RETRACTION METHODS IN FIXED POINT THEORY

Andrei Horvat-Marc
Babeș-Bolyai University
Cluj-Napoca, Romania

Abstract. To obtain fixed point theorems for nonself-mappings there are two possibilities. One consists in using continuation methods of Leray-Schauder type. Roughly speaking, by means of a continuation theorem we can obtain a solution of a given equation starting from one of the solutions of a more simpler equations (see [21]). The other way makes use of the retraction mapping principle. This technique was presented by I.A. Rus in [29].

In this report we adopt the way of a retraction mapping principle. Our goal is to show that under suitable geometrical conditions, continuation theorems of Leray-Schauder type can be alternatively obtained by means of the retraction mapping principle. We shall consider only the boundary conditions of Leray-Schauder, Browder-Petryshyn and Cramer-Ray and we shall restrict ourselves to the case of Banach spaces and vector lattices.

Keywords: fixed point structures, Hilbert space, retractible mapping

AMS Subject Classification: 47H10, 54H25

1 Fixed point structures

Let $X$ be a nonempty set and $Y \in P(X)$, where $P(X)$ denote the set of all nonempty subset of $X$. We denote by $M(X)$ the set of all mapping $f : X \to X$.

Definition 1.1. (see [28]) A triple $(X, S, M)$ is a fixed point structure if
(i) $S \subset P(X)$ is a nonempty subset of $P(X)$;
(ii) $M : P(X) \to \bigcup_{Y \in P(X)} M(Y)$, $Y \subset M(Y)$ is a mapping such that, if $Z \subset Y$ then $M(Z) \supset \{f|_Z : f \in M(Y) \text{ and } f(Z) \subset Z\}$;
(iii) Every $Y \in S$ has the fixed point property with respect to $M(Y)$.

Example 1.1. Let $X$ is a nonempty set, $S = \{\{x\} : x \in X\}$ and $M(Y) = M(Y)$.

Example 1.2. (Knaster, Tarski, Birkhoff) $(X, \leq)$ is a complete lattice, $S = \{Y \in P(X) : (Y, \leq) \text{ is a complete sublattice of } X\}$ and $M(Y) = \{f : Y \to Y : f \text{ is order-preserving mapping}\}$.

Example 1.3. (Banach, Caccioppoli) $(X, d)$ is a complete metric space, $S = P_d(X)$ and $M(Y) = \{f : Y \to Y : f \text{ is a contraction}\}$.

Example 1.4. (Nemytzki, Edelstein) $(X, d)$ is a complete metric space, $S = P_{cp}(X)$ and $M(Y) = \{f : Y \to Y : f \text{ is a contractive mapping}\}$. 
Example 1.5. (Schauder) $X$ is a Banach space, $S = P_{cv}(X)$ and $M(Y) = C(Y,Y)$.

Example 1.6. (Dotson) $X$ is a Banach space, $S = P_{cp,cl}(X)$ and $M(Y) = \{ f : Y \to Y : f$ is a nonexpansive mapping $\}$.

Example 1.7. (Browder) $X$ is a Hilbert space, $S = P_{b,cl,cv}(X)$ and $M(Y) = \{ f : Y \to Y : f$ is a nonexpansive mapping $\}$.

Example 1.8. (Tychonov) $X$ is a Banach space, $S = P_{wcp,cv}(X)$ and $M(Y) = \{ f : Y \to Y : f$ is weakly continuous $\}$.

Example 1.9. (Schauder) $X$ is a Banach space, $S = P_{b,cl,cv}(X)$ and $M(Y) = \{ f : Y \to Y : f$ is completely continuous $\}$.

Example 1.10. (Tychonov) $X$ is a locally convex space, $S = P_{cp,cv}(X)$ and $M(Y) = C(Y,Y)$.

If more generally we let $X$ be a Banach space, $S = P_{cl,cv}(X)$ and $M(Y) = \{ f : Y \to Y : f$ is weakly continuous $\}$, then the triple $(X,S,M)$ is a fixed point structure in a generalized sense, when (ii) does not hold (see [17]).

2 The retraction notion

Let $X$ be a nonempty set and $Y \subset X$ a nonempty subset of $X$.

Definition 2.1. ([9]) A mapping $\rho : X \to Y$ is called a retraction of $X$ onto $Y$ if and only if $\rho|_Y = 1_Y$, i.e. $\rho(x) = x$ for any $x \in Y$.

If $X$ has a certain structure, the mapping $\rho$ must be compatible with that structure. For example a retraction of a topological space will be assumed to be continuous.

2.1 An example of retraction in Hilbert spaces

In this paragraph we consider $H$ be a Hilbert space and $K \subset H$ a nonempty, convex and closed subset, i.e. $K \in P_{cv,cl}(X)$. We will show that $P : H \to K$ the projection mapping of $H$ onto $K$, is a retraction. At first we present some additional results.

Theorem 2.2.1. Let $K \subset H$ be a nonempty, convex and closed subset of $H$, and $u \in H$. Let

$$d = \inf_{v \in K} \| u - v \| = d(u,K).$$

Then there exists a unique element $w \in K$ with $d = \| u - w \| = d(u,K)$.

Proof. For any $v \in K$, we have $\| u - v \| \geq 0$, so for a given $u \in H$, the set of real numbers $\{ \| u - v \| : v \in H \}$ is lower bounded by zero. So $d = \inf_{v \in K} \| u - v \| = d(u,K)$.

Let $(v_n)_{n \geq 1} \subset K$ be a sequence of points from $H$ such that $\| u - v_n \| \to d$, as $n \to \infty$. Since $K$ is convex and $v_n, v_m \in K$ for any $m, n \in N$, we have $\lambda v_m + (1 - \lambda)v_n \in K$ for all $m, n \in N$ and $0 \leq \lambda \leq 1$. Put $\lambda = \frac{1}{2}$. Then $\frac{v_n + v_m}{2} \in K$, so

$$\left\| u - \frac{v_n + v_m}{2} \right\| \geq d.$$

Recall the parallelogram's equality

$$\| x + y \|^2 + \| x - y \|^2 = 2(\| x \|^2 + \| y \|^2)$$

for all $x, y \in H$. 

Andrei Horvat-Marc
We consider $x = u - v_m$ and $y = u - v_n$. Hence
\[
\|v_n - v_m\|^2 = 2\left(\|u - v_m\|^2 + \|u - v_n\|^2\right) - 4\left\|u - \frac{v_m + v_n}{2}\right\|^2.
\]
Then
\[
\|v_n - v_m\|^2 \leq 2\left(\|u - v_m\|^2 + \|u - v_n\|^2\right) - 4d^2.
\]
When $m, n \to \infty$, we obtain $\|v_n - v_m\| \to 0$. This implies that the sequence $(v_n)_{n \geq 1} \subset K$ is fundamental, so it has a limit $w$. Since $(v_n)_{n \geq 1} \subset K$ and $K$ is closed, it follows that $w = \lim_{n \to \infty} v_n \in K$. Hence $\|u - v_n\| \to \|u - w\| = d$ as $n \to \infty$.

In this way, we have shown that there exists $w \in K$ such that
\[
\|u - w\| = d = \inf_{v \in K} \|u - v\|.
\]

For the uniqueness, we assume that there exists $q \in K$, $q \neq w$ such that $\|u - w\| = d = \|u - q\|$. Since $K$ is convex, we have $\frac{q + w}{2} \in K$, hence
\[
d = \inf_{v \in K} \|u - v\| \leq \left\| u + \frac{q + w}{2} \right\| = \left\| \frac{1}{2}u + \frac{1}{2}w\right\| \leq \frac{1}{2}\|u - w\| + \frac{1}{2}\|u - q\| = d
\]
and
\[
d = \left\| u - \frac{q + w}{2} \right\|.
\]
From the parallelogram’s equality, for $x = u - w$ and $y = u - q$ we obtain
\[
\|w - q\|^2 = 2\left(\|u - w\|^2 + \|u - q\|^2\right) - 4\left\|u - \frac{w + q}{2}\right\|^2 = 2(d^2 + d^2) - 4d^2 = 0.
\]
So $\|w - q\| = 0$, which is equivalent to $w = q$. □

Now we formulate

**Definition 2.2.1.** Let $H$ be a Hilbert space, $K \subset H$ a nonempty, convex and closed subset of $X$. Let $P : H \to K$ be the mapping giving by $P(u) = v$, where $w \in K$ is such as
\[
\|u - w\| = d = \inf_{v \in K} \|u - v\|.
\]
The mapping $P$ is called the metric projection of $H$ onto $K$.

We have the following results (see [12]).

**Theorem 2.2.2.** Let $H$ be a Hilbert space, $K \subset H$ a nonempty, convex and closed subset of $X$. The following statements are equivalent:

(i) $w \in K$, $\|u - w\| \leq \|u - v\|$ for every $v \in K$;

(ii) $w \in K$, $\text{Re} \ (u - w, v - w) \leq 0$ for every $v \in K$;

(iii) $w \in K$, $\text{Re} \ (u - v, w - v) \geq 0$ for every $v \in K$.

**Theorem 2.2.3.** Let $H$ be a Hilbert space, $K \subset H$ a nonempty, convex and closed subset of $X$. The metric projection of $X$ onto $K$ is a nonexpansive mapping, i.e.
\[
\|P(u) - P(v)\| \leq \|u - v\|, \ \forall \ u, v \in H.
\]
A consequence of this theorem is the continuity of $P$. Indeed, for any $u \in H$ and any sequence $(u_n)_{n \geq 1} \subset H$ which is norm convergent at $u$, we have $\|P(u) - P(u_n)\| \leq \|u - u_n\|$. Since $\|u_n - u\| \to 0$, as $n \to \infty$, it follows that 

$$\|P(u_n) - P(u)\| \to 0, \quad \text{as } n \to \infty,$$

i.e. $P$ is continuous.

Thus we may conclude that the mapping given by Definition 3.2.1 is a topological retraction of $H$ onto $K$.

**Remark 2.2.1.** For the uniqueness of the element $w \in K$ satisfying $d = \|u - w\| = d(u, K)$ the parallelogram’s equality is an important tool. This is in connexion with the structure of Hilbert space. Thus, Definition 3.2.1 cannot be given for an arbitrary Banach space. However, if $K$ is a nonempty, closed, convex set of an uniformly convex Banach space the metric projection $P$ is univoque and continuous (see [23]).

**Definition 2.2.2.** Let $X$ be a space with the norm $\| \cdot \|$ and $Y \subset X$ a closed subspace of $X$. A linear continuous mapping $P : X \to Y$ is called projection mapping of $X$ onto $Y$ if it is a surjection and $P(y) = y$ for any $y \in Y$.

**Definition 2.2.3.** A closed subspace $Y$ of a Banach space $X$ is called complementably if there exists a projection of $X$ onto $Y$.

**Theorem 2.2.4.** (see [16]) If any closed subspace of a Banach space $X$ is complementably, then $X$ is isomorphic with a Hilbert space.

**Examples.**

- $c_0$ is not complementably in $l^\infty$
- $C[0,1]$ is not complementably in $L^\infty(0,1)$.

### 2.2 An example of retraction onto Banach spaces

Let $X$ be a Banach space, $U \subset X$ a nonempty, convex and closed subset of $X$ and $u_0 \in \text{int}U$.

For every pair $u, v \in X$, the set $[u, v] = \{w \in X : w = (1 - \lambda)u + \lambda v, \, \lambda \in [0,1]\}$ is called the segment between $u$ and $v$. For any $u \in X$ we make the notation $Z(u) = [u, u_0] \cap \partial U$. Now, we define the mapping $\varphi : X \to \mathbb{R}$ by

$$\varphi(u) = \begin{cases} 
\|u - u_0\| & \text{if } Z(u) = \emptyset \\
\max_{v \in Z(u)} \|v - u_0\| & \text{if } Z(u) \neq \emptyset
\end{cases}$$

By means of this mapping we construct the operator $\rho : X \to \overline{U}$, where

$$\rho(u) = \frac{\varphi(u)}{\|u - u_0\|} u + \left(1 + \frac{\varphi(u)}{\|u - u_0\|}\right) u_0$$

(2.3.1)

This mapping is a retraction. Indeed, if $u \in \text{int}U$ then $Z(u) = \emptyset$, so $\varphi(u) = \|u - u_0\|$ and this implies $\rho(u) = u$. If $u \in \partial U$ then $\varphi(u) = \|u - u_0\|$ and again $\rho(u) = u$. Hence $\rho(u) = u$ for any $u \in \overline{U}$. If $u \not\in \overline{U}$ then $Z(u) \neq \emptyset$ and $\varphi(u) < \|u - u_0\|$.
So \( \frac{\varphi(u)}{\|u - u_0\|} \in (0, 1) \) and consequently \( \rho(u) \in [u, u_0] \), i.e. the image of any point \( u \in X \setminus U \) by \( \rho \) lies on the segment \([u, u_0]\).

Moreover, we have

\[
\|\rho(u) - u_0\| = \left\| \frac{\varphi(u)}{\|u - u_0\|} u + \left(1 - \frac{\varphi(u)}{\|u - u_0\|}\right) u_0 - u_0 \right\| = \left\| \frac{\varphi(u)}{\|u - u_0\|} u - \frac{\varphi(u)}{\|u - u_0\|} u_0 \right\| = \varphi(u).
\]

In conclusion, if \( u \in X \setminus U \) then \( \rho(u) \) is the intersection point of the segment \([u, u_0]\) with \( \partial U \), which is the most nearly by \( u \). So \( \rho \) is a continuous retraction.

If \( U = B(u_0, r) = \{ u \in X : \|u - u_0\| < r \} \subset X \) the mapping \( \rho : X \to \overline{U} \) is giving by

\[
\rho(u) = \begin{cases} 
  u & \text{if } u \in \overline{U} \\
  \frac{r}{\|u - u_0\|} u + \left(1 - \frac{r}{\|u - u_0\|}\right) u_0 & \text{if } u \notin \overline{U}
\end{cases}
\]

and it is call "the radial retraction".

### 2.3 An example of retraction onto ordered spaces

Let \( X \) be a real vectorial space. \( X \) is a vector lattice (ordered space) if \( X \) is lattice and

i) for any \( z \in X \), \( x \leq y \) then \( x + z \leq y + z \)

ii) if \( x \geq 0 \) and \( \lambda \geq 0 \) then \( \lambda x \geq 0 \).

In any ordered space \( X \), denote by \([x, y] = \{ z \in X : x \leq z \leq y \}\) the interval with respect to order (ordered interval).

The set \( X_+ = \{ x \in X : x \geq 0 \} \) is called the cone of positifs elements of vectorial lattice \( X \).

For every \( x \in X \), the element \( x_+ = x \lor 0 \) is called the positive part of \( x \) and \( x_- = (-x) \lor 0 = (-x)_+ \) the negative part. The element \(|x| = x_+ - x_- \) means the absolute value of \( x \). For any \( x \in X \) we have \( x = x_+ - x_- \).

Let \( v \in X_+ \). Demote with \( Y = [-v, v] \) and define the application \( \varphi : X \to Y \),

\[
\varphi(u) = \begin{cases} 
  u & \text{if } u \in Y \\
  \sup\{0, u_+\} \cap [0, v] - \sup\{0, u_-\} \cap [0, v] & \text{if } u \notin Y
\end{cases}
\]

We make the notations

\[
Y_+ = [0, v] \quad U_+ = [0, u_+] \quad \text{and} \quad U_- = [0, u_-]
\]

The application \( \varphi \) is a retraction of \( X \) onto \( Y \) which is compatible with structure of space \( X \), i.e. it is continuous and for every \( u_1, u_2 \in X \), \( u_1 \leq u_2 \) we have \( \varphi(u_1) \leq \varphi(u_2) \).
Indeed, let \( u_1, u_2 \in X \) with \( u_1 \leq u_2 \).

1. Suppose that \(-v \leq u_1 \leq u_2 \leq v\), i.e. \( u_1, u_2 \in Y \). Then \( \varphi(u_1) = u_1 \leq u_2 = \varphi(u_2) \).

2. If \( u_1 \in Y \) and \( u_2 \not\in Y \) the \( \varphi(u_1) = u_1 \) and \( \varphi(u_2) = \sup\{U_{2+} \cap Y_{+}\} - \sup\{U_{2-} \cap Y_{+}\} \).

From \( u_1 \leq u_2 \) we have \( u_{1+} \leq u_{2+} \) and \( u_{2-} \leq u_{1-} \). Since \( u_1 \in Y \) we obtain \( u_{1+} \leq v \) and \( u_{1-} \leq v \). We have \( u_{1+} \leq u_{2+} \) and \( u_{1+} \leq v \), hence

\[
\varphi(u_1) = u_1 = \sup\{U_{2+} \cap Y_{+}\}.
\]

From \( u_{2-} \leq u_{1-} \) and \( u_{1-} \leq v \) results

\[
u_{1-} \geq \sup\{U_{2-} \cap Y_{+}\}.
\]

Then

\[
\varphi(u_1) = u_1 = u_{1+} - u_{1-} \leq \sup\{U_{2+} \cap Y_{+}\} - \sup\{U_{2-} \cap Y_{+}\} = \varphi(u_2).
\]

If \( u_1 \not\in Y \) and \( u_2 \in Y \) the proof is similary.

3. If \( u_1 \not\in Y \) and \( u_2 \not\in Y \) then

\[
\varphi(u_i) = \sup\{U_{i+} \cap Y_{+}\} - \sup\{U_{i-} \cap Y_{+}\}, \quad i = 1,2.
\]

Since \( u_1 \leq u_2 \) we have \( u_{1+} \leq u_{2+} \) and \( u_{2-} \leq u_{1-} \). Then \( U_{1+} \subset U_{2+} \) and \( U_{2-} \subset U_{1-} \). Results

\[
\sup\{U_{1+} \cap Y_{+}\} \leq \sup\{U_{2+} \cap Y_{+}\}
\]

and

\[
\sup\{U_{2-} \cap Y_{+}\} \leq \sup\{U_{1-} \cap Y_{+}\}.
\]

Finally, we have

\[
\varphi(u_1) = \sup\{U_{1+} \cap Y_{+}\} - \sup\{U_{1-} \cap Y_{+}\} \leq \sup\{U_{2+} \cap Y_{+}\} - \sup\{U_{2-} \cap Y_{+}\} = \varphi(u_2).
\]

In conclusion, for any \( u_1, u_2 \in X \) with \( u_1 \leq u_2 \) we have \( \varphi(u_1) \leq \varphi(u_2) \). In other words \( \varphi \) is increasing.

3 Boundary conditions

We recall Leray-Schauder boundary condition and show its equivalence to those of Browder-Petryshyn and Cramer-Ray when the domain is a ball. For all there definitions \( U \) is a subset of a Banach space \( X \), \( u_0 \in \text{int} \, U \) and \( T : U \rightarrow X \) is a mapping.

For \( r > 0 \) and \( u \in X \) we let \( B(u, r) \) be the open ball of \( X \) of radius \( r \) and center \( u \), i.e.

\[
B(u, r) = \{ v \in X : \| u - v \| < r \}.
\]
For every pair $u, v \in X$, the set $[u, v] = \{ w \in X : w = (1 - \lambda)u + \lambda v, \lambda \in [0, 1] \}$ is called the segment between $u$ and $v$.

We shall assume $u_0 \in \text{int } U$.

**Definition 3.1.** (Leray-Schauder, see [15]) Let $u \in \partial U$. $T$ satisfies the Leray-Schauder boundary condition (LSB) at $u$ relative to $U$ if and only if

\[
(1 - \lambda)u_0 + \lambda T(u) \neq u \quad \text{for every } \lambda \in [0, 1].
\]

**Remark 3.1.** The definition has the equivalent form

\[
T(u) - u_0 \neq k(u - u_0) \quad \text{for } \lambda \in [0, 1].
\]

In fact Definition 3.1 says that $T$ satisfies LSB at $u$ if and only if the point $u$ doesn’t lie on the segment $[u_0, T(u)]$.

**Definition 3.2.** (Browder-Petryshyn, see [8]) Let $u \in U$ with $u \neq T(u)$. $T$ satisfies the Browder-Petryshyn condition (BP) at $u$ relative to $U$ if and only if

\[
B(T(u), ||T(u) - u||) \cap U \neq \emptyset.
\]

**Remark 3.2.** (i) The relation (3) is equivalent to the existence of an element $v \in U$ such that

\[
||T(u) - v|| < ||T(u) - u||.
\]

(ii) Obviously, if $T(u) \in U$ or $u \in \text{int } U$, then $T$ satisfies BP at $u$ relative to $U$.

**Definition 3.3.** (Cramer-Ray, see [22]) Let $u \in U$ with $u \neq T(u)$. $T$ satisfies the Cramer-Ray condition (CR) at $u$ relative to $U$ if and only if

\[
\liminf_{h \to 0^+} \frac{d((1 - h)u + hT(u), U)}{h} < ||u - T(u)||.
\]

**Lemma 3.1.** Let $U$ be convex and $u \in U$ with $u \neq T(u)$. $T$ satisfies CR at $u$ if and only if there exists $v \in U$ and $0 < h \leq 1$ such that

\[
\frac{||(1 - h)u + hT(u) - v||}{h} < ||u - T(u)||.
\]

**Proof.** $\Rightarrow$) Obvious.

$\Leftarrow$) Without loss of generality, choose $0 < k < 1$ such that

\[
\frac{||(1 - h)u + hT(u) - v||}{h} < k||u - T(u)||.
\]

For each $a \in (0, 1)$ let $z(a) = u + a(v - u)$. Since $z(a) \in [u, v]$ and $U$ is convex we have $z(a) \in U$. Now, it suffices to show that for any $a \in (0, 1)$, $z(a)$ satisfies

\[
\frac{||(1 - ah)u + ahT(u) - z(a)||}{ah} \leq k||u - T(u)||.
\]
Since
\[
\frac{\| (1 - ah)u + ahT(u) - z(a) \|}{ah} = \frac{\| u - ahu + ahT(u) - u - a(v - u) \|}{ah} = \\
\frac{\| (1 - h)u + hT(u) - v \|}{h} \leq k\| u - T(u) \|.
\]
Thus the lemma is proved. □

**Remark 3.3.** If \( X \) is a Hilbert space, with inner product \((\cdot, \cdot)\), it is possible to introduce the Leray-Schauder condition (LS), see [31], in the following way:
Let \( u \in U \) with \( u \neq T(u) \) and
\[
LS(u, T(u)) = \{ v \in X : \text{Re} (T(u) - u, v - u) > 0 \}.
\]
The mapping \( T \) satisfies (LS) at \( u \) relative to \( U \) if and only if
\[
(6) \hspace{1cm} LS \ast u, T(u) \cap U \neq \emptyset.
\]
If \( U \) is convex and \( u \in U \) with \( u \neq T(u) \) then (see [31])
\[
T \text{ satisfies LS at } u \text{ if and only if } T \text{ satisfies BP at } u
\]
and
\[
T \text{ satisfies LS at } u \text{ if and only if } T \text{ satisfies CR at } u.
\]

**Proposition 3.1.** Let \( X \) be a Banach space, \( U = B(u_0, r) \) and \( u \in \partial U \) such that \( u \neq T(u) \). \( T \) satisfies LSB at \( u \) if and only if \( T \) satisfies BP at \( u \).

**Proof.** \( \Leftarrow \) Assume that \( T \) satisfies BP and we wish \( T \) satisfies LSB. We know that
\[
\| T(u) - u_0 \| \leq \| T(u) - v \| + \| v - u_0 \|
\]
for any \( v \in U \). If \( T \) satisfies BP at \( u \) then conform of remark 3.2 exists \( v \in U \) such that
\[
\| T(u) - v \| < \| T(u) - u \|. \]
Since \( u \in \partial U \) we have
\[
\| u_0 - v \| < \| u_0 - u \| = r.
\]
So
\[
\| T(u) - u_0 \| < \| T(u) - u \| + \| u - u_0 \|.
\]
In conclusion \( u \notin [u_0, T(u)] \), i.e. \( T \) satisfies LSB.

\( \Rightarrow \) Assume that \( T \) satisfies LSB and we wish \( T \) satisfies BP. Without loss of generality we can consider \( \| u_0 - T(u) \| > r \). Affirm that
\[
v = \frac{r}{\| T(u) - u_0 \|} T(u) + \left( 1 - \frac{r}{\| T(u) - u_0 \|} \right) u_0 \in U \cap B(T(u), \| T(u) - u \|).
\]
Indeed, we have\[\|v - u_0\| = \|T(u) - u_0\| \cdot T(u) + \left(1 - \frac{r}{\|T(u) - u_0\|}\right) u_0 - u_0 = r\]
hence \(v \in U\).

On the other side\[\|T(u) - v\| = \left|1 - \frac{r}{\|T(u) - u_0\|}\right| \|T(u) - u_0\| = \|T(u) - u_0\| - \|v - u_0\|.
\]
Since \(T\) satisfies LSB results\[\|T(u) - u_0\| < \|T(u) - u\| - \|v - u_0\|.
\]
Then\[\|T(u) - v\| < \|T(u) - u\|, \text{i.e. } v \in B(T(u), \|T(u) - u\|). \quad \square \]
If \(U \neq B(u_0, r)\), the last proposition is not true.

**Example 3.1.** Let \(X = \mathbb{R}^2\), with euclidian’s norm and \(U = \{(x, y) \in \mathbb{R}^2, |x| \leq 1, |y| \leq 1\}\), i.e. \(U\) is the square with vertex \((1,1), (-1,1), (-1,-1), (1,-1)\). Choose \(u_0 = (0,0)\), \(u = \left(1, \frac{1}{n}\right)\) with \(n > 1\) and suppose that exists a mapping \(T : U \rightarrow \mathbb{R}^2\) such that \(T(u) = \left(k, \frac{k}{n}\right)\), for \(k > 1\). Under of this assumption, we have \(T(u) = ku\) for \(k > 1\), so remark 1.1 said \(T\) does not satisfies LSB. But for \(k > \frac{n + 1}{2}\), \(T\) satisfies BP.

Now, we fix the point \(v = (1,1)\) and obtain\[\|T(u) - v\|^2 = (k - 1)^2 + \left(\frac{k}{n} - 1\right)^2 = \frac{n^2(k - 1)^2 + (k - n)^2}{n^2}.
\]
Moreover\[\|T(u) - u\|^2 = (k - 1)^2 + \left(\frac{k}{n} - 1\right)^2 = \frac{(n^2 + 1)(k - 1)^2}{n^2}.
\]
The mapping \(T\) satisfies BP is equivalent with\[\|T(u) - v\| < \|T(u) - u\|,
\]that is to say\[(n^2 + 1)(k - 1)^2 > n^2(k - 1)^2 + (k - n)^2 \Rightarrow (k - 1)^2 > (k - n)^2 \Rightarrow 2k(n - 1) > n^2 - 1 \Rightarrow k > \frac{n + 1}{2}.
\]
So for \(k > \frac{n + 1}{2}\), \(T\) satisfies BP, but \(T\) not satisfies LSB.
3.1 Conditions of retractibility

In following, we denote by $F_f$ the set of fixed point of the mapping $f$.

**Definition 3.2.1.** ([9]) A mapping $f : Y \rightarrow X$ is retractible onto $Y$ if there is a
retraction $\rho : X \rightarrow Y$ such that $F_{\rho \circ f} = F_f$.

Condition (i) $F_{\rho \circ f} = F_f$ is equivalent with:

(ii) if $x \in \rho(f(Y) \setminus Y)$, then $f(x) \notin \rho^{-1}(x) \setminus \{x\}$.

Indeed, theorem 1.1 from [7] - the retraction mapping principle - shows that condi-
tion (ii) implies (i); now we suppose $F_{\rho \circ f} = F_f$ and there exists $f(x) \in \rho(f(Y) \setminus Y)$
such that $f(x) \in \rho^{-1}(x) \setminus \{x\}$. Hence $x \notin F_f$, but on the other side $x = \rho(f(x))$, i.e.
$x \in F_{\rho \circ f}$. This is a contradiction, so (i) implies (ii). In conclusion Definition 3.2.1 is
equivalent with the definition given by Brown (see [7]).

**Example 2.1.** (Poincaré, Bohr, Leray-Schauder, Rothe, Altman, Furi-Vignoli,...) Let $X$ be a Banach space and $Y = \overline{B}(0, R) \subset X$. If $f : \overline{B}(0, R) \rightarrow X$ is such that $\|x\| = R$, $f(x) = \lambda x$ implies $\lambda < 1$, then $f$ is retractible onto $\overline{B}(0, R)$ with respect to
the radial retraction $\rho : X \rightarrow \overline{B}(0, R)$.

**Example 2.2.** (Altman) Let $X$ be a Banach space and $f : X \rightarrow X$ a norme con-
traction mapping. Then there exists $R > 0$ such that $f : \overline{B}(0, R) \rightarrow X$ is retractible
onto $\overline{B}(0, R)$ with respect to the radial retraction.

**Example 2.3.** (Halpern-Beroman) Let $X$ be a strictly convex normed linear
space. Let $Y \subset X$ be a compact convex subset of $X$ and $\rho : X \rightarrow Y$ the
metric projection onto $Y$. If $f : Y \rightarrow X$ is nowhere normal-outward, then $f$ is retractible
onto $Y$ with respect to $\rho$.

**Example 2.4.** Let $X$ be a subset of $X$ and $\rho : X \rightarrow Y$ a retraction.
If $f : Y \rightarrow X$ is such that $x \in Y \setminus F_f$ implies $f(x) \in X \setminus \rho^{-1}(x)$, then $f$ is retractible
onto $Y$ with respect to $\rho$.

In this paragraph we will give some theorems with form: if $T$ satisfies a kind of
boundary conditions then $T$ is retractible.

**Theorem 3.2.1.** Let $X$ be a Hilbert space, $U \in P_{\text{cv,cl}}(X)$. If the mapping $T : U \rightarrow X$ satisfies BP for any $u \in \partial U$ then $T$ is retractible onto $U$ with respect to
the projection mapping of $X$ to $U$.

**Proof.** Here $\rho = P$ denote the metric projection. Assume that $F_{\rho \circ f} \neq F_f$. Let
$u \in F_{\rho \circ f} \setminus F_f$. Then $u = P(T(u))$ and $u \in \partial U$. This is equivalent with $T(u) \neq u$ and
$0 < \|u - T(u)\| < \|T(u) - v\|$, for any $v \in U$. Results a contradiction with $T$ satisfies BP condition. □

Let $X$ be a Hilbert space, $U \subset X$ convex, $u \in U$ with $u \neq T(u)$. From Remark
3.3 results $T$ satisfies BP at $u$ iff $T$ satisfies CR at $u$. Then we have

**Theorem 3.2.2.** Let $X$ be a Hilbert space, $U \in P_{\text{cv,cl}}(X)$. If the mapping $T : U \rightarrow X$ satisfies CR for any $u \in \partial U$, then $T$ is retractible onto $U$ with respect to
the metric projection of $X$ onto $U$.

For a Banach space $X$ we will consider the retraction $\rho$ given by relation (2.3.1).

**Theorem 3.2.3.** Let $X$ be a Banach space, $U \in P_{\text{cv,cl}}(X)$, $u_0 \in \text{int} U$ and the
mapping $T : U \rightarrow X$. If $T$ satisfies LSB for any $u \in \partial U$, then $T$ is retractible onto $U$
with respect of the retraction $\rho$.

**Proof.** Assume that $F_{\rho \circ f} \neq F_f$. Let $u \in F_{\rho \circ f} \setminus F_f = \emptyset$, i.e. $T(u) \neq u$ and
Let $X$ be a Banach space and $u_0 \in X$. If the mapping $T : \overline{B}(u_0, r) \to X$ satisfies BP for any $u \in \partial \overline{B}(u_0, r)$ then $T$ is retractible onto $\overline{B}(u_0, r)$ with respect to the radial retraction.

Theorem 3.2.5. Let $X$ be a vector lattice (ordered space), $v \in X_+$ and $T : [-v, v] \to X$ be an operator.

If $T(u) \not\in Y$ implies

$$\sup\{0, T(u)_+ \cap [0, v]\} - \sup\{0, T(u)_- \cap [0, v]\} \neq u$$

then $T$ is retractible onto $[-v, v]$ with respect of retraction $\varphi$ given by relation (2.4.1).

Proof. Let $u \in F_{\varphi \circ T} \setminus F_T \neq \emptyset$. Then $u = (\varphi \circ T)(u)$ and $u \neq T(u)$. Results $T(u) \not\in [-v, v]$ so

$$u = \varphi(T(u)) = \sup\{0, T(u)_+ \cap [0, v]\} - \sup\{0, T(u)_- \cap [0, v]\}.$$

We get a contradiction, hence $F_{\varphi \circ T} \subset F_T$. This implies $F_{\varphi \circ T} = F_T$, i.e. $T$ is retractible onto $[-v, v]$ with respect to $\varphi$. □

4 Fixed points of retractible mappings

4.1

Let us starting with

Lemma 4.1. (see [29]) Let $(X, S, M)$ be a fixed point structure. Let $Y \in S$ and $\rho : X \to Y$ a retraction. Let $f : Y \to X$ be such that

(i) $\rho \circ f \in M(Y)$

(ii) $f$ is retractible onto $Y$ by $\rho$.

Then $F_f \neq \emptyset$.

Proof. From (i) we obtain $F_{\rho \circ f} \neq \emptyset$ and from (ii) we have $F_{\rho \circ f} = F_f$. Results $F_f \neq \emptyset$. □

4.2

Theorem 4.2.1. Let $X$ be a Hilbert space, $U \in P_{cv,cl,b}(X)$ and $T : U \to X$ is a nonexpansive mapping. If $T$ satisfies BP for any $u \in \partial U$, then $F_f \neq \emptyset$.

Proof. We take $(X, S, M)$ as in example 1.7 and $\rho$ the projection mapping of $X$ onto $Y$. Since $\rho$ and $T$ is nonexpansive mapping hence (i) from lemma 4.1 is verified. By Theorem 3.2.1 we have $T$ is retractible onto $U$ with respect to the metric projection, then (ii) is satisfied. □

Obviously, we have

Theorem 4.2.2. Let $X$ be a Hilbert space, $U \in P_{cv,cl,b}(X)$ and $T : U \to X$ is a nonexpansive mapping. If $T$ satisfies CR for any $u \in \partial U$, then $F_f \neq \emptyset$. 

4.3 A Leray-Schauder type theorem

Let $X$ be a Banach space, $Y \in P_{cl,cv}(X)$ and $x_0 \in \text{int } Y$. A mapping $T : Y \to Y$ is said to be a Mönch operator if and only if $T$ is continuous and for any $C \in P_b(X)$ satisfies $\overline{C} \subset \overline{\varphi}\{x_0\} \cup T(C)$ we have that $\overline{C}$ is compact. In what follows we denote by $\alpha$ a measure of noncompactness on $X$.

**Remark 4.3.1.** If $T : Y \to Y$ is $\alpha$-condensing (i.e. $T$ is continuous and for any $C \in P_b(X)$ with $\alpha(C) \neq 0$ we have $\alpha(T(C)) < \alpha(C)$) then $T$ is a Mönch operator. Indeed, for $C \in P_b(X)$, since $\overline{C} \subset \overline{\varphi}\{x_0\} \cup T(C)$ we have $\alpha(\overline{C}) < \alpha(\overline{\varphi}\{x_0\} \cup T(C)) = \alpha(T(C)) < \alpha(C)$. Hence $\alpha(C) = 0$, that is $\overline{C}$ is compact.

**Remark 4.3.2.** If $T : Y \to Y$ is a $(\alpha, a)$-contraction (i.e. $T$ is continuous and there is $a \in [0, 1)$ such that for any $C \in P_b(X)$ we have $\alpha(T(C)) < a\alpha(C)$) then $T$ is a Mönch operator. Indeed, if $C \in P_b(X)$ satisfies $\overline{C} \subset \overline{\varphi}\{x_0\} \cup T(C)$, then

$$\alpha(\overline{C}) < a\alpha(\overline{\varphi}\{x_0\} \cup T(C)) = \alpha(T(C)) < a\alpha(C).$$

Hence $\alpha(C)(1 - a) < 0$. Thus $a > 1$. This is a contradiction with $a \in [0, 1)$, so $\alpha(C) = 0$.

**Remark 4.3.3.** If $T : Y \to Y$ is complet continuous (i.e. $T$ is continuous and for any $C \in P_b(X)$, $T(\overline{C})$ is compact), then $T$ is a Mönch operator. Indeed, if $C \in P_b(X)$ and $\overline{C} \subset \overline{\varphi}\{x_0\} \cup T(C)$ then

$$\alpha(\overline{C}) < a\alpha(\overline{\varphi}\{x_0\} \cup T(C)) = \alpha(T(C)) = 0,$$ 

i.e. $\overline{C}$ is compact.

Now we present a new proof of a result by Mönch [17], in the particular case that the domain of the operator is convex.

**Theorem 4.3.1.** Let $X$ be a Banach space, $Y \in P_{cl,cv}(X)$, $x_0 \in \text{int } Y$ and $T : Y \to X$ a Mönch operator. If $T$ satisfies LSB for any $x \in \partial Y$, then $F_T \neq \emptyset$.

**Proof.** Let $\rho : X \to Y$ be the retraction given by (2.3.1). Obviously, $\rho \circ T : Y \to Y$ is continuous, and $T$ is retractible onto $Y$ by $\rho$. We wish to prove that $\rho \circ T : Y \to Y$ is a Mönch operator. For this, let $C \in P_b(X)$ such that $\overline{C} \subset \overline{\varphi}\{x_0\} \cup (\rho \circ T)(C)$. By the definition of $\rho$, we have

$$(\rho \circ T)(C) \subset \overline{\varphi}\{x_0\} \cup T(C).$$

Then

$$\overline{C} \subset \overline{\varphi}\{x_0\} \cup (\rho \circ T)(C) \subset \overline{\varphi}\{x_0\} \cup T(C).$$

Since $T$ is a Mönch operator, we have $\overline{C}$ compact. Hence $\rho \circ T$ is a Mönch operator.

Using Remark 4.3.1, 4.3.2 and 4.3.3 we can derive from Theorem 4.3.1 the following results:

**Theorem 4.3.2.** Let $X$ be a Banach space, $Y \in P_{cl,cv}(X)$, $x_0 \in \text{int } Y$. If $T : Y \to X$ is $\alpha$-condensing and $T$ satisfies LSB for any $x \in \partial Y$, then $F_T \neq \emptyset$.

**Theorem 4.3.3.** Let $X$ be a Banach space, $Y \in P_{cl,cv}(X)$, $x_0 \in \text{int } Y$. If $T : Y \to X$ is a $(\alpha, a)$-contraction and $T$ satisfies LSB for any $x \in \partial Y$, then $F_T \neq \emptyset$. 
Theorem 4.3.4. (The classical principle of Leray-Schauder, see [15]) Let $X$ be a Banach space, $Y \in P_{clcv}(X)$, $x_0 \in \text{int} Y$. If $T^* Y \to X$ is completely continuous and $T$ satisfies LSB for any $x \in \partial Y$, then $F_T \neq \emptyset$.

If $Y = \overline{B}(x_0, R)$, then $T$ satisfies LSB if and only if $T$ satisfies BP. Thus, we have:

Theorem 4.3.5. Let $X$ be a Banach space, and $T : Y = \overline{B}(x_0, R) \to X$ a Mönch operator. If $T$ satisfies BP for any $x \in \partial Y$, then $F_T \neq \emptyset$.

4.4

We have the following result (see [30])

Theorem 4.4.1. Let $(X, d, \leq)$ be an ordered metric space, $f : X \to X$ an operator and $x, y \in X$ such that $x < y$, $x \leq f(x)$ and $f(y) \leq y$.

Assume that

(i) $f$ is increasing;

(ii) $f$ is weakly Picard operator.

Then

a) $x \leq f^\infty(x) \leq f^\infty(y) \leq y$

b) $f^\infty(x)$ is the minimal fixed point of $f$ in $F_T \cap [x, y]$ and $f^\infty(y)$ is the maximal fixed point of $f$ in $F_T \cap [x, y]$.

Now we can prove the most important result of this paragraph.

Theorem 4.4.2. Let $X$ be an ordered space, $v \in X$, and let the operator $T : [-v, v] \to X$ be continuous and increasing. If $T(u) \not\in [-v, v]$ implies

$$\sup\{[0, T(u)_+] \cap [0, v]\} - \sup\{[0, T(u)_-] \cap [0, v]\} \neq u$$

for any $u \in [-v, v]$, then there exists $\underline{u}$ and $\overline{u}$, the minimal solution, respectively the maximal solution of the equation $T(u) = u$.

Proof. We can define the operator $h : [-v, v] \to [-v, v]$, $h = \varphi \circ T$ with $\varphi$ the retraction giving by (2.4.1). We have $F_h = F_T$ and application $h$ is continuous and increasing. Much more $-v \leq h(v)$ and $h(v) \leq v$. So, hypothesis from theorem 4.4.1 is satisfied. Then

$$-v \leq h^\infty(-v) \leq h^\infty(v) \leq v$$

and $h^\infty(-v) = \underline{u}$ is the minimal fixed point of $h$ in $[-v, v]$, $h^\infty(v) = \overline{u}$ is the maximal fixed point of $h$ in $[-v, v]$. Since $\underline{u}, \overline{u} \in F_h$, hence $\underline{u}, \overline{u} \in F_T$ and $\underline{u} \leq u \leq \overline{u}$ for every $y \in F_T$. $\square$

For a similar result when $T$ is decreasing see [20].

References


