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WEAKLY PICARD OPERATORS AND APPLICATIONS

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Abstract. In this paper we present the basic results of the weakly Picard operators theory and we apply these results to some problems of the theory of differential and integral equations. In this way we unify a number of classical results and give new results.

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1 Introduction

Beginning with 1983 ([36]) we developed the theory of weakly Picard operators ([37], [38], [41]-[47]). The purpose of this paper is to present the basic results of this theory, to give some applications and to formulate some open problems.

2 Weakly Picard operators on metric space

Let (X, d) be a metric space and $A : X \to X$ an operator. In this paper we shall use the following notations:

$P(X) := \{Y \subset X|\ Y \neq \emptyset\}$;

$F_A := \{x \in X|\ A(x) = x\}$ - the fixed point set of A;

$I(A) := \{Y \in P(X)|\ A(Y) \subset Y\}$ - the family of the nonempty invariant subsets of A;

$A^{n+1} := A \circ A^n,\ A^0 = 1_X,\ A^1 = A,\ n \in N.$

Definition 2.1 ([36], [37], [47]). An operator A is weakly Picard operator (WPO) if the sequence

$$(A^n(x))_{n \in N}$$

converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of A.

Definition 2.2 ([36], [37], [47]). If the operator A is WPO and $F_A = \{x^*\}$, then by definition A is a Picard operator (PO).

Remark 2.1. If A is a PO, then A is a Bessaga operator, i.e.,

$$F_{A^n} = F_A = \{x^*\}, \text{ for all } n \in N^*.$$

Remark 2.2. If A is WPO, then (see [37])

$$F_{A^n} = F_A \neq \emptyset, \text{ for all } n \in N^*.$$

Remark 2.3 (see [34]). An operator A is PO if and only if $F_A = \{x^*\}$ and $\{x^*\}$ is a global atractor for the discrete dynamic generated by the operator A.

Remark 2.4. For some example and properties of POs and WPOs see [36], [37], [47], [33], [38], [40]-[46], [48].

Remark 2.5. To establish if a given operator is or isn't PO or WPO is a very difficult problem. For example we have:

Discrete Markus-Yamabe conjecture ([10], [45]). Let A be a C^1 function from R^n into itself such that $A(0) = 0$ and for all $x \in R^n$, $JA(x)$ (the Jacobian matrix of A at x) has all its eigenvalues with modulus less than one. Then A is a Picard function.

Belitskii-Lyubich conjecture (see [50]). Let X be a Banach space, $\Omega \subset X$ an open subset and $A : \Omega \to X$ be a compact and continuously differentiable in Ω. Suppose D is a nonempty bounded convex open subset of X such that $A(\overline{D}) \subset \overline{D} \subset \Omega$ and $\sup\limits_{x \in \overline{D}} (A'(x)) < 1$ (r stand for the spectral radius). Then the operator $A : \overline{D} \to \overline{D}$ is a Picard operator.

Definition 2.3 ([36], [37], [47]). If A is WPO, then we consider the operator A^∞ defined by

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n \to \infty} A^n(x).$$

We remark that $A^\infty(X) = F_A$ and $\omega_A(x) = \{A^\infty(x)\}$ (see [33], [34]).

Definition 2.4 (see [47], [48]). Let A be an WPO and $c > 0$. The operator A is c-WPO iff

$$d(x, A^\infty(x)) \leq cd(x, A(x), \ \forall\ x \in X.$$

Example 2.1. Let (X, d) be a complete metric space and $A : X \to X$ an a-contraction. Then the operator A is c-WPO with $c = (1 - a)^{-1}$.

Example 2.2. Let (X, d) be a complete metric space and $A : X \to X$. We suppose that there exists $a \in [0, 1[$ such that

$$d(A^2(x), A(x)) \leq ad(x, A(x), \ \forall\ x \in X.$$

Then A is c-WPO with $c = (1 - a)^{-1}$.

Example 2.3 (A generic example of WPO). Let (X_i, d_i), $i \in I$ a family of metric space, $A_i : X_i \to X_i$, a family of POs and x_i^* the unique fixed point of A_i. Let $X := \bigcup\limits_{i \in I} X_i$ be the disjoint union of the family $(X_i)_{i \in I}$. Let

$$d : X \times X \to R_+, \quad d(x, y) := \begin{cases} d_i(x, y) \text{ if } x, y \in X_i, \ i \in I \\ d_i(x, x_i^*) + d_j(y, x_j^*) + 1, \text{ if } i \neq j, \ x \in X_i, \ y \in X_j \end{cases}$$

a metric on X. Then (see [37]) the operator A is a WPO. Moreover we have the following characterization of the WPOs.

Theorem 2.1 ([37]). *Let (X,d) be a metric space and $A : X \to X$ an operator. The operator A is WPO (c-WPO) if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda \in \Lambda} X_\lambda$$

such that

(a) $X_\lambda \in I(A),\ \lambda \in \Lambda$;

(b) $A|X_\lambda : X_\lambda \to X_\lambda$ is a Picard (c-Picard) operator, for all $\lambda \in \Lambda$.

Remark 2.6. It is clear that

(i) $cardF_A = card\Lambda$;

(ii) if $\Lambda_1 \subset \Lambda$, then

$$card\left(F_A \cap \left(\bigcup_{\lambda \in \Lambda_1} X_\lambda\right)\right) = card\Lambda_1.$$

For the class of c-WPOs we have the following data dependence result

Theorem 2.2 ([48]). *Let (X,d) be a metric space and $A_i : X \to X$, $i = 1,2$. We suppose that*

(i) the operator A_i is $c_i - WPO$, $i = 1,2$;

(ii) there exists $\eta > 0$ such that

$$d(A_1(x), A_2(x)) \le \eta,\ \forall\ x \in X.$$

Then

$$H(F_{A_1}, F_{A_2}) \le \eta \max(c_1, c_2).$$

Here H stands for Hausdorff-Pompeiu functional.

3 WPOs on ordered metric spaces

Let X be a nonempty set, d a metric on X, and $\le$ an ordered relation on X. If $\le$, as a subset of $X \times X$, is closed, then by definition $(X,d,\le)$ is an ordered metric space. We have

Lemma 3.1 ([33], [47]). *Let $(X,d,\le)$ be an ordered metric space and $A : X \to X$ an operator such that:*

(i) A is monotone increasing;

(ii) A is WPO.

Then the operator A^∞ is monotone increasing.

Lemma 3.2 (abstract comparison lemma), *Let $(X,d,\le)$ be an ordered metric space and $A,B,C : X \to X$ be such that:*

(i) $A \le B \le C$;

(ii) the operators A,B,C are WPOs;

(iii) the operator B is monotone increasing.
Then

$$x \leq y \leq z \ \Rightarrow \ A^{\infty}(x) \leq B^{\infty}(y) \leq C^{\infty}(z).$$

Remark 3.1. Let A, B, C as in the Lemma 3.2. Moreover, we suppose that $F_B = \{x_B^*\}$, i.e., B is a Picard operator. Then we have

$$A^{\infty}(x) \leq x_B^* \leq C^{\infty}(x), \ \forall \ x \in X.$$

But $A^{\infty}(X) = F_A$, $C^{\infty}(X) = F_C$. Thus we have

$$F_A \leq x_B^* \leq F_C.$$

Lemma 3.3 (Abstract Gronwall lemma; [38]-[40], [45], [46], [34]). *Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator. We suppose that:*
(i) A is Picard operator;
(ii) A is monotone increasing.
If we denote by x_A^*, the unique fixed point of A, then
(a) $x \leq A(x) \ \Rightarrow \ x \leq x_A^*$;
(b) $x \geq A(x) \ \Rightarrow \ x \geq x_A^*$.
Let

$$(UF)_A := \{x \in X | \ A(x) \leq x\}$$

and

$$(LF)_A := \{x \in X | \ A(x) \geq x\}.$$

It is clear that if the operator A is monotone increasing, then ([14])

$$(UF)_A \in I(A) \text{ and } (LF)_A \in I(A).$$

From Lemma 3.3 we have
Lemma 3.4. *Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an increasing operator. If*

$$A|_{(UF)_A \cup (LF)_A} \text{ is Picard operator,}$$

then

$$(LF)_A \leq x_A^* \leq (UF)_A.$$

Remark 3.2. Lemma 3.3 generalizes Proposition 7.15 from [63] and Lemma 1 from [64] where on considered the case of linear bounded operator in ordered Banach spaces. For other generalizations of Gronwall lemma see [3], [5], [8], [9], [11], [12], [15]-[20], [22]-[25], [29], [40], [49], [52], [53], [57], [60], [62].

Lemma 3.5 (see [47], [45], [34]). *Let $(X, d, \leq)$ be an ordered metric space, $A : X \to X$ an operator and $x, y \in X$ such that*

$$x < y, \quad x \leq A(x), \quad y \geq A(y).$$

We suppose that
(i) A is WPO;
(ii) A is monotone increasing.
Then
(a) $x \le A^\infty(x) \le A^\infty(y) \le y$;
(b) $A^\infty(x)$ is the minimal fixed point of A in $[x, y]$ and $A^\infty(y)$ is the maximal fixed point of A in $[x, y]$.

For some results related to this lemma see [2], [9], [19], [20], [22], [49].

4 Fiber WPOs problem

Let (X, d) and (Y, ρ) be two metric spaces. Let $A : X \times Y \to X \times Y$ be such that $A(x, y) = (B(x), C(x, y))$. We suppose that
(i) B is a PO (WPO);
(ii) $C(x, \cdot)$ is a PO (WPO), for all $x \in X$.
Is the operator A a Picard operator?
We have (see [41]-[45])

Theorem 4.1. *Let (X, d) and (Y, ρ) be two metric space and $A = (B, C)$ a triangular operator. We suppose that*
(i) (Y, ρ) is a complete metric space;
(ii) the operator $B : X \to X$ is WPO;
(iii) there exists $a \in [0, 1[$ such that $C(x, \cdot)$ is an a-contraction, for all $x \in X$;
(iv) if $(x^*, y^*) \in F_A$, then $C(\cdot, y^*)$ is continuous in x^*.
Then the operator A is WPO. If B is PO, then A is PO.

For other results for the fiber WPOs problem see [21], [4], [28], [54], [56].

In what follow we shall give some applications of these abstract results.

5 The Cauchy problem

Let X be an ordered Banach space and $f \in C([a, b] \times X, X)$. Then the following problems are equivalent

$$x' = f(t, x), \quad t \in [a, b], \quad x \in C^1([a, b], X), \tag{5.1}$$

and

$$x(t) = x(a) + \int_a^t f(s, x(s))ds, \quad t \in [a, b], \quad x \in C([a, b], X). \tag{5.2}$$

Consider the operator

$$A_f : C([a, b], X) \to C([a, b], X)$$

defined by

$$A_f(x)(t) := x(a) + \int_a^t f(s, x(s))ds.$$

Let $\lambda \in X$ and $X_\lambda := \{x \in C([a,b],X) |\ x(a) = \lambda\}$. Then

$$C([a,b],X) = \bigcup_{\lambda \in X} X_\lambda$$

is a partition of $C([a,b],X)$ and $X_\lambda \in I(A_f)$, for all $\lambda \in X$.

We have

Theorem 5.1. *We suppose that*

(i) there exists $l > 0$ such that

$$\|f(t,u) - f(t,v)\| \leq l\|u - v\|,\ \forall\ t \in [a,b],\ u,v \in X;$$

(ii) $f(t,\cdot)$ is monotone increasing for all $t \in [a,b]$.

Let $x, y \in C^1([a,b],X)$ be two solutions of the equation (5.1). If $x(a) \leq y(a)$, then $x \leq y$.

Proof. From (i) we have that the operator

$$A_f|_{X_\lambda} : X_\lambda \to X_\lambda$$

is a PO, for all $\lambda \in X$ (see [3], [14], [39]). So the operator A_f is WPO.

From the condition (ii), the operator A_f is monotone increasing. If $u \in X$, then denote by $\widetilde{u}$ the constant operator

$$\widetilde{u} : C([a,b],X) \to C([a,b],X)$$

defined by

$$\widetilde{u}(t) = u,\ \forall\ t \in [a,b].$$

It is clear that

$$\widetilde{x(a)} \in X_{x(a)} \text{ and } \widetilde{y(a)} \in X_{y(a)}.$$

By Lemma 3.1 we have that

$$x(a) \leq y(a)\ \Rightarrow\ A_f^\infty(\widetilde{x(a)}) \leq A_f^\infty(\widetilde{y(a)}).$$

But $x = A_f^\infty(\widetilde{x(a)})$ and $y = A_f^\infty(\widetilde{y}(a))$. So, $x \leq y$.

Theorem 5.2. *Consider the following differential equations*

$$x' = f_i(t,x), \quad t \in [a,b], \quad i = 1,2,3. \tag{i}$$

We suppose that

(i) $f_i \in C([a,b] \times X, X)$, $i = 1,2,3$ and $f_1 \leq f_2 \leq f_3$;

(ii) there exists $l_i > 0$ such that

$$\|f_i(t,u) - f_i(t,v)\| \leq l_i\|u - v\|,\ \forall\ t \in [a,b],\ u,v \in X,\ i = 1,2,3;$$

(iii) $f_2(t,\cdot)$ is monotone increasing.

Let x_i be a solution of the equation (i), $i = 1,2,3$. If $x_1(a) \leq x_2(a) \leq x_3(a)$, then $x_1 \leq x_2 \leq x_3$.

Proof. We consider the operator A_{f_i}, $i = 1,2,3$ (see the proof of the Theorem 5.1). These operator are WPOs. Condition (iii) implies that the operator A_{f_2} is monotone increasing. Consider the partition of $C([a,b],X)$ as in the proof of the Theorem 5.1. We remark that

$$x_i \in X_{\widetilde{x_i(a)}}, \quad i = 1,2,3.$$

So,

$$x_i = A_{f_i}^{\infty}(\widetilde{x_i(a)}).$$

Now the proof follows from the Lemma 3.2.

Remark 5.1. For other proofs of the above results see [16], [60], [2].

Remark 5.2. Similar results in the case of Carathéodory solution can be given.

Remark 5.3. Let f be as in the Theorem 5.1. Let $c \leq a$, $g \in C([a,b],[c,b])$ such that $g(t) \leq t$, for all $t \in [a,b]$. Consider the following differential equation with deviating argument

$$x'(t) = f(t, x(g(t))), \quad t \in [a,b]. \tag{5.3}$$

By definition a function

$$x \in C([c,b],X) \cap C^1([a,b],X)$$

is a solution of (5.3) if it satisfies the relation (5.3).

By a similar technique as in the case of the equation (5.1) we have

Theorem 5.3. *Let $x, y \in C([c,b],X) \cap C^1([a,b],X)$ be two solutions of the equation (5.3). If $x(t) \leq y(t)$ for all $t \in [c,a]$, then $x \leq y$.*

Theorem 5.4. *Let f_i be as in the Theorem 5.2, and g as in the Theorem 5.3. Let x_i be a solution of the equation*

$$x_i'(t) = f_i(t, x_i(g(t))), \quad t \in [a,b], \quad i = 1,2,3.$$

If

$$x_1(t) \leq x_2(t) \leq x_3(t), \ \forall \ t \in [c,a],$$

then

$$x_1 \leq x_2 \leq x_3.$$

Remark 5.4. From the abstract Gronwall lemma we have

Theorem 5.5. *Let f and g be as in the Theorem 5.3. Let x be a solution of the equation (5.3) and y a solution of the inequality*

$$y'(t) \leq f(t, y(g(t))), \ \forall \ t \in [a,b].$$

Then

$$y|_{[c,a]} \leq x|_{[c,a]} \ \Rightarrow \ y \leq x.$$

6 Boundary value problem

Let $f \in C([a,b] \times R)$. Then the following problems are equivalent (see [7], [18], [31], [39])

$$-x'' = f(t,x), \quad t \in [a,b]; \quad x \in C^2[a,b] \tag{6.1}$$

and

$$x(t) = \frac{t-a}{b-a}x(b) + \frac{b-t}{b-a}x(a) + \int_a^b G(t,s)f(s,x(s))ds, \quad t \in [a,b], \quad x \in C[a,b]. \tag{6.2}$$

Here G stands for the Green functions.

Consider the operator $A_f : C[a,b] \to C[a,b]$, $A_f(x)(t) :=$ second part of the integral equation (6.2).

Let $\lambda, \mu \in R$ and $X_{\lambda,\mu} := \{x \in C[a,b] | \ x(a) = \lambda, \ x(b) = \mu\}$. Then

$$C[a,b] = \bigcup_{\lambda,\mu} X_{\lambda,\mu}$$

is a partition of $C[a,b]$ and $X_{\lambda,\mu} \in I(A_f)$, for all $\lambda, \mu \in R$.

We have

Theorem 6.1. *We suppose that*

(i) there exists $l > 0$ such that

$$|f(t,u) - f(t,v)| \le l|u - v|, \ \forall \ t \in [a,b], \ u,v \in R;$$

(ii) $l \int_a^b G(t,s)ds \le q < 1, \ \forall \ t \in [a,b]$;

(iii) $f(t,\cdot)$ is monotone increasing for all $t \in [a,b]$.

Let $x, y \in C^2[a,b]$ be two solution of the equation (6.1). If $x(a) \le y(a)$, $x(b) \le y(b)$, then $x \le y$.

Proof. From the conditions (i) and (ii) the operator

$$A_f|_{X_{\lambda,\mu}} : X_{\lambda,\mu} \to X_{\lambda,\mu}$$

is a PO, for all $\lambda, \mu \in R$. So the operator A_f is WPO.

From the condition (iii), the operator A_f is monotone increasing.

For $x \in C[a,b]$ we denote by $\widetilde{x}$ the function defined by

$$\widetilde{x}(t) = \frac{b-t}{b-a}x(a) + \frac{t-a}{b-a}x(b), \quad t \in [a,b].$$

It is clear that $\widetilde{x} \in X_{x(a),x(b)}$.

By Lemma 3.1 we have that

$$x(a) \le y(a), \quad x(b) \le y(b) \ \Rightarrow \ A_f^\infty(\widetilde{x}) \le A_f^\infty(\widetilde{y}).$$

But $x = A_f^\infty(\widetilde{x})$ and $y = A_f^\infty(\widetilde{y})$.

So, $x \le y$.

Theorem 6.2. *Let $f_i \in C([a,b] \times R)$ satisfy the condition (i) and (ii) in the Theorem 6.1. We suppose that*

- *$f_2(t, \cdot)$ is monotone increasing;*
- *$f_1 \leq f_2 \leq f_3$.*

Let x_i be a solution of the equation

$$-x'' = f_i(t, x), \quad t \in [a, b]; \quad i = 1, 2, 3.$$

If

$$x_1(a) \leq x_2(a) \leq x_3(a), \quad x_1(b) \leq x_2(b) \leq x_3(b),$$

then

$$x_1 \leq x_2 \leq x_3.$$

Proof. The proof follows from Lemma 3.2.

Theorem 6.3. *Let f be as in the Theorem 6.1. Let x be a solution of the equation (6.1) and y a solution of the inequality*

$$-y'' \leq f(t, y).$$

Then

$$y(a) \leq x(a), \quad y(b) \leq x(b) \;\Rightarrow\; y \leq x.$$

Proof. The proof follows from Lemma 3.3.

Remark 6.1. In the case of the Dirichlet problem (see [15])

$$-\Delta u = f(x, u), \quad f \in C(\overline{\Omega} \times R)$$

$$u|_{\partial\Omega} = \varphi, \quad \varphi \in C(\partial\Omega)$$

the corresponding WPO is the following

$$A_f(u)(x) := \int\limits_{\Omega} G(x, s) f(s, u(s)) ds + \int\limits_{\partial\Omega} \frac{\partial G(x, s)}{\partial n_s} u(s) d\sigma_s.$$

Remark 6.2. In the case of the Darboux problem (see [26])

$$\frac{\partial^2 u}{\partial x \partial y} = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right), \quad 0 < x < a, \quad 0 < y < b$$

$$\begin{cases} u(x, 0) = \varphi(x), & x \in [0, a], \\ u(0, y) = \psi(y), & y \in [0, b] \end{cases}$$

($f \in C([0, a] \times [0, b] \times R^3)$, $f(x, y, \cdot, \cdot, \cdot) \in Lip$, $\varphi \in C[0, a]$, $\psi \in C[0, b]$) the corresponding WPO is

$$A(u, v, w) = (A_1(u, v, w), A_2(u, v, w), A_3(u, v, w)),$$

where

$$A_1(u,v,w)(x,y) := u(x,0) + u(0,y) + \int_0^x \int_0^y f(s,t,u(s,t),v(s,t),w(s,t))dsdt,$$

$$A_2(u,v,w)(x,y) := v(x,0) + \int_0^y f(x,t,u(x,t),v(x,t),w(x,t))dt,$$

$$A_3(u,v,w)(x,y) := w(0,y) + \int_0^x f(s,y,u(s,y),v(s,y),w(s,y))ds.$$

7 Integral equations

Some applications of WPOs technique to integral equations are given in [40], [43]-[46], [48]-[50], [30]. In what follow we consider the integral equation

$$u(x) = h(x,u(a)) + \int_{a_1}^{x_1} \dots \int_{a_m}^{x_m} K(x,s,u(s))ds, \quad x \in \prod_{i=1}^{i=m} [a_i,b_i]. \tag{8.1}$$

We denote $\overline{D} := [a_1,b_1] \times \cdots \times [a_m,b_m]$. We have

Theorem 8.1. *We suppose that*

(i) $h \in C(\overline{D} \times R)$ *and* $K \in C(\overline{D} \times \overline{D} \times R)$;

(ii) $h(a,\alpha) = \alpha$, *for all* $\alpha \in R$;

(iii) $h(x,\cdot)$ *and* $K(x,s,\cdot)$ *are monotone increasing for all* $x,s \in \overline{D}$;

(iv) there exists $L_K > 0$ *such that*

$$|K(x,s,u_1) - K(x,s,u_2)| \leq L_K|u_1 - u_2|,$$

for all $x,s \in \overline{D}$ *and* $u_1,u_2 \in R$.

In these condition the equation (8.1) has in $C(\overline{D})$ *an infinity of solutions. Moreover if* u *and* v *are two solutions of the equation then*

$$u(a) \leq v(a) \;\Rightarrow\; u \leq v.$$

Proof. Consider the operator $A_{h,K} : C(\overline{D}) \to C(\overline{D})$, $A_{h,K}(u)(x) :=$ second part of (8.1).

Let $\lambda \in R$ and $X_\lambda := \{u \in C(\overline{D}) |\ u(a) = \lambda\}$. Then

$$C(\overline{D}) = \bigcup_{\lambda \in R} X_\lambda$$

is a partition of $C[a,b]$ and $X_\lambda \in I(A)$, for all $\lambda \in R$. From the condition (iv), the operator

$$A_{h,K}|_{X_\lambda} : X_\lambda \to X_\lambda$$

is a PO, for all $\lambda \in R$. So the operator $A_{h,K}$ is WPO. From the condition (iii), the operator $A_{h,K}$ is monotone increasing.

By Lemma 3.1 we have that

$$u(a) \leq v(a) \ \Rightarrow \ A^{\infty}_{h,K}(\widetilde{u(a)}) \leq A^{\infty}_{h,K}(\widetilde{v(a)}).$$

But

$$u = A^{\infty}_{h,K}(\widetilde{u(a)} \quad \text{and} \quad v = A^{\infty}_{h,K}(\widetilde{v(a)}).$$

So, $u \leq v$.

Theorem 8.2. *Let $h_i \in C(\overline{D} \times R)$ and $K_i \in C(\overline{D} \times \overline{D} \times R)$, $i = 1, 2, 3$ satisfy the conditions (i), (ii) and (iv) in the Theorem 8.1.*

We suppose that

- *$h_2(x, \cdot)$ and $K_2(x, s, \cdot)$ are monotone increasing, for all $x, s \in \overline{D}$;*
- *$h_1 \leq h_2 \leq h_3$ and $K_1 \leq K_2 \leq K_3$.*

Let u_i be a solution of the equation (8.1) corresponding to h_i and K_i.

Then

$$u_1(a) \leq u_2(a) \leq u_3(a) \ \Rightarrow \ u_1 \leq u_2 \leq u_3.$$

Proof. The proof follows from Lemma 3.2.

Theorem 8.3. *Let h and K be as in the Theorem 8.1. Let u be a solution of (8.1) and v a solution of the inequality*

$$v(x) \leq h(x, v(a)) + \int_{a_1}^{x_1} \cdots \int_{a_m}^{x_m} K(x, s, v(s))ds.$$

Then

$$v(a) \leq u(a) \ \Rightarrow \ v \leq u.$$

Proof. The proof follows from Lemma 3.3.

8 Difference equations

Let (X, d) be a metric space and $f : X^k \to X$ an operator ($k \in N^*$, $k \geq 2$). Consider the following difference equation

$$x_{n+k} = f(x_n, \ldots, x_{n+k-1}), \quad n \in N, \tag{9.1}$$

with the initial values $x_0, x_1, \ldots, x_{k-1} \in X$.

We consider the following operator

$$A_f : X^k \to X^k, \quad (u_1, \ldots, u_k) \mapsto (u_2, \ldots, u_k, f(u_1, \ldots, u_k)).$$

We have

Theorem 9.1 (see [46]). *The equation (9.1) has a global asymptotic stable equilibrium iff the operator A_f is a Picard operator.*

Remark 9.1. From Lemma 3.1, 3.2 and 3.3, and the Theorem 9.1 we have some Gronwall type lemmas and comparison theorems for the difference equations ([1]).

9 Smooth dependence of solutions on parameters

The fiber WPOs theorem is very useful for proving solutions of some operatorial equations to be differentiable with respect to parameters. For example:

- (J. Sotomayor) differentiablity with respect to initial data for the solution of the Cauchy problem

$$x' = f(t,x), \quad x(t_0) = x_0, \quad f : \Omega \to R^m, \quad \Omega \subset R^{m+1};$$

- (I.A. Rus) differentiability with respect to λ for the solution of the integral equation

$$x(t) = 1 + \lambda \int_t^1 x(s)x(s-t)ds, \quad y \in [0,1];$$

- (A. Tămăşan) differentiability with respect to lag function for pantograph equation

$$x'(t) = f(t, x(t), x(\lambda t)), \quad t > 0, \quad 0 < \lambda < 1,$$
$$x(0) = 0$$

- (V. Mureşan) differentiability with respect to a parameter for the solution of a Volterra-Sobolev integral equation;

- (G. Dezsö) differentiability with respect to a parameter for the solution of Darboux-Ionescu problem;

- (I.A. Rus) differentiability with respect to a parameter for the solution of the following integral equation

$$x(t) = \int_{t-\tau}^{t} f(x, s(s)); \lambda)ds$$

  modeling population growth in a periodic environment.

10 Open problems

The above considerations give rise to the following open problems.

Problem 11.1. Let (X,d) be a metric space. Which are the conditions on X and $A : X \to X$ such that A is WPO $\Leftrightarrow$ A is asymptotically regular?

References: [45], [47], [38], [35], [63].

Problem 11.2a. Let $(X, +, R, \|\cdot\|)$ a Banach space and $A \in C^1(X,X)$. Are the following conditions equivalent?

(i) The operator A is WPO.

(ii) The operator $A'(x)$ is WPO, for all $x \in X$.

Problem 11.2b. Let $(X, +, R, \|\cdot\|)$ a Banach space and $A \in C^1(X,X)$. Are the following conditions equivalent?

(i) The operator A is PO.

(ii) The operator $A'(x)$ is PO, for all $x \in X$.

References: [38], [47], [10], [16], [45].

Problem 11.3. Let (X, d) be a complete metric space and $A : X \to X$. Give some metric conditions on A which imply that A is WPO.

References: [45], [47], [35], [36], [38].

Problem 11.4. Let (X, d) be a metric space and $f, g : X \to X$ two weakly Picard operators. If exists $\eta > 0$ such that

$$d(f(x), g(x)) \leq \eta, \ \forall \ x \in X,$$

estimate $H(F_f, F_g)$.

References: [38], [46], [45], [48].

Problem 11.5. Fiber WPOs problem. Let (X, d) and (Y, ρ) be two metric spaces. Let $A : X \times Y \to X \times Y$ be such that $A(x, y) = (B(x), C(x, y))$. We suppose that

(i) B is WPO;

(ii) $C(x, \cdot)$ is WPO, $\forall \ x \in X$.

Is the operator A WPO?

References: see §4 of this paper.

Problem 11.6. Let (X, d) be a complete metric space. Let $A_1, \ldots, A_m$ be some continuous WPOs. These operators generate the following operator (fractal operator!)

$$T_A : P_{cp}(X) \to P_{cp}(X), \quad Y \mapsto A_1(Y) \cup \cdots \cup A_m(Y).$$

Is the operator

$$T_A : (P_{cp}(X), H) \to (P_{cp}(X), H)$$

a WPO?

References: [45], [30].

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