 Main results

Definition 2.1 Let $T^*(n)$ denote the product of all unitary divisors of $n$:
$$T^*(n) = \prod_{d|n, (d, \frac{n}{d}) = 1} d.$$  
Let $T^{*k}(n)$ the $k$th iterate of $T^*(n)$:
$$T^{*k}(n) = T^*\left(T^{*(k-1)}(n)\right), \quad k \geq 1.$$
Definition 2.2 The number $n > 1$ is multiplicatively unitary perfect (or shortly $m$ unitary perfect) if
\[ T^*(n) = n^2, \]
multiplicatively unitary super-perfect (or $m$ unitary super-perfect), if
\[ T^*(T^*(n)) = n^2, \]
and multiplicatively unitary $k$-l perfect (or $m$ unitary $k - l$ perfect), if
\[ T^*k(n) = n^l. \]

First we prove the following result:

Lemma 2.1 For $n \geq 1$
\[ T^*(n) = n^{\frac{\tau^*(n)}{2}} \]
where $\tau^*(n)$ denotes the number of unitary divisors of $n$.

Proof. If $d_1, d_2, \ldots, d_k$ are all unitary divisors of $n$, then
\[ \{d_1, d_2, \ldots, d_k\} = \left\{ \frac{n}{d_1}, \frac{n}{d_2}, \ldots, \frac{n}{d_k} \right\}, \]
implying that
\[ d_1d_2\ldots d_k = \frac{n}{d_1}\cdot \frac{n}{d_2}\cdot \ldots\cdot \frac{n}{d_k}, \]
\[ T(n) = n^{k/2} \]
where $k = \tau^*(n)$ denotes the number of unitary divisors of $n$. \hfill \Box

Remark
The $T^*(n)$ function not multiplicative and not additive function.

Theorem 2.1 All $m$-perfect numbers $n$ have the following form: $n = p_1^{\alpha_1}p_2^{\alpha_2}$ where $p_1, p_2$ are distinct primes and $\alpha_1, \alpha_2 > 0$ natural numbers or $\alpha_1 = \alpha_2 = 0$.

There are no $n > 1$ $m$-super-perfect numbers.

Proof. The $n = 1$ solution of the equation. If we assume that $n > 1$ we have $n = p_1^{\alpha_1}\ldots p_k^{\alpha_k}$ be the prime factorisation of $n > 1$. It is well-known that
\[ \tau^*(n) = 2^{\omega(n)} = 2^k, \]
where $\omega(n)$ is the number of distinct prime divisors of $n$ (see [7]).
We have by Lemma 2.1 that $n$ multiplicatively unitary perfect iff
\[ n^{\frac{\tau^*(n)}{2}} = n^2 \]
or
\[ \tau^*(n) = 4, \quad \omega(n) = 2, \]
which equivalent to the form \( n = p_1^{a_1} p_2^{a_2} \).

Let \( n > 1 \) a natural number. We have

\[
T^* (T^* (n)) = (T^* (n)) \frac{\tau^* (T^* (n))}{2} = n \frac{\tau^* (n)}{2} \frac{\tau^* (n)}{2}
\]

(2.1)

because \( n > 1 \)

\[
\tau^* (T^* (n)) = \tau^* \left( n \frac{\tau^* (n)}{2} \right) = \tau^* (n).
\]

(2.2)

From (2.1) and (2.2) we have that \( n \) unitary \( m \) superperfect number iff

\[
n \left( \frac{\tau^* (n)}{2} \right)^2 = n^2,
\]

or

\[
(\tau^* (n))^2 = 8
\]

which is impossible.

We now investigate the problem of the existence of \( k - l \) unitary \( m \) perfect numbers.

First we prove that:

**Lemma 2.2** If \( n \) a natural number such that \( \omega(n) \geq l + 1 \) we have:

\[
T^* k (n) \geq n^{2k}.
\]

**Proof.** Let \( m \) be a smallest natural number with \( \omega(m) = l + 1 \) Because for \( n \geq m \) with \( \omega(n) \geq l + 1 \)

\[
T^* (n) = n^{2\omega(n)} \geq m^{2\omega(m)} = T^* (m)
\]

we have

\[
T^* (n) = n^{2\omega(n)}
\]

the function \( T^* (n) \) increasing function and therefore \( T^* k (n) \) increasing too.

Let \( m \) be a smallest natural number with \( \omega(n) = l + 1 \). We have \( n \geq m \) and therefore

\[
T^* k (n) \geq T^* k (m).
\]

(2.3)

By induction and (2.1), (2.2) we have

\[
T^* k (n) = n \left( \frac{\tau^* (n)}{2} \right)^k.
\]

(2.4)

But \( \omega(n) = l + 1 \) implies that

\[
\tau^* (m) = 2^{l+1}.
\]

(2.5)

From (2.3), (2.4), (2.5) we have

\[
T^* k (n) \geq T^* k (m) \geq n^{2k}.
\]

This lemma implies the following result:
Theorem 2.2 If \( l \neq 2^{mk} \) there are no \( k-l \) unitary perfect numbers.  
If \( l = 2^{mk} \) every \( n \) natural number with \( \omega(n) = m \) \( k-l \) unitary \( m \) perfect number.

Theorem 2.3 For every even perfect number and every \( k \) exist \( l \) such that \( n \) unitary \( k-l \) perfect number.

Proof. By Lemma 1.1, \( n \) even perfect iff \( n = 2^{p-1}q \), where \( q = 2^p - 1 \) is prime which implies that \( \omega(n) = 2 \), \( \tau^*(n) = 4 \) and 
\[
T^k(n) = n \left( \frac{\tau^*(n)}{2} \right)^k = n^{2k} = n^l
\]
with \( l = 2^k \).

References


