Seminar on Fixed Point Theory Cluj-Napoca, **Volume 1, 2000, 39-54**
http://www.math.ubbcluj.ro/~nodeacj/journal.htm

RETRACTION METHODS IN FIXED POINT THEORY

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Abstract. To obtain fixed point theorems for nonself-mappings there are two possibilities. One consists in using continuation methods of Leray-Schauder type. Roughly speaking, by means of a continuation theorem we can obtain a solution of a given equation starting from one of the solutions of a more simpler equations (see [21]). The other way makes use of the retraction mapping principle. This technique was presented by I.A. Rus in [29].

In this report we adopt the way of a retraction mapping principle. Our goal is to show that under suitable geometrical conditions, continuation theorems of Leray-Schauder type can be alternatively obtained by means of the retraction mapping principle. We shall consider only the boundary conditions of Leray-Schauder, Browder-Petryshyn and Cramer-Ray and we shall restrict ourselves to the case of Banach spaces and vector lattices.

Keywords: fixed point structures, Hilbert space, retractible mapping
AMS Subject Classification: 47H10, 54H25

1 Fixed point structures

Let X be a nonempty set and $Y \in P(X)$, where $P(X)$ denote the set of all nonempty subset of X. We denote by $M(X)$ the set of all mapping $f : X \to X$.

Definition 1.1. (see [28]) A triple (X, S, M) is a fixed point structure if

(i) $S \subset P(X)$ is a nonempty subset of $P(X)$;

(ii) $M : P(X) \to \bigcup_{Y \in P(X)} M(Y)$, $Y \subset M(Y)$ is a mapping such that, if $Z \subset Y$ then

$$M(Z) \supset \{f|_Z : \ f \in M(Y) \text{ and } f(Z) \subset Z\};$$

(iii) Every $Y \in S$ has the fixed point property with respect to $M(Y)$.

Example 1.1. Let X is a nonempty set, $S = \{\{x\} : \ x \in X\}$ and $M(Y) = M(Y)$.

Example 1.2. (Knaster, Tarski, Birkhoff) $(X, \leq)$ is a complete lattice, $S = \{Y \in P(X) : \ (Y, \leq)$ is a complete sublattice of $X\}$ and $M(Y) = \{f : Y \to Y : \ f$ is order-preserving mapping$\}$.

Example 1.3. (Banach, Caccioppoli) (X, d) is a complete metric space, $S = P_d(X)$ and $M(Y) = \{f : Y \to Y : \ f$ is a contraction$\}$.

Example 1.4. (Nemytzki, Edelstein) (X, d) is a complete metric space, $S = P_{cp}(X)$ and $M(Y) = \{f : Y \to Y : \ f$ is a contractive mapping$\}$.

Example 1.5. (Schauder) X is a Banach space, $S = P_{cp,cv}(X)$ and $M(Y) = C(Y,Y)$.

Example 1.6. (Dotson) X is a Banach space, $S = P_{cp,cl}(X)$ and $M(Y) = \{f : Y \to Y :\ f$ is a nonexpansive mapping$\}$.

Example 1.7. (Browder) X is a Hilber space, $S = P_{b,cl,cv}(X)$ and $M(Y) = \{f : Y \to Y :\ f$ is a nonexpansive mapping$\}$.

Example 1.8. (Tychonov) X is a Banach space, $S = P_{wcp,cv}(X)$ and $M(Y) = \{f : Y \to Y :\ f$ is weakly continuous$\}$.

Example 1.9. (Schauder) X is a Banach space, $S = P_{b,cl,cv}(X)$ and $M(Y) = \{f : Y \to Y :\ f$ is completely continuous$\}$.

Example 1.10. (Tychonov) X is a locally convex space, $S = P_{cp,cv}(X)$ and $M(Y) = C(Y,Y)$.

If more generally we let X be a Banach space, $S = P_{cl,cv}(X)$ and $M(Y) = \{f : Y \to Y :\ f$ is continuous and there is $x_0 \in Y$ such that for any $C \in P_b(Y)$ relation $\overline{C} \subset \overline{cv}\{\{x_0\}Yf(C)\}$ implies $\overline{C}$ compact$\}$, then the triple (X,S,M) is a fixed point structure in a generalized sense, when (ii) does not hold (see [17]).

2 The retraction notion

Let X be a nonempty set and $Y \subset X$ a nonempty subset of X.

Definition 2.1. ([9]) A mapping $\rho : X \to Y$ is called a retraction of X onto Y if and only if $\rho|_Y = 1_Y$, i.e. $\rho(x) = x$ for any $x \in Y$.

If X has a certain structure, the mapping ρ must be compatible with that structure. For example a retraction of a topological space will be assumed to be continuous.

2.1 An example of retraction in Hilbert spaces

In this paragraph we consider H be a Hilbert space and $K \subset H$ a nonempty, convex and closed subset, i.e. $K \in P_{cv,cl}(X)$. We will show that $P : H \to K$ the projection mapping of H onto K, is a retraction. At first we present some additional results.

Theorem 2.2.1. *Let $K \subset H$ be a nonempty, convex and closed subset of H, and $u \in H$. Let*

$$d = \inf_{v \in K} \|u - v\| = d(u,K).$$

Then there exists a unique element $w \in K$ with $d = \|u - w\| = d(u,K)$.

Proof. For any $v \in K$, we have $\|u - v\| \geq 0$, so for a given $u \in H$, the set of real numbers $\{\|u - v\| :\ v \in H\}$ is lower bounded by zero. So $d = \inf_{v \in K} \|u - v\| = d(u,K)$.

Let $(v_n)_{n \geq 1} \subset K$ be a sequences of points from H such that $\|u - v_n\| \to d$, as $n \to \infty$. Since K is convex and $v_n, v_m \in K$ for any $m,n \in N$, we have $\lambda v_m + (1-\lambda)v_n \in K$ for all $m,n \in N$ and $0 \leq \lambda \leq 1$. Put $\lambda = \frac{1}{2}$. Then $\frac{v_n + v_m}{2} \in K$, so $\left\| u - \frac{v_n + v_m}{2} \right\| \geq d$. Recall the parallelogram's equality

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \text{ for all } x, y \in H.$$

We consider $x = u - v_m$ and $y = u - v_n$. Hence

$$\|v_n - v_m\|^2 = 2(\|u - v_m\|^2 + \|u - v_n\|^2) - 4\left\|u - \frac{v_m + v_n}{2}\right\|^2.$$

Then

$$\|v_n - v_m\|^2 \le 2(\|u - v_m\|^2 + \|u - v_n\|^2) - 4d^2.$$

When $m, n \to \infty$, we obtain $\|v_n - v_m\| \to 0$. This implies that the sequence $(v_n)_{n\ge 1} \subset K$ is fundamental, so it has a limit w. Since $(v_n)_{n\ge 1} \subset K$ and K is closed, it follows that $w = \lim\limits_{n\to\infty} v_n \in K$. Hence $\|u - v_n\| \to \|u - w\| = d$ as $n \to \infty$.

In this way, we have shown that there exists $w \in K$ such that

$$\|u - w\| = d = \inf_{v\in K} \|u - v\|.$$

For the uniqueness, we assume that there exists $q \in K$, $q \ne w$ such that $\|u-w\| = d = \|u - q\|$. Since K is convex, we have $\dfrac{q+w}{2} \in K$, hence

$$d = \inf_{v\in K} \|u-v\| \le \left\|u + \frac{q+w}{2}\right\| = \left\|\frac{1}{2}(u-w) + \frac{1}{2}(u-q)\right\| \le \frac{1}{2}\|u-w\| + \frac{1}{2}\|u-q\| = d$$

and

$$d = \left\|u - \frac{q+w}{2}\right\|.$$

From the parallelogram's equality, for $x = u - w$ and $y = u - q$ we obtain

$$\|w - q\|^2 = 2(\|u - w\|^2 + \|u - q\|^2) - 4\left\|u - \frac{w+q}{2}\right\|^2 = 2(d^2 + d^2) - 4d^2 = 0.$$

So $\|w - q\| = 0$, which is equivalent to $w = q$. □

Now we formulate

Definition 2.2.1. Let H be a Hilbert space, $K \subset H$ a nonempty, convex and closed subset of X. Let $P : H \to K$ be the mapping giving by $P(u) = w$, where $w \in K$ is such as

$$\|u - w\| = d = \inf_{v\in K} \|u - v\|.$$

The mapping P is called the metric projection of H onto K.

We have the following results (see [12]).

Theorem 2.2.2. *Let H be a Hilbert space, $K \subset H$ a nonempty, convex and closed subset of X. The following statements are equivalent:*

(i) $w \in K,\ \|u - w\| \le \|u - v\|$ *for every* $v \in K$*;*

(ii) $w \in K,\ \mathrm{Re}\ (u - w, v - w) \le 0$ *for every* $v \in K$*;*

(iii) $w \in K,\ \mathrm{Re}\ (u - v, w - v) \ge 0$ *for every* $v \in K$*.*

Theorem 2.2.3. *Let H be a Hilbert space, $K \subset H$ a nonempty, convex and closed subset of X. The metric projection of X onto K is a nonexpansive mapping, i.e.*

$$\|P(u) - P(v)\| \le \|u - v\|,\ \forall\ u, v \in H.$$

A consequence of this theorem is the continuity of P. Indeed, for any $u \in H$ and any sequence $(u_n)_{n\geq 1} \subset H$ which is norm convergent at u, we have $\|P(u) - P(u_n)\| \leq \|u - u_n\|$. Since $\|u_n - u\| to 0$, as $n \to \infty$, it follows that

$$\|P(u_n) - P(u)\| \to 0, \text{ as } n \to \infty,$$

i.e. P is continuous.

Thus we may conclude that the mapping given by Definition 3.2.1 is a topological retraction of H onto K.

Remark 2.2.1. For the uniqueness of the element $w \in K$ satisfying $d = \|u-w\| = d(u, K)$ the parallelogram's equality is an important tool. This is in connexion with the structure of Hilbert space. Thus, Definition 3.2.1 cannot be given for an arbitrary Banach space. However, if K is a nonempty, closed, convex set of an uniformly convex Banach space the metric projection P is univoque and continuous (see [23]).

Definition 2.2.2. Let X be a space with the norm $\|\cdot\|$ and $Y \subset X$ a closed subspace of X. A linear continuous mapping $P : X \to Y$ is called projection mapping of X onto Y if it is a surjection and $P(y) = y$ for any $y \in Y$.

Definition 2.2.3. A closed subspace Y of a Banach space X is called complementabely if there exists a projection of X onto Y.

Theorem 2.2.4. (see [16]) *If any closed subspace of a Banach space X is complementabely, then X is isomorph with a Hilbert space.*

Examples.

c_0 is not complementabely in l^∞

$C[0, 1]$ is not complementabely in $L^\infty(0, 1)$.

2.2 An example of retraction onto Banach spaces

Let X be a Banach space, $U \subset X$ a nonempty, convex and closed subset of X and $u_0 \in intU$.

For every pair $u, v \in X$, the set $[u, v] = \{w \in X : \ w = (1 - \lambda)u + \lambda v, \ \lambda \in [0, 1]\}$ is called the segment between u and v. For any $u \in X$ we make the notation $Z(u) = [u, u_0] \cap \partial U$. Now, we define the mapping $\varphi : X \to \mathbb{R}$ by

$$\varphi(u) = \begin{cases} \|u - u_0\| & \text{if} \quad Z(u) = \emptyset \\ \max\limits_{v \in Z(u)} \|v - u_0\| & \text{if} \quad Z(u) \neq \emptyset \end{cases}$$

By means of this mapping we construct the operator $\rho : X \to \overline{U}$, where

$$\rho(u) = \frac{\varphi(u)}{\|u - u_0\|} u + \left(1 + \frac{\varphi(u)}{\|u - u_0\|}\right) u_0 \tag{2.3.1}$$

This mapping is a retraction. Indeed, if $u \in int\,U$ then $Z(u) = \emptyset$, so $\varphi(u) = \|u - u_0\|$ and this implies $\rho(u) = u$. If $u \in \partial U$ then $\varphi(u) = \|u - u_0\|$ and again $\rho(u) = u$. Hence $\rho(u) = u$ for any $u \in \overline{U}$. If $u \notin \overline{U}$ then $Z(u) \neq \emptyset$ and $\varphi(u) < \|u - u_0\|$.

So $\dfrac{\varphi(u)}{\|u-u_0\|} \in (0,1)$ and consequently $\rho(u) \in [u, u_0]$, i.e. the image of any point $u \in X \setminus \overline{U}$ by ρ lies on the segment $[u, u_0]$.

Moreover, we have

$$\|\rho(u) - u_0\| = \left\| \frac{\varphi(u)}{\|u-u_0\|} u + \left(1 - \frac{\varphi(u)}{\|u-u_0\|}\right) u_0 - u_0 \right\| =$$

$$= \left\| \frac{\varphi(u)}{\|u-u_0\|} u + - \frac{\varphi(u)}{\|u-u_0\|} u_0 \right\| = \varphi(u).$$

In conclusion, if $u \in X \setminus \overline{U}$ then $\rho(u)$ is the intersection point of the segment $[u, u_0]$ with ∂U, which is the most nearly by u. So ρ is a continuous retraction.

If $U = B(u_0, r) = \{u \in X : \ \|u - u_0\| < r\} \subset X$ the mapping $\rho : X \to \overline{U}$ is giving by

$$\rho(u) = \begin{cases} u & \text{if} \quad u \in \overline{U} \\ \dfrac{r}{\|u-u_0\|} u + \left(1 - \dfrac{r}{\|u-u_0\|}\right) u_0 & \text{if} \quad u \notin \overline{U} \end{cases}$$

and it is call "the radial retraction".

2.3 An example of retraction onto ordered spaces

Let X be a real vectorial space. X is a vector lattice (ordered space) if X is lattice and

i) for any $z \in X$, $x \leq y$ then $x + z \leq y + z$
ii) if $x \geq 0$ and $\lambda \geq 0$ then $\lambda x \geq 0$.

In any ordered space X, denote by

$$[x, y] = \{z \in X : \ x \leq z \leq y\}$$

the interval with respect to order (ordered interval).

The set $X_+ = \{x \in X : \ x \geq 0\}$ is called the cone of positifs elements of vectorial lattice X.

For every $x \in X$, the element $x_+ = x \vee 0$ is called the positive part of x and $x_- = (-x) \vee 0 = (-x)_+$ the negative part. The element $|x| = x_+ + x_-$ means the absolute value of x. For any $x \in X$ we have $x = x_+ - x_-$.

Let $v \in X_+$. Demote with $Y = [-v, v]$ and define the application $\varphi : X \to Y$,

$$\varphi(u) = \begin{cases} u & \text{if} \quad u \in Y \\ \sup\{[0, u_+] \cap [0, v]\} - \sup\{[0, u_-] \cap [0, v]\} & \text{if} \quad u \notin Y \end{cases} \tag{2.4.1}$$

We make the notations

$$Y_+ = [0, v] \quad U_+ = [0, u_+] \quad \text{and} \quad U_- = [0, u_-]$$

The application φ is a retraction of X onto Y which is compatible with structure of space X, i.e. it is continuous and for every $u_1, u_2 \in X$, $u_1 \leq u_2$ we have $\varphi(u_1) \leq \varphi(u_2)$.

Indeed, let $u_1, u_2 \in X$ with $u_1 \leq u_2$.

1. Suppose that $-v \leq u_1 \leq u_2 \leq v$, i.e. $u_1, u_2 \in Y$. Then $\varphi(u_1) = u_1 \leq u_2 = \varphi(u_2)$.

2. If $u_1 \in Y$ and $u_2 \notin Y$ the $\varphi(u_1) = u_1$ and $\varphi(u_2) = \sup\{U_{2+} \cap Y_+\} - \sup\{U_{2-} \cap Y_+\}$.

From $u_1 \leq u_2$ we have $u_{1+} \leq u_{2+}$ and $u_{2-} \leq u_{1-}$. Since $u_1 \in Y$ we obtain $u_{1+} \leq v$ and $u_{1-} \leq v$. We have $u_{1+} \leq u_{2+}$ and $u_{1+} \leq v$, hence

$$u_{1+} \leq \sup\{U_{2+} \cap Y_+\}.$$

From $u_{2-} \leq u_{1-}$ and $u_{1-} \leq v$ results

$$u_{1-} \geq \sup\{U_{2-} \cap Y+\}.$$

Then

$$\varphi(u_1) = u_1 = u_{1+} - u_{1-} \leq \sup\{U_{2+} \cap Y_+\} - \sup\{U_{2-} \cap Y_+\} = \varphi(u_2).$$

If $u_1 \notin Y$ and $u_2 \in Y$ the proof is similary.

3. If $u_1 \notin Y$ and $u_2 \notin Y$ then

$$\varphi(u_i) = \sup\{U_{i+} \cap Y_+\} - \sup\{U_{i-} \cap Y_+\}, \quad i = \overline{1,2}.$$

Since $u_1 \leq u_2$ we have $u_{1+} \leq u_{2+}$ and $u_{2-} \leq u_{1-}$. Then $U_{1+} \subset U_{2+}$ and $U_{2-} \subset U_{1-}$. Results

$$\sup\{U_{1+} \cap Y_+\} \leq \sup\{U_{2+} \cap Y_+\}$$

and

$$\sup\{U_{2-} \cap Y_+\} \leq \sup\{U_{1-} \cap Y_+\}.$$

Finally, we have

$$\varphi(u_1) = \sup\{U_{1+} \cap Y_+\} - \sup\{U_{1-} \cap Y_+\} \leq$$

$$\leq \sup\{U_{2+} \cap Y_+\} - \sup\{U_{2-} \cap Y_+\} = \varphi(u_2).$$

In conclusion, for any $u_1, u_2 \in X$ with $u_1 \leq u_2$ we have $\varphi(u_1) \leq \varphi(u_2)$. In other words φ is increasing.

3 Boundary conditions

We recall Leray-Schauder boundary condition and show its equivalence to those of Browder-Petryshyn and Cramer-Ray when the domain is a ball. For all there definitions U is a subset of a Banach space X, $u_0 \in int\, U$ and $T : U \to X$ is a mapping.

For $r > 0$ and $u \in X$ we let $B(u, r)$ be the open ball of X of radius r and center u, i.e.

$$B(u, r) = \{v \in X : \|u - v\| < r\}.$$

For every pair $u, v \in X$, the set $[u, v] = \{w \in X : \ w = (1-\lambda)u + \lambda v, \ \lambda \in [0,1]\}$ is called the segment between u and v.

We shall assume $u_0 \in int\, U$.

Definition 3.1. (Leray-Schauder, see [15]) Let $u \in \partial U$. T satisfies the Leray-Schauder boundary condition (LSB) at u relative to U if and only if

$$(1-\lambda)u_0 + \lambda T(u) \neq u \text{ for every } \lambda \in [0,1]. \tag{1}$$

Remark 3.1. The definition has the equivalent form

$$T(u) - u_0 \neq k(u - u_0) \text{ for } \lambda \in [0,1]. \tag{2}$$

In fact Definition 3.1 says that T satisfies LSB at u if and only if the point u doesn't lie on the segment $[u_0, T(u)]$.

Definition 3.2. (Browder-Petryshyn, see [8]) Let $u \in U$ with $u \neq T(u)$. T satisfies the Browder-Petryshyn condition (BP) at u relative to U if anf only if

$$B(T(u), \|T(u) - u\|) \cap U \neq \emptyset. \tag{3}$$

Remark 3.2. (i) The relation (3) is equivalent to the existence of an element $v \in U$ such that

$$\|T(u) - v\| < \|T(u) - u\|.$$

(ii) Obviously, if $T(u) \in U$ or $u \in int\, U$, then T satisfies BP at u relative to U.

Definition 3.3. (Cramer-Ray, see [22]) Let $u \in U$ with $u \neq T(u)$. T satisfies the Cramer-Ray condition (CR) at u relative to U if and only if

$$\liminf_{h \to 0^+} \frac{d((1-h)u + hT(u), U)}{h} < \|u - T(u)\|. \tag{4}$$

Lemma 3.1. *Let U be convex and $u \in U$ with $u \neq T(u)$. T satisfies CR at u if and only if there exists $v \in U$ and $0 < h \leq 1$ such that*

$$\frac{\|(1-h)u + hT(u) - v\|}{h} < \|u - T(u)\|. \tag{5}$$

Proof. $\Rightarrow$) Obvious.

$\Leftarrow$) Without loss of generality, choose $0 < k < 1$ such that

$$\frac{\|(1-h)u + hT(u) - v\|}{h} < k\|u - T(u)\|.$$

For each $a \in (0,1)$ let $z(a) = u + a(v-u)$. Since $z(a) \in [u, v]$ and U is convex we have $z(a) \in U$. Now, it suffices to show that for any $a \in (0,1)$, $z(a)$ satisfies

$$\frac{\|(1-ah)u + ahT(u) - z(a)\|}{ah} \leq k\|u - T(u)\|.$$

Since

$$\frac{\|(1-ah)u+ahT(u)-z(a)\|}{ah}=\frac{\|u-ahu+ahT(u)-u-a(v-u)\|}{ah}=$$

$$=\frac{\|(1-h)u+hT(u)-v\|}{h}\leq k\|u-T(u)\|.$$

Thus the lemma is proved. □

Remark 3.3. If X is a Hilbert space, with inner product $(\cdot,\cdot)$, it is possible to introduce the Leray-Schauder condition (LS), see [31], in the following way:

Let $u\in U$ with $u\neq T(u)$ and

$$LS(u,T(u))=\{v\in X:\ \text{Re }(T(u)-u,v-u)>0\}.$$

The mapping T satisfies (LS) at u relative to U if and only if

$$LS*u,T(u))\cap U\neq\emptyset. \tag{6}$$

If U is convex and $u\in U$ with $u\neq T(u)$ then (see [31])

T satisfies LS at u if and only if T satisfies BP at u

and

T satisfies LS at u if and only if T satisfies CR at u.

Proposition 3.1. *Let X be a Banach space, $U=\overline{B}(u_0,r)$ and $u\in\partial U$ such that $u\neq T(u)$. T satisfies LSB at u if and only if T satisfies BP at u.*

Proof. $\Leftarrow$) Assume that T satisfies BP and we wish T satisfies LSB. We know that

$$\|T(u)-u_0\|\leq\|T(u)-v\|+\|v-u_0\|$$

for any $v\in U$. If T satisfies BP at u then conform of remark 3.2 exists $v\in U$ such that

$$\|T(u)-v\|<\|T(u)-u\|.$$

Since $u\in\partial U$ we have

$$\|u_0-v\|<\|u_0-u\|=r.$$

So

$$\|T(u)-u_0\|<\|T(u)-u\|+\|u-u_0\|.$$

In conclusion $u\notin[u_0,T(u)]$, i.e. T satisfies LSB.

$\Rightarrow$) Assume that T satisfies LSB and we wish T satisfies BP. Without loss of generality we can consider $\|u_0-T(u)\|>r$. Affirm that

$$v=\frac{r}{\|T(u)-u_0\|}T(u)+\left(1-\frac{r}{\|T(u)-u_0\|}\right)u_0\in U\cap B(T(u),\|T(u)-u\|).$$

Indeed, we have

$$\|v - u_0\| = \left\| \frac{r}{\|T(u) - u_0\|} T(u) + \left(1 - \frac{r}{\|T(u) - u_0\|}\right) u_0 - u_0 \right\| = r$$

hence $v \in U$.

On the other side

$$\|T(u) - v\| = \left|1 - \frac{r}{\|T(u) - u_0\|}\right| \|T(u) - u_0\| = \|T(u) - u_0\| - \|v - u_0\|.$$

Since T satisfies LSB results

$$\|T(u) - u_0\| < \|T(u) - u\| + \|u - u_0\|.$$

Then

$$\|T(u) - v\| < \|T(u) - u\| + \|u - u_0\| - \|v - v_0\| = \|u - T(u)\|,$$

since $\|u_0 - v\| = \|u_0 - u\| = r$.

Then $\|T(u) - v\| < \|T(u) - u\|$, i.e. $v \in B(T(u), \|T(u) - u\|)$. □

If $U \neq \overline{B}(u_0, r)$, the last proposition is not true.

Example 3.1. Let $X = \mathbb{R}^2$, with euclidian's norm and

$$U = \{(x, y) \in \mathbb{R}^2,\ |x| \leq 1,\ |y| \leq 1\},$$

i.e. U is the square with vertex $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$. Choose $u_0 = (0, 0)$, $u = \left(1, \frac{1}{n}\right)$ with $n > 1$ and suppose that exists a mapping $T : U \to \mathbb{R}^2$ such that $T(u) = \left(k, \frac{k}{n}\right)$, for $k > 1$. Under of this assumption, we have $T(u) = ku$ for $k > 1$, so remark 1.1 said T does not satisfies LSB. Bur for $k > \frac{n+1}{2}$, T satisfies BP.

Now, we fix the point $v = (1, 1)$ and obtain

$$\|T(u) - v\|^2 = (k - 1)^2 + \left(\frac{k}{n} - 1\right)^2 = \frac{n^2(k - 1)^2 + (k - n)^2}{n^2}.$$

Moreover

$$\|T(u) - u\|^2 = (k - 1)^2 + \left(\frac{k}{n} - \frac{1}{n}\right)^2 = \frac{(n^2 + 1)(k - 1)^2}{n^2}.$$

The mapping T satisfies BP is equivalent with

$$\|T(u) - v\| < \|T(u) - u\|,$$

that is to say

$$\begin{gathered} (n^2 + 1)(k - 1)^2 > n^2(k - 1)^2 + (k - n)^2 \\ (k - 1)^2 > (k - n)^2 \\ 2k(n - 1) > n^2 - 1 \\ k > \frac{n + 1}{2}. \end{gathered}$$

So for $k > \frac{n+1}{2}$, T satisfies BP, but T not satisfies LSB.

3.1 Conditions of retractibility

In following, we denote by F_f the set of fixed point of the mapping f.

Definition 3.2.1. ([9]) A mapping $f : Y \to X$ is retractible onto Y if there is a retraction $\rho : X \to Y$ such that $F_{\rho\circ f} = F_f$.

Condition (i) $F_{\rho\circ f} = F_f$ is equivalent with:

(ii) if $x \in \rho(f(Y) \setminus Y)$, then $f(x) \notin \rho^{-1}(x) \setminus \{x\}$.

Indeed, theorem 1.1 from [7] - the retraction mapping principle - shows that condition (ii) implies (i); now we suppose $F_{\rho\circ f} = F_f$ and there exists $x \in \rho(f(Y) \setminus Y)$ such that $f(x) \in \rho^{-1}(x) \setminus \{x\}$. Hence $x \notin F_f$, but on the other side $x = \rho(f(x))$, i.e. $x \in F_{\rho\circ f}$. This is a contradiction, so (i) implies (ii). In conclusion Definition 3.2.1 is equivalent with the definition given by Brown (see [7]).

Example 2.1. (Poincaré, Bohl, Leray-Schauder, Rothe, Altman, Furi-Vignoli,...) Let X be a Banach space and $Y = \overline{B}(0, R) \subset X$. If $f : \overline{B}(0, R) \to X$ is such that $\|x\| = R$, $f(x) = \lambda x$ implies $\lambda \leq 1$, then f is retractible onto $\overline{B}(0, R)$ with respect to the radial retraction $\rho : X \to \overline{B}(0, R)$.

Example 2.2. (Altman) Let X be a Banach space and $f : X \to X$ a norme contraction mapping. Then there exists $R > 0$ such that $f : \overline{B}(0, R) \to X$ is retractible onto $\overline{B}(0, R)$ with respect to the radial retraction.

Example 2.3. (Halpern-Beroman) Let X be a strictly convex normed linear space. Let $Y \subset X$ be a compact convex subset of X and $\rho : X \to Y$ the metric projection onto Y. If $f : Y \to X$ is nowhere normal-outward, then f is retractible onto Y with respect to ρ.

Example 2.4. Let X be a set, $Y \subset X$ a subset of X and $\rho : X \to Y$ a retraction. If $f : Y \to X$ is such that $x \in Y \setminus F_f$ implies $f(x) \in X \setminus \rho^{-1}(x)$, then f is retractible onto Y with respect to ρ.

In this paragraph we will give some theorems with form: if T satisfies a kind of boundary conditions then T is retractible.

Theorem 3.2.1. *Let X be a Hilbert space, $U \in P_{cv,cl}(X)$. If the mapping $T : U \to X$ satisfies BP for any $u \in \partial U$ then T is retractible onto U with respect to the projection mapping of X to U.*

Proof. Here $\rho = P$ denote the metric projection. Assume that $F_{\rho\circ f} \neq F_f$. Let $u \in F_{P\circ\Gamma} \setminus F_\Gamma \neq \emptyset$. Then $u = P(T(u))$ and $u \in \partial U$. This is equivalent with $T(u) \neq u$ and $0 < \|u - T(u)\| < \|T(u) - v\|$, for any $v \in U$. Results a contradiction with T satisfies BP condition. □

Let X be a Hilbert space, $U \subset X$ convex, $u \in U$ with $u \neq T(u)$. From Remark 3.3 results T satisfies BP at u iff T satisfies CR at u. Then we have

Theorem 3.2.2. *Let X be a Hilbert space, $U \in P_{cv,cl}(X)$. If the mapping $T : U \to X$ satisfies CR for any $u \in \partial U$, then T is retractible onto U with respect to the metric projection of X onto U.*

For a Banach space X we will consider the retraction ρ given by relation (2.3.1).

Theorem 3.2.3. *Let X be a Banach space, $U \in P_{cv,cl}(X)$, $u_0 \in int\,U$ and the mapping $T : U \to X$. If T satisfies LSB for any $u \in \partial U$, then T is retractible onto U with respect of the retraction ρ.*

Proof. Assume that $F_{\rho\circ f} \neq F_f$. Let $u \in F_{P\circ\Gamma} \setminus F_\Gamma \neq \emptyset$, i.e. $T(u) \neq u$ and

$u = \rho(T(u)) \in \partial U$. From definition of ρ results that there exists $k \in (0,1)$ such that $u = kT(u) + (1-k)u_0$. We get a contraction with T satisfies LSB for $u \in \partial U$. In conclusion T is retractible onto U with respect to ρ. □

By Proposition 3.1 if $U = \overline{B}(u_0, r)$, then T satisfies LSB at u is equivalent with T satisfies BP. Then we have

Theorem 3.2.4. *Let X be a Banach space and $u_0 \in X$. If the mapping $T : \overline{B}(u_0, r) \to X$ satisfies BP for any $u \in \partial\overline{B}(u_0, r)$ then T is retractible onto $\overline{B}(u_0, r)$ with respect to the radial retraction.*

Theorem 3.2.5. *Let X be a vector lattice (ordered space), $v \in X_+$ and $T : [-v, v] \to X$ be an operator.*

If $T(u) \notin Y$ implies

$$\sup\{[0, T(u)_+] \cap [0, v]\} - \sup\{[0, T(u)_- \cap [0, v]\} \neq u$$

then T is retractible onto $[-v, v]$ with respect of retraction φ given by relation (2.4.1).

Proof. Let $u \in F_{\varphi \circ T} \setminus F_T \neq \emptyset$. Then $u = (\varphi \circ T)(u)$ and $u \neq T(u)$. Results $T(u) \notin [-v, v]$ so

$$u = \varphi(T(u)) = \sup\{[0, T(u)_+] \cap [0, v]\} - \sup\{[0, T(u)_- \cap [0, v]\}.$$

We get a contradiction, hence $F_{\varphi\circ\Gamma} \subset F_\Gamma$. This implies $F_{\varphi\circ\Gamma} = F_\Gamma$, i.e. T is retractible onto $[-v, v]$ with respect to φ. □

4 Fixed points of retractible mappings

4.1

Let us starting with

Lemma 4.1. (see [29]) *Let (X, S, M) be a fixed point structure. Let $Y \in S$ and $\rho : X \to Y$ a retraction. Let $f : Y \to X$ be such that*

(i) $\rho \circ f \in M(Y)$

(ii) f is retractible onto Y by ρ.

Then $F_f \neq \emptyset$.

Proof. From (i) we obtain $F_{\rho\circ f} \neq \emptyset$ and from (ii) we have $F_{\rho\circ f} = F_f$. Results $F_f \neq \emptyset$. □

4.2

Theorem 4.2.1. *Let X be a Hilbert space, $U \in P_{cv,cl,b}(X)$ and $T : U \to X$ is a nonexpansive mapping. If T satisfies BP for any $u \in \partial U$, then $F_f \neq \emptyset$.*

Proof. We take (X, S, M) as in example 1.7 and ρ the projection mapping of X onto Y. Since ρ and T is nonexpansive mapping hence (i) from lemma 4.1 is verified. By Theorem 3.2.1 we have T is retractible onto U with respect to the metric projection, then (ii) is satisfied. □

Obviously, we have

Theorem 4.2.2. *Let X be a Hilbert space, $U \in P_{cv,cl,b}(X)$ and $T : U \to X$ is a nonexpansive mapping. If T satisfies CR for any $u \in \partial U$, then $F_f \neq \emptyset$.*

4.3 A Leray-Schauder type theorem

Let X be a Banach space, $Y \in P_{cl,cv}(X)$ and $x_0 \in int\, Y$. A mapping $T : Y \to Y$ is said to be a Mönch operator if and only if T is continuous and for any $C \in P_b(X)$ satisfies $\overline{C} \subset \overline{cv}\{\{x_0\} \cup T(C)\}$ we have that $\overline{C}$ is compact. In what follows we denote by α a measure of noncompactness on X.

Remark 4.3.1. If $T : Y \to Y$ is α-condensing (i.e. T is continuous and for any $C \in P_b(X)$ with $a(C) \neq 0$ we have $\alpha(T(C)) < \alpha(C)$) then T is a Mönch operator. Indeed, for $C \in P_b(X)$, since $\overline{C} \subset \overline{cv}\{\{x_0\} \cup T(C)\}$ we have $\alpha(\overline{C}) < \alpha(\overline{c}\{\{x_0\} \cup T(C)\}) = \alpha(T(C)) < \alpha(C)$. Hence $\alpha(C) = 0$, that is $\overline{C}$ is compact.

Remark 4.3.2. If $T : Y \to Y$ is a (α, a)-contraction (i.e. T is continuous and there is $a \in [0, 1)$ such that for any $C \in P_b(X)$ we have $\alpha(T(C)) < a\alpha(C)$) then T is a Mönch operator. Indeed, if $C \in P_b(X)$ satisfies $\overline{C} \subset \overline{cv}\{\{x_0\} \cup T(C)\}$, then

$$\alpha(\overline{C}) < \alpha(\overline{cv}\{\{x_0\} \cup T(C)\}) = \alpha(T(C)) < a\alpha(C).$$

Hence $\alpha(C)(1 - a) < 0$. Thus $a > 1$. This is a contradiction with $a \in [0, 1)$, so $\alpha(C) = 0$.

Remark 4.3.3. If $T : Y \to Y$ is complet continuous (i.e. T is continuous and for any $C \in P_b(X)$, $\overline{T(C)}$ is compact), then T is a Mönch operator. Indeed, if $C \in P_b(X)$ and $\overline{C} \subset \overline{cv}\{\{x_0\} \cup T(C)\}$ then

$$\alpha(\overline{C}) < \alpha(\overline{cv}\{\{x_0\} \cup T(C)\}) = \alpha(T(C)) = 0,$$

i.e. $\overline{C}$ is compact.

Now we present a new proof of a result by Mönch [17], in the particular case that the domain of the operator is convex.

Theorem 4.3.1. *Let X be a Banach space, $Y \in P_{cl,cv}(X)$, $x_0 \in int\, Y$ and $T : Y \to X$ a Mönch operator. If T satisfies LSB for any $x \in \partial Y$, then $F_T \neq \emptyset$.*

Proof. Let $\rho : X \to Y$ be the retraction given by (2.3.1). Obviously, $\rho \circ T : Y \to Y$ is continuous, and T is retractible onto Y by ρ. We wish to prove that $\rho \circ T : Y \to Y$ is a Mönch operator. For this, let $C \in P_b(X)$ such that $\overline{C} \subset \overline{cv}\{\{x_0\} \cup (\rho \circ T)(C)\}$. By the definition of ρ, we have

$$(\rho \circ T)(C) \subset \overline{cv}\{\{x_0\} \cup T(C)\}.$$

Then

$$\overline{C} \subset \overline{cv}\{\{x_0\} \cup (\rho \circ T)(C)\} \subset \overline{cv}\{\{x_0\} \cup T(C)\}.$$

Since T is a Mönch operator, we have $\overline{C}$ compact. Hence $\rho \circ T$ is a Mönch operator. □

Using Remark 4.3.1, 4.3.2 and 4.3.3 we can derive from Theorem 4.3.1 the following results:

Theorem 4.3.2. *Let X be a Banach space, $Y \in P_{cl,cv}(X)$, $x_0 \in int\, Y$. If $T : Y \to X$ is α-condensing and T satisfies LSB for any $x \in \partial Y$, then $F_T \neq \emptyset$.*

Theorem 4.3.3. *Let X be a Banach space, $Y \in P_{cl,cv}(X)$, $x_0 \in int\, Y$. If $T : Y \to X$ is a (α, a)-contraction and T satisfies LSB for any $x \in \partial Y$, then $F_T \neq \emptyset$.*

Theorem 4.3.4. (The classical principle of Leray-Schauder, see [15]) *Let X be a Banach space, $Y \in P_{clcv,}(X)$, $x_0 \in int\,Y$. If T"$Y \to X$ is completely continuous and T satisfies LSB for any $x \in \partial Y$, then $F_T \neq \emptyset$.*

If $Y = \overline{B}(x_0, R)$, then T satisfies LSB if and only if T satisfies BP. Thus, we have:

Theorem 4.3.5. *Let X be a Banach space, and $T : Y = \overline{B}(x_0, R) \to X$ a Mönch operator. If T satisfies BP for any $x \in \partial Y$, then $F_T \neq \emptyset$.*

4.4

We have the following result (see [30])

Theorem 4.4.1. *Let $(X, d, \leq)$ be an ordered metric space, $f : X \to X$ an operator and $x, y \in X$ such that $x < y$, $x \leq f(x)$ and $f(y) \leq y$.*

Assume that

(i) f is increasing;

(ii) f is weakly Picard operator.

Then

a) $x \leq f^\infty(x) \leq f^\infty(y) \leq y$

b) $f^\infty(x)$ is the minimal fixed point of f in $F_f \cap [x, y]$ and $f^\infty(y)$ is the maximal fixed point of f in $F_f \cap [x, y]$.

Now we can prove the most important result of this paragraph.

Theorem 4.4.2. *Let X be an ordered space, $v \in X$, and let the operator $T : [-v, v] \to X$ be continuous and increasing. If $T(u) \notin [-v, v]$ implies*

$$\sup\{[0, T(u)_+] \cap [0, v]\} - \sup\{[0, T(u)_-] \cap [0, v]\} \neq u$$

for any $u \in [-v, v]$, then there exists $\underline{u}$ and $\overline{u}$, the minimal solution, respectively the maximal solution of the equation $T(u) = u$.

Proof. We can define the operator $h : [-v, v] \to [-v, v]$, $h = \varphi \circ T$ with φ the retraction giving by (2.4.1). We have $F_h = F_T$ and application h is continuous and increasing. Much more $-v \leq h(v)$ and $h(v) \leq v$. So, hypothesis from theorem 4.4.1 is satisfied. Then

$$-v \leq h^\infty(-v) \leq h^\infty(v) \leq v$$

and $h^\infty(-v) = \underline{u}$ is the minimal fixed point of h in $[-v, v]$, $h^\infty(v) = \overline{u}$ is the maximal fixed point of h in $[-v, v]$. Since $\underline{u}, \overline{u} \in F_h$, hence $\underline{u}, \overline{u} \in F)T$ and $\underline{u} \leq u \leq \overline{u}$ for every $y \in F_T$. □

For a similar result when T is decreasing see [20].

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