FIXED POINT THEOREMS FOR MULTIVALUED EXPANSIVE OPERATORS

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Abstract. The main goal of this paper is to study the existence and data dependence of the fixed points for a class of generalized expansive multifunctions.

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1 Introduction

In some papers from the 80’s, some fixed point theorems for expansive-type single-valued mappings are proved (see [1], [2], [6] etc.). The main purpose of this note is to extend the above mentioned result to the multivalued case. The date dependence of the fixed points set is also studied.

Let $(X, d)$ be a metric space. Throughout the paper we use the following symbols:

- $P(X) := \{ Y \subset X \mid Y \neq \emptyset \}$,
- $P_p(X) := \{ Y \in P(X) \mid Y \text{ has the property } "p" \}$, where "p" could be: $cl = \text{closed}$, $cp = \text{compact}$, $b = \text{bounded}$, etc.

We consider now the following functionals:

- $D : P(X) \times P(X) \to \mathbb{R}_+$, $D(A, B) = \inf \{ d(a, b) \mid a \in A, b \in B \}$;
- $\delta : P_b(X) \times P_b(X) \to \mathbb{R}_+$, $\delta(A, B) = \sup \{ d(a, b) \mid a \in A, b \in B \}$;
- $\rho : P_b(X) \times P_b(X) \to \mathbb{R}_+$, $\rho(A, B) = \sup \{ D(a, B) \mid a \in A \}$;
- $H : P_b(X) \times P_b(X) \to \mathbb{R}_+$, $H(A, B) = \max \{ \rho(A, B), \rho(B, A) \}$.

It is well known that $(P_b, cl(X), H)$ is a metric space and if $(X, d)$ is complete then $(P_b, cl(X), H)$ is complete too. Also, the following properties are true (see [3]).

**Lemma 1.1.** Let $(X, d)$ be a metric space, $A, B \in P(X)$ and $\eta > 0$. If

i) for each $a \in A$ there is $b \in B$ such that $d(a, b) \leq \eta$ and

ii) for each $b \in B$ there is $a \in A$ such that $d(a, b) \leq \eta$.

Then $H(A, B) \leq \eta$.

**Lemma 1.2.** Let $(X, d)$ be a metric space, $A, B \in P(X)$ and $q \in \mathbb{R}$, $q > 1$. Then for every $a \in A$ there exists $b \in B$ such that $d(a, b) \leq qH(A, B)$.

Let $T : X \to P(X)$ be a multivalued operator. Then $x^* \in X$ is a fixed point for $T$ if $x^* \in T(x^*)$. The set of all fixed points will be denoted by $F_T$. If $x^* \in X$ has the property $\{x^*\} = T(x^*)$, then $x^*$ is said to be a strict fixed point for $T$ and the
symbol \((SF)_T\) denotes the strict fixed points set of \(T\). The multivalued operator \(T\) is surjective if and only if \(T(X) := \bigcup_{x \in X} T(x) = X\).

2 Main results

The first main result of the paper is the following existence theorems for a class of expansive multivalued operator.

**Theorem 2.1.** Let \((X, d)\) be a complete metric space and \(T : X \rightarrow P_{cl}(X)\) be a surjective multivalued operator. If there exist \(a, b, c \in \mathbb{R}^+\) \((b < 1 \text{ and } a + b + c > 1)\) such that

\[
d(y_1, y_2) \geq ad(x_1, x_2) + bd(x_1, y_1) + cd(x_2, y_2), \text{ for each } x_i \in X
\]

and each \(y_i \in T(x_i), \ i \in \{1, 2\}, \) with \(y_1 \neq y_2\).

Then \(F_T \neq \emptyset\).

Moreover, if \(a > 1\) then \(F_T = \{x^*\}\).

**Proof.** Let \(x_0 \in X\) be arbitrarily. Because \(T\) is surjective we can find \(x_1 \in X\) such that \(x_0 \in T(x_1)\). Using the same argument, we obtain a sequence \((x_n)_{n \in \mathbb{N}}\) such that \(x_{n-1} \in T(x_n)\) for each \(n \in \mathbb{N}, \ n \geq 1\).

If there exists \(m \in \mathbb{N}, \ m \geq 1\) such that \(x_m = x_{m-1}\) then \(x_m \in F_T\) and the proof is complete. Let us suppose now that \(x_n \neq x_{n-1}, \) for each \(n \in \mathbb{N}, \ n \geq 1\).

From (1) we deduce:

\[
d(x_{n-1}, x_n) \geq ad(x_n, x_{n+1}) + bd(x_n, x_{n-1}) + cd(x_{n+1}, x_n), \text{ for } n \geq 1
\]

and hence

\[
d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n), \text{ where } k = \frac{1 - b}{a + c} < 1.
\]

Obviously, we get that

\[
d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)
\]

and after a simple computation we obtain

\[
d(x_n, x_{n+m}) \leq \frac{k^n}{1 - k} d(x_0, x_1).
\]

It follows that \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence and hence convergent in the complete metric space \((X, d)\). Let \(x^* = \lim_{n \to \infty} x_n\). We will prove that \(x^* \in F_T\). For this purpose, let us consider an element \(y^* \in T^{-1}(x^*)\). We have successively:

\[
d(x_n, x^*) \geq ad(x_{n+1}, y^*) + bd(x_{n+1}, x_n) + cd(y^*, x^*), \text{ for } n \geq 1.
\]

Taking \(n \to \infty\) we obtain \(0 \geq (a + c)d(y^*, x^*)\) and hence \(x^* = y^*\), proving that \(x^* \in T(x^*)\).
For the uniqueness of the fixed point, let us suppose, by contradiction, that there exists \( x_1^* \) and \( x_2^* \) two distinct fixed points for \( T \). Then
\[
d(x_1^*, x_2^*) \geq ad(x_1^*, x_2^*) + bd(x_1^*, x_1^*) + cd(x_2^*, x_2^*) = ad(x_1^*, x_2^*).
\]

Because \( a > 1 \) we get the desired contradiction. Hence \( F_T = \{ x^* \} \). \(\Box\)

**Example.** Let \( X = [0, 4] \) and \( d : X \times X \to \mathbb{R} \) the metric given by the following formula:
\[
d(t_1, t_2) = \begin{cases} 
1, & \text{if } (t_1, t_2) \in [0, 2) \times [0, 2) \cup [2, 4) \times [0, 1) \cup [0, 1) \times [2, 4] \\
\frac{3}{2}, & \text{if } (t_1, t_2) \in [1, 2) \times [2, 4) \cup [2, 4] \times [1, 2) \\
2, & \text{if } (t_1, t_2) \in [2, 4] \times [2, 4], (t_1 \neq t_2) \\
0, & \text{if } t_1 = t_2
\end{cases}
\]

We consider now the multivalued operator \( T : [0, 4] \to P([0, 4]) \) given by:
\[
T(x) = \begin{cases} 
[3, 4], & \text{if } x \in [0, 1) \\
[2, 3], & \text{if } x \in [1, 2) \\
\left\{ \frac{3}{2} \right\}, & \text{if } x \in [2, 4]
\end{cases}
\]

Then \( T \) satisfy the contractive condition from Theorem 2.1. (with \( a = \frac{1}{2}, b = \frac{1}{4} \) and \( c = \frac{1}{2} \)) and the fixed points set \( F_T = \left\{ \frac{3}{2} \right\} \).

Next, we shall discuss data dependence problem for the set of all fixed points of such multivalued expansive-type operators.

The second main result is:

**Theorem 2.2.** Let \((X,d)\) be a complete metric space and \( T_1, T_2 : X \to P_{b,cl}(X) \) two surjective multivalued operators. We suppose:

- i) there exist \( a_i, b_i, c_i \in \mathbb{R}_+ \), with \( b_i < 1 \) and \( c_i > 1 \) such that
  \[
d(y_1, y_2) \geq a_i d(x_1, x_2) + b_i d(x_1, y_1) + c_i d(x_2, y_2),
\]
  for each \( x_1, x_2 \in X \) and \( (y_1, y_2) \in T_i(x_1) \times T_i(x_2), i \in \{1, 2\} \) with \( y_1 \neq y_2 \).

- ii) there exists \( \eta > 0 \) such that \( \delta(T_i^{-1}(y), T_i^{-1}(y)) \leq \eta \), for each \( y \in X \).

Then

- a) \( F_{T_i} \in P_{cl}(X) \), for \( i \in \{1, 2\} \)

- b) \( H(F_{T_1}, F_{T_2}) \leq \frac{\eta}{1 - \max\{k_1, k_2\}} \), where \( k_i = \frac{1 - b_i}{a_i + c_i} \) for \( i \in \{1, 2\} \).

**Proof.** a) From Theorem 2.1 we get \( F_{T_i} \in P(X) \), for \( i \in \{1, 2\} \). We shall prove that \( F_T \) is closed, where \( T \) is \( T_1 \) or \( T_2 \). Let \((x_n)_{n \in \mathbb{N}} \subseteq F_T \) such that \( \lim_{n \to \infty} x_n = x^* \). We suppose, by contradiction, that \( x^* \notin F_T \), i.e. \( x^* \notin T(x^*) \). Then, for \( x_n \in T(x_n) \) and every \( y \in T(x^*) \), we have:
\[
d(x_n, y) \geq ad(x_n, x^*) + bd(x_n, x_n) + cd(x^*, y).
\]
Taking now \( \inf_{y \in T(x^*)} \) in the previous relation, we get:

\[ D(x_n, T(x^*)) \geq a d(x_n, x^*) + c D(x^*, T(x^*)). \]

When \( n \) tends to infinite we obtain \((c - 1) D(x^*, T(x^*)) \leq 0\) and hence \( D(x^*, T(x^*)) = 0 \). So \( x^* \in T(x^*) \), that is a contradiction.

b) For the second part, let us consider any \( x_0 \in F_{T_1} \), i.e. \( x_0 \in T_1(x_0) \). Obviously \( x_0 \in T_1^{-1}(x_0) \). Let us observe that for each \( x \in T_2^{-1}(x_0) \) we have

\[ d(x_0, x) \leq \delta(T_1^{-1}(x_0), T_2^{-1}(x_0)) \leq \eta. \]

On the other hand, from the surjectivity of \( T_2 \) we can deduce that there exists \( x_1 \in X \) such that \( x_0 \in T_2(x_1) \) or equivalently \( x_1 \in T_2^{-1}(x_0) \). Obviously, \( d(x_0, x_1) \leq \eta \).

Using the same construction as in the proof of Theorem 2.1 we obtain the sequence \( (x_n)_{n \in \mathbb{N}} \) having the properties:

\( \alpha \) \( x_{n-1} \in T_2(x_n) \); \( n \in \mathbb{N} \), \( n \geq 1 \)

\( \beta \) \( d(x_n, x_{n+m}) \leq \frac{k^2}{1 - k_2^2} d(x_0, x_1) \); \( n \in \mathbb{N} \), \( m \in \mathbb{N} \), \( m \geq 1 \).

As before, the sequence \( (x_n)_{n \in \mathbb{N}} \) is convergent in \( X \) and its limit \( x^* \) is a fixed point for \( T_2 \). For \( m \to \infty \) the relation \( (\beta) \) becomes:

\[ d(x_n, x^*) \leq \frac{k^2}{1 - k_2} d(x_0, x_1) \leq \frac{k^2}{1 - k_2} \eta, \text{ for } n \in \mathbb{N}. \]

If we consider \( n = 0 \) we get

\[ d(x_0, x^*) \leq \frac{\eta}{1 - k_2}. \] (2)

Let us consider now a fixed point \( y_0 \in F_{T_2} \). Following the same method we obtain that there exist \( y^* \in F_{T_1} \) such that

\[ d(y_0, y^*) \leq \frac{\eta}{1 - k_1}. \] (3)

From (2) and (3), by using Lemma 1.1, the conclusion follows. \( \square \)

If we consider \( b = c = 0 \) then Theorem 2.1 becomes:

**Corollary 2.3.** (the dual form of the Avramescu-Markin-Nadler theorem)

Let \( (X, d) \) be a complete metric space and \( T : X \to P_{b,cl}(X) \) be a surjective multivalued operator. If there exists \( a \in \mathbb{R} \), \( a > 1 \) such that:

\[ d(y_1, y_2) \geq ad(x_1, x_2), \text{ for each } x_i \in X \text{ and each } y_i \in T(x_i), \text{ for } i \in \{1, 2\} \text{ and } y_1 \neq y_2, \]

then \( F_T = \{x^*\} \).

**Remark 2.4.** If \( T \) is a singlevalued operator, then we get Theorem 3 in [1] (the dual form of the Banach contraction principle).

Another result of this type is:

**Theorem 2.5.** Let \( (X, d) \) be a complete metric space and \( T_1, T_2 : X \to P_{b,cl}(X) \) be two surjective multivalued operators.
We suppose that:
i) there exist $k_1, k_2 \in \mathbb{R}$, $k_1 > 1$, $k_2 > 1$ such that
\[ d^2(y_1, y_2) \geq k_1 \min\{d^2(x_1, y_1), d^2(x_2, y_2), d(x_1, y_1)d(x_1, x_2), d(x_2, y_2)d(x_1, x_2)\}, \quad (4) \]

for each $x_1, x_2 \in X$ and each $(y_1, y_2) \in T_i(x_1) \times T_i(x_2)$, for $i \in \{1, 2\}$.

ii) there exists $\eta > 0$ such that for each $y \in X$
\[ \delta(T_i^{-1}(y), T_i^{-1}(y)) \leq \eta. \]

Then:
\begin{enumerate}
  \item \( F_{T_i} \in P_{cl}(X) \)
  \item \( H(F_{T_1}, F_{T_2}) \leq \frac{1}{1 - \max\{k_1^{-1}, k_2^{-1}\} - \eta}. \)
\end{enumerate}

**Proof.**

a) Using the surjectivity of $T_i = T$, we can construct a sequence \((x_n)_{n \in \mathbb{N}}\) such that $x_n \neq x_{n-1}$ and $x_{n-1} \in T(x_n)$, for $n \in \mathbb{N}$, $n \geq 1$.

From (4) we have:
\[
\begin{align*}
  d^2(x_{n-1}, x_n) &\geq k \min\{d^2(x_n, x_{n-1}), d^2(x_{n+1}, x_n), \\
  &\quad d(x_n, x_{n-1})d(x_n, x_{n+1}), d(x_{n+1}, x_n)d(x_n, x_{n-1})\} = \\
  &= k \min\{d^2(x_{n+1}, x_n), d(x_n, x_{n-1})d(x_n, x_{n+1})\} = \\
  &= kd(x_n, x_{n+1}) \min\{d(x_{n+1}, x_n), d(x_n, x_{n-1})\}.
\end{align*}
\]

The following alternative is now possible:

I. $d^2(x_{n-1}, x_n) \geq kd^2(x_{n+1}, x_n)$ and hence
\[
  d(x_n, x_{n+1}) \leq \frac{1}{\sqrt{k}}d(x_{n-1}, x_n),
\]

for $n \in \mathbb{N}$, $n \geq 1$.

II. $d^2(x_{n-1}, x_n) \geq kd(x_n, x_{n+1})d(x_n, x_{n-1})$ that means
\[
  d(x_n, x_{n+1}) \leq \frac{1}{k}d(x_{n-1}, x_n) \leq \frac{1}{\sqrt{k}}d(x_{n-1}, x_n),
\]

for $n \in \mathbb{N}$, $n \geq 1$.

From the both cases, it results:
\[
  d(x_n, x_{n+1}) \leq \left(\frac{1}{\sqrt{k}}\right)^n d(x_0, x_1).
\]

Obviously, \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence and hence it is convergent.

Let us denote $x^* = \lim_{n \to \infty} x_n$. For $x^* \in X$, we consider $y^* \in T^{-1}(x^*)$ and from (4) we have
\[
  d^2(x_n, x^*) \geq k \min\{d^2(x_{n+1}, x_n), d^2(y^*, x^*), d(x_{n+1}, x_n)d(x_{n+1}, y^*), d(y^*, x^*)d(x_{n+1}, y^*)\}.
\]
For \( n \to \infty \), we conclude \( 0 \geq kd(x^*, y^*) \) and so \( x^* = y^* \in T^{-1}(x^*) \), proving that \( x^* \in F_T \).

b) For the second part, the proof goes similar with part b) in Theorem 2.2. □

**Remark 2.6.** If \( T_i \ (i \in \{1, 2\}) \) are singlevalued operator, then from Theorem 2.5 we get also Theorem 1 in Popa [2].

**Remark 2.7.** For other fixed point and date dependence theorems see also Rus-Petrușel-Săntămărian [5].

**Remark 2.8.** It is an open question to prove some strict fixed point results for such expansive-type multifunctions.

**References**


