ABOUT DIFFERENCE-DIFFERENTIAL EQUATIONS WHICH APPEAR IN NUMBER THEORY

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Abstract. In this paper we prove, that the solution of the equation
\[(Q_n(t)p(t))' = kp(t) - kp(t + \alpha)\]
is unique if the solution that have normalized polynomial-like boundary condition.

Keywords: Dickmann function, difference-differential equations

AMS Subject Classification: 39A10, 34K06

1. Introduction

In the sieve theories there occur a pair of difference-differential equations with
retarded arguments. These functions appear in several asymptotic formulas.
For example let \( P_1(n) \) the largest prime factor of \( n \) and \( P_2(n) \) the second largest
prime factor of \( n \). Then we have (Wheller [4]):
\[
\sum_{1 \leq n \leq x \atop P_2(n) \leq (P_1(n))^\frac{1}{5}} 1 = e^{\gamma} p(u) x + O\left(\frac{x}{\log x}\right)
\]
where \( \gamma \) denotes the Euler’s constant and \( p(u) \) the Dickman function, which satisfies
the following difference-differential equation:
\[up'(u) + p(u - 1) = 0 \quad (u > 1)\]
or
\[(up(u))' = p(u) - p(u - 1)\]

obtained some properties of these functions.
In this paper we have studied the following generalized difference-differential equa-
tions:
\[(Q_n(t)p(t))' = kp(t) - kp(t + \alpha)\]
where \(Q_n(t)\) is a polynomial of degree \(n\) with positive coefficients and \(k, \alpha\) are positive constants.

We prove that the solution of this equation unique (Theorem 2.2) and positive (Theorem 2.3) under some conditions.

2. Main results

**Theorem 2.1.**
Let \(Q_n(t)\) be a polynomial of degree \(n\) with positive coefficients and \(k, \alpha\) positive constants.

If \(p \in C^1((a, \infty))\) satisfies

\[(Q_n(t)p(t))' = kp(t) - kp(t + \alpha)\]  \tag{1}

and

\[p(t) \sim t^a\]  \tag{2}

then \(a = -n\).

**Proof**
Integrating the difference-differential equations we have:

\[Q_n(t)p(t) + k \int_t^{t+\alpha} p(s) \, ds = c\]

where \(c\) is some constant. By (2) we have:

\[t^{n+a} + kt^a \to c, \quad t \to \infty.\]

It follows that \(a + n = 0\) or \(a = -n\).

This argument shows that \(c = 1\), so we have a following integral equation:

\[Q_n(t)p(t) + k \int_t^{t+\alpha} p(s) \, ds = 1, \quad t > 0.\]  \tag{3}

Now we show the uniqueness of \(p\).
Theorem 2.2.
For each $k > 0$ there exists at most one function $p \in C^1 ((0, \infty))$ which satisfies (1) and (2).

Proof
Let $\tilde{p} := p_1 - p_2$ where $p_1$ and $p_2$ satisfy (1) and (2) for the same value of $k$. Each of $p_1$ and $p_2$ satisfies (3) and hence

$$Q_n(t) \tilde{p}(t) = -k \int_0^t \tilde{p}(s) \, ds$$

Also by (2) we have

$$Q_n(t) \tilde{p}(t) \longrightarrow 0$$

or

$$\tilde{p}(t) = o(t^n).$$

Suppose that

$$|\tilde{p}(t)| < M \cdot t^{-n}$$

for some positive number $M$ and $t \geq t_0$. On this range we have:

$$|Q_n(t) \tilde{p}(t)| \leq k \cdot M \int_0^t s^{-n} \, ds < k \cdot M \cdot t^{-n}$$

and so

$$t^n |\tilde{p}(t)| < k \cdot M \cdot t^{-n}$$

$$|\tilde{p}(t)| < k \cdot M \cdot t^{-2n} \leq \frac{1}{2} M \cdot t^{-n}$$

if $u_0 \geq \max \left\{u_0, (2k)^{\frac{1}{n}}\right\}$.

It follows in same way that

$$|\tilde{p}(t)| < \left(\frac{1}{2}\right)^l \cdot M \cdot t^{-n} \quad \forall l \geq 1$$

and if $l \longrightarrow \infty$ we have $|\tilde{p}(t)| = 0$ for all $t$.

Theorem 2.3
If $p(t) > 0 \quad \forall t \in (0, 1]$ and $Q'_n(0) > k$ the solution $p \in C^1 ((0, \infty))$ of (1) and (2) satisfies the following properties:

$$p(t) > 0 \quad \forall t > 0$$
Proof
If exist, let \( \tau := \sup \{ t \mid p(t) \leq 0 \} \).
If \( \tau \) is finite, then \( \tau > 1 \) since \( p \) is continuous and satisfies \( p(t) > 0 \) for \( 0 \leq t \leq 1 \). We have \( p(t) > 0 \) for \( t > \tau \) and by (1) we can then write

\[
Q_n(t)p'(t) = -kp(t + \alpha) - (Q'_n(t) - k)p(t)
\]

By the \( Q'_n(0) > k \) implies, that the right hand side negative. This shows that \( p'(t) < 0 \) or the function \( p \) is strict decreasing:

\[
p(t) < p(\tau) \leq 0 \quad \forall t > \tau
\]

which contradiction with (2).
If \( \tau \) is infinite, exist \( (t_k)_{k \geq 1} \) such that \( t_k \rightarrow \infty \) if \( k \rightarrow \infty \) and

\[
p(t_k) \leq 0 \quad \forall k \geq 1.
\]
This implies that

\[
(t_k)^n p(t_k) \leq 0
\]
which contradiction with (2).
This shows that \( \tau \) not exist and \( p(t) > 0 \quad \forall t \in [0, \infty) \).
The \( p'(t) < 0 \) follows by (4).

References


