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FIXED POINT THEORY

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Contents

Introduction

One of the most dynamic area of research of the last 60 years, with a lot of applications in various fields of pure and applied mathematics, as well as, in physical, economic or life sciences, is without doubt **the fixed point theory**. Not only solutions of several classes of equations or inclusions, but also equilibrium states of an economy, optimization processus solution, fractals, closed orbits in a system of mutually gravitating bodies, etc. are fixed points of an appropriate operator.

The dynamic of this topic is reflected, at least, by the following arguments:

★ **Over 120 books (monographs, lecture notes, proceedings) on fixed point theory and its applications:**

F.E. Browder (1948), M.A. Krasnoselskii (1962), F.F. Bonsal (1962), J. Cronin (1964), T. van der Walt (1963), J. Reinermann (1970), R.F. Brown (1971), I.A. Rus (1971), V.I. Istrăţescu (1973), I.A. Rus (1973), M. Hegedüs (1973), H. Amann (1974), S.P. Singh (1974), D.R. Smart (1974), K. Deimling (1974), M. Hegedüs (1974), M.A. Krasnoselskii and P. Zabrejko (1975), M. van de Vel (1975), D. Fromholzer et al. (1975), F.E. Browder (1976), B.C. Eaves (1976), L. Górniewicz (1976), A.A. Ivanov (1976), S. Swaminatham (1976), M.J. Todd (1976), J.W. de Bakker (1976), T. Riedrich (1976), R. Gaines and J. Mawhin (1977), M.L. Balinski and R.W. Cottle (1978), C. Eisenack and C. Fenske (1978), O. Hadžić (1978), M. Hegedüs (1978), N. Lloyd (1978), H.-O. Peitgen and H.-O. Walther (1979), I.A. Rus (1979), I.A. Rus (1979), J. Banas and K. Goebel (1980), S. Czerwik (1980), W. Forster (1980), A.J.J. Talman (1980), G. van der Laan (1980), St.M. Robinson (1980), E. Fadell and G. Fournier (1981), V.I. Istrăţescu (1981), J. Dugundji and A. Granas (1982),

R. Wegrzyk (1982), B.J. Jiang (1983), I.A. Rus (1983), R.C. Sine (1983), D. L. Goncalves and J.C. de Souza Kiihl (1983), K. Goebel and S. Reich, M. Lösch (1984), O. Hadžić (1984), K.C. Border (1985), J. Mawhin (1985), R.D. Nussbaum (1985), J. Bewersdorff (1985), E. Zeidler (1985), M.F. Iwano (1985), F.E. Browder (1986), F. Robert (1986), A. Dold (1986), K. Schilling (1986), M.R. Tasković (1986), R. Kuczumow (1987), B. Blümel (1987), R F. Brown (1988), H. Ulrich (1988), D. Guo and V. Lakshmikantham (1989), T.-H. Kiang (1989), B. J. Jiang (1989), Yu.A. Shashkin (1989), A.G. Aksoy and M.A. Khamsi (1990), K. Goebel and W.A. Kirk (1990), M.A. Théra and J.-B. Baillon (1991), G. Sommaruga-Rosolemos (1991), K.K. Tan (1992), R.R. Akhmerov, M.I. Kamenskii, A.S. Potapov, A. E. Rodkina and B. N. Sadovskii (1992), L. Schwartz (1994), J. Jaworowski, W.A. Kirk and S. Park (1995), J. Oprea (1995), O. Hadžić (1995), W.V. Petryshyn (1995), J.J. Duistermaat (1996), V.F. Démyanov (1996), T. Dominguez Benavides (1996), V. Berinde (1997), S.P. Singh, B. Watson and P. Srivastava (1997), J.M. Ayerbe Toledano, T. Dominguez Benavides and G. López Acedo (1997), F.H. Clarke, Yu.S. Ledyaev and R.J. Stern (1997), V. Radu, C. Grecu, A. Pogan, L. Radu and T. Venţe, (1998), Z. Yang (1999), N. Negoescu (1999), L. Górniewicz (1999), R.P. Agarwal and D. O'Regan (2000), Y.J. Cho (2000), D. Butnariu and A.N. Iusem (2000), W. Takahashi (2000), M.A. Khamsi and W.A. Kirk (2001), R.P. Agarwal, M. Meehan and D. O'Regan (2001), D. O'Regan and R. Precup (2001), W.A. Kirk and B. Sims (2001), O. Hadžić and E. Pap (2001), A. Buică (2001), I.A. Rus (2001), K. Goebel (2002), A. Petruşel (2002), M.A. Şerban (2002), A. Muntean (2002), A. Bege (2002), A. Petruşel (2002), V. Berinde (2002), A. Petruşel, G. Petruşel and I.A. Rus (2002), J. Andres and L. Górniewicz (2003), A. Granas and J. Dugundji (2003), S.B. Nadler jr. (2003), Z. Denkowski, S. Migórski and N.S. Papageorgiou, A. Fryszkowski (2004), D. Guo, Y.J. Cho and J. Zhu (2004), D. Miklaszewski (2005), R.F. Brown, M. Furi, L. Górniewicz and B. Jiang (Eds.) (2005), S. Reich and D. Shoikhet (2005), M. Balaj (2006), T.A. Burton (2006), L. Górniewicz (2006), C. Vladimirescu and C. Avramescu (2006), L. Gasiński and N.S. Papageorgiou (2006), I.A. Rus (2006), G. Moţ, A. Petruşel and G. Petruşel (2007), R. Skiba (2007), V. Berinde (2007), T.A. Burton (2008), E.U. Tarafdar and M.S.R.

Chowdhury (2008), V.G. Angelov (2008).

★ **Over 12,000 papers on fixed point theory from 1940 until now.**

★ **Almost 4,000 papers on fixed point theory only between 2000-2008.**

★ Except these theoretical books and papers, there are **more than 2,000 books, monographs and proceedings and over 40,000 papers**, which use the abstract theory of fixed point for various problems of pure, applied and computational mathematics.

★ The field of the fixed point theory is today vast and open to lots of techniques and ideas. A large number of applications are also developed in various directions.

Let us present some topics of the fixed point theory:

A. Topics in terms of structured sets:

- ⧫ Fixed Point Theory in Sets
- ⧫ Fixed Point Theory in Ordered Sets
- ⧫ Fixed Point Theory in Groups
- ⧫ Fixed Point Theory in Rings
- ⧫ Fixed Point Theory in Algebras
- ⧫ Fixed Point Theory in Universal Algebras
- ⧫ Fixed Point Theory in Categories
- ⧫ Fixed Point Theory in Metric Spaces
- ⧫ Fixed Point Theory in Generalized Metric Spaces
- ⧫ Fixed Point Theory in Geodesic Spaces
- ⧫ Fixed Point Theory in Gauge Spaces
- ⧫ Fixed Point Theory in Hilbert Spaces
- ⧫ Fixed Point Theory in Banach Spaces
- ⧫ Fixed Point Theory in Banach Algebras
- ⧫ Fixed Point Theory in Locally Convex Spaces
- ⧫ Fixed Point Theory in Linear Topological Spaces
- ⧫ Fixed Point Theory in Topological Spaces
- ⧫ Fixed Point Theory in Algebraic Topology

♦ Fixed Point Theory on Manifolds

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B. Topics in terms of some classes of operators:

♦ Fixed Point Theory for Increasing Operators
♦ Fixed Point Theory for Decreasing Operators
♦ Fixed Point Theory for Progressive Operators
♦ Fixed Point Theory for Continuous Operators
♦ Fixed Point Theory for Operators with Closed Graph
♦ Fixed Point Theory for Open Operators
♦ Fixed Point Theory for Closed Operators
♦ Fixed Point Theory for Differentiable Operators
♦ Fixed Point Theory for Holomorphic Operators
♦ Fixed Point Theory for Generalized Contractions
♦ Fixed Point Theory for Nonexpansive Operators
♦ Fixed Point Theory for Asymptotically Nonexpansive Operators
♦ Fixed Point Theory for Rotative Operators
♦ Fixed Point Theory for Isometries
♦ Fixed Point Theory for Delating Operators
♦ Fixed Point Theory for Accretive Operators
♦ Fixed Point Theory for Pseudocontractive Operators
♦ Fixed Point Theory for Monotone Operators
♦ Fixed Point Theory for Acyclic Operators
♦ Fixed Point Theory for Symplectic Operators

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C. Topics in deep connection to fixed point theory:

♦ Coincidence Point Theory
♦ Zero Point Theory
♦ Surjectivity Theory
♦ Spectral Theory
♦ Bifurcation Theory
♦ Topological Degree Theory
♦ Dynamical System Theory

- ⧫ Invariant Subsets
- ⧫ Convexity Structures
- ⧫ Geometry of the Banach Space
- ⧫ Measure of Noncompactness
- ⧫ Measure of Nonconvexity
- ⧫ Complexity of Computation
- ⧫ Ramsey Theory
- ⧫ Extremal Element Theory

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D. Topics generated by some classical results:

- ⧫ Borsuk-Ulam Type Theorems
- ⧫ Tarski-Kantorovich Type Theorems
- ⧫ Schauder-Tychonoff Type Theorems
- ⧫ Darbo Type Theorems
- ⧫ Sadovskii Type Theorems
- ⧫ Caristi-Kirk Type Theorems
- ⧫ Caristi-Browder Type Theorems
- ⧫ Browder-Ghöde-Kirk Type Theorems
- ⧫ Browder Type Theorems
- ⧫ Frum-Ketkov Type Theorems
- ⧫ Krasnoselskii Type Theorems
- ⧫ Leray-Schauder Type Theorems
- ⧫ Granas Type Theorems
- ⧫ Knaster-Kuratowski-Mazurkiewicz Type Theorems
- ⧫ Ky Fan Type Lemmas
- ⧫ Markov-Kakutani Type Theorems
- ⧫ Lefschetz Type Theorems
- ⧫ Nielsen Type Theorems
- ⧫ Poincaré-Birkhoff Type Theorems
- ⧫ Rabinowitz-Nussbaum Type Theorems

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E. Other topics:

- ⧫ Periodic Point Theory

- ♦ Almost Fixed Point Theory
- ♦ Common Fixed Point Theory
- ♦ Fixed Point Algorithms
- ♦ Mathematics of Fractals

F. Applications of the fixed point theory to:

- ♦ Equations in $\mathbb{R}^n$
- ♦ Equations in $\mathbb{C}^n$
- ♦ Matrix Equations
- ♦ Functional Equations
- ♦ Ordinary Differential Equations
- ♦ Partial Differential Equations
- ♦ Integral Equations
- ♦ Functional-Differential Equations
- ♦ Functional-Partial Differential Equations
- ♦ Functional-Integral Equations
- ♦ Differential Inclusions
- ♦ Integral Inclusions
- ♦ Mathematical Economics
- ♦ Informatics

G. The topics of the **Handbook of Metric Fixed Point Theory (W.A. Kirk and B. Sims - Eds.)** R[1] are the following:

- ♦ Contraction Operators and Extensions
- ♦ Fixed Point Free Operators
- ♦ Nonexpansive Operators
- ♦ Geometric Theory of Banach Spaces and Fixed Points
- ♦ Fixed Point Theory in Terms of Measure of Noncompactness
- ♦ Fixed Point Theory in l^1 and c_0
- ♦ Fixed Point Theory of Nonself Nonexpansive Operators
- ♦ Fixed Point Theory of Rotative Operators
- ♦ Fixed Point Theory in Banach Function Lattices
- ♦ Fixed Point Theory in Hyperconvex Spaces
- ♦ Fixed Point Theory of Holomorphic Operators
- ♦ Fixed Points and Semigroups of Nonlinear Operators

- ⧫ Generic Aspects of Metric Fixed Point Theory
- ⧫ Minimal Displacement Problem
- ⧫ Retractions and Fixed Points
- ⧫ Order-Theoretic Aspects of Metric Fixed Point Theory
- ⧫ Fixed Point Theory of Multivalued Operators

H. The topics of the **Handbook of Topological Fixed Point Theory (R.F. Brown, M. Furi, L. Górniewicz and B. Jiang - Eds.)** R[1] are the following:

⧫ I. Homological Methods in Fixed Point Theory (coincidence theory, Lefschetz fixed point theorem, Nielsen classes, homotopy minimal periods, periodic points and braid theory, fixed point theory of multivalued weighted operators, fixed point theory for homogeneous spaces)

⧫ II. Equivariant Fixed Point Theory (equivariant fixed point, equivariant degree theory, bifurcation of solutions of $SO(2)$-symmetric nonlinear problems with variational structure)

⧫ III. Nielsen Theory (Nielsen theory, applications of Nielsen theory, algebraic and fibre techniques for calculating the Nielsen number, Wecken theorem, relative Nielsen theory)

⧫ IV. Applications (applications to differential equations and inclusions, applications to multivalued dynamical systems, Poincaré translation operator on differentiable manifolds, Wazewski method)

I. The topics of the book **Principles and Applications of Fixed Point Theory (Ioan A. Rus)** B[73] are the following:

⧫ I. Fixed Point Theory

- The fixed point set
- Tarski's fixed point theorem
- Bourbaki's fixed point theorem
- Contraction principle
- Perov's fixed point theorem
- Luxemburg-Jung's fixed point theorem
- Brouwer's fixed point theorem
- Schauder's fixed point theorem

- Tychonoff 's fixed point theorem
- Browder-Ghöde-Kirk's fixed point theorem
- Fixed point theorems for multivalued operators
- Problems and results in fixed point theory (the method of successive approximations, measures of noncompactness, topological degree, the fixed point set, sequences of operators and fixed points, data dependence of fixed points, operators on cartesian product, fixed point theorems in $\mathbb{R}^n$, fixed point theorems in $\mathbb{C}^n$, common fixed point theory, coincidence point theory, almost fixed points, fixed point theory in categories).

♦ II. Applications of the Fixed Point Theory

- Equations in $\mathbb{R}^n$
- Equations in $s(\mathbb{R})$
- Functional Equations
- Integral Equations
- Functional-Differential Equations
- Partial Differential Equations
- Equations in Applied Mathematics

J. There also exists a project of **M.S. Khamsi: Fixed Point Theory and its Applications on the Web.** The topics considered there are:

♦ The Contraction Principle

♦ Nonexpansive Mappings in Hilbert Spaces

♦ Nonexpansive Mappings in Banach Spaces

♦ Orbit, Omega-set

♦ Ergodic Theorems

♦ Approximation Techniques

♦ Non-classical Banach Spaces (Orlicz spaces, James' spaces, Tsirelson' spaces)

♦ Metric Spaces

♦ Measure of Non-compactness

♦ Caristi's Fixed Point Theorem

♦ Bifurcation Theory

♦ Multivalued Mappings

♦ Generalized Structures (Ordered Set, Generalized Metric Spaces, Modular Spaces)

⧫ Topological Fixed Point Theory (Brouwer's Theorem, Minimax Theorems, KKM-Maps, Degree Theory, Sperner's Lemma, Discrete Brouwer's Theorem, Leray-Schauder's Fixed Point Theorem, Degree Theory, ANR' Sets, Nielsen Theorems, Lefschetz Fixed Point Theorems, Bifurcation Theory, Complementarity Problems, Renorming Techniques)

★ **Fixed Point Theory-An International Journal on Fixed Point Theory, Computation and Applications** is the first journal entirely devoted to fixed point theory and its applications. Actually, the academic year 1999-2000 marked the 30-th anniversary of the Seminar on Fixed Point Theory Cluj-Napoca. This research seminar started in 1969 at the initiative and under the guidance of Professor Ioan A. Rus from Babeş-Bolyai University of Cluj-Napoca. The yearly publication of the Seminar was *Seminar on Fixed Point Theory, Preprint no. 3.* The journal *Seminar on Fixed Point Theory Cluj-Napoca* (between 2000 and 2002) and *Fixed Point Theory* (since 2003) are continuations of this publication. The Editorial Board of the journal *Fixed Point Theory* is the following: Ioan A. Rus (Editor-in-Chief), Adrian Petruşel (Managing Editor), George Isac, Radu Precup (Editors), Jan Andres, Vasil Angelov, Jürgen Appell, Vasile Berinde, Theodore A. Burton, Dan Butnariu, Constantin Corduneanu, Tomas Dominguez Benavides, Marlène Frigon, Vasile Glăvan, Kazimierz Goebel, Lech Górniewicz, Kiyoshi Iseki, Genaro López Acedo, Enrique Llorens Fuster, William Art Kirk, Valeri Obukhovskii, Donal O'Regan, Viorel Radu, Simeon Reich, Biagio Ricceri, S.P. Singh, Wataru Takahashi, Mihai Turinici, Hong-Kun Xu (Editorial Board). The journal *Fixed Point Theory* publishes important research and expository papers devoted to the theory, computation and applications of the fixed points.

Since then, other three journals on fixed point theory appeared in the mathematics literature:

• **Fixed Point Theory and Applications** (since 2004). The Editorial Board of the journal is composed by: R.P. Agarwal (Editor-in-Chief), Mohamed Amine Khamsi, Thomas Bartsch, Hichem Ben-El-Mechaiekh, Jonathan

M. Borwein, Robert F. Brown, Tomas Dominguez Benavides, Patrick M. Fitzpatrick, Hélène Frankowska, Massimo Furi, Lech Górniewicz, Djairo Guedes de Figueiredo, Evelyn Hart, Jerzy Jezierski, William A. Kirk, V. Lakshmikantham, Anthony To-Ming Lau, Jean Mawhin, Huang Nanjing, Roger D. Nussbaum, Donal O'Regan, Simeon Reich, Billy E. Rhoades, Klaus Schmitt, Brailey Sims, Tomonari Suzuki, Andrzej Szulkin, Wataru Takahashi, J.R.L. Webb, Fabio Zanolin (Associate Editors). The aim of this journal is "to report new fixed point theorems and their applications where the essentiality of the fixed point theorems is highlighted. Fixed point theorems give the conditions under which maps (single or multivalued) have solutions. The theory itself is a beautiful mixture of analysis, topology, and geometry. Over the last 50 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. In particular fixed point techniques have been applied in such diverse fields as biology, chemistry, economics, engineering, game theory, and physics."

• **Journal of Fixed Point Theory and Applications** (since 2007). The Editorial Board of this journal is the following: Andrzej Granas (Editor-in-Chief), Gilles Gauthier (Managing Editor); Section Editors: Michael Crabb (Algebraic and Geometric Topology), Octav Cornea (Symplectic Topology and Global Analysis), Krystyna Kuperberg (Dynamical Systems), Norman Dancer (Nonlinear Analysis), Simeon Reich (Classical Topics), Fon Che Liu (Games, Economics and Computation Theory), Richard S. Palais (Surveys and Research Expository Papers), Alberto Abbondandolo (Short Communications and Open Problems); Editorial Advisory Board: Haim Brezis, Felix Browder, Yvonne Choquet-Bruhat, Albrecht Dold Alexander Ioffe, Anatole Katok, Paul Malliavin, Victor Maslov Isaac Namioka, Paul Rabinowitz, Czeslaw Ryll-Nardzewski, Albert Schwarz Anatoli Skorokhod; Associate Editors: Hichem Ben-El-Mechaiekh, Vieri Benci, Robert Cauty, Kung-Ching Chang, Bernard Cornet, Edward Fadell, John Franks, Marlène Frigon, Kazimierz Geba, Peter Gilkey, Ronald B. Guenther, Charles Horvath, Jacek Jachymski, Jan Jaworowski, Boju Jiang, Sam B. Nadler jr., Roger Nussbaum, Kaoru Ono, Heinz-Otto Peitgen, Grzegorz Rosenberg, Yuli Rudyak, Slawomir Rybicki, Matthias Schwarz, Alexander N. Sharkovsky, Michael Shub, Evgenij

G. Sklyarenko, Gencho Skordev, Heinrich Steinlein, Andrzej Szulkin, Sergei Tabachnikov, Wataru Takahashi, John Toland, Aleksy Tralle, Gerard van der Laan, Victor Zvyagin.

A short description of this journal reads as follows: "This journal publishes high-quality, peer-reviewed research papers in all disciplines in which the use of tools of the fixed point theory plays an essential role. It details new developments in fixed point theory as well as in related topological methods and examines ramifications to symplectic topology, dynamical systems and global analysis. In addition, the Journal of Fixed Point Theory and Applications presents significant applications in nonlinear analysis, mathematical economics and computation theory. It also features contributions to important problems in geometry, fluid dynamics and mathematical physics."

The journal is organized into eight sections:

- Algebraic and Geometric Topology
- Dynamical Systems
- Symplectic Topology and Global Analysis
- Nonlinear Analysis
- Classical Topics
- Games, Economics and Computation Theory
- Surveys and Research Expository Papers
- Short Communications and Open Problems.

• **JP Journal of Fixed Point Theory and Applications** (since 2007). The Editorial Board of this journal is the following: K.K. Azad (Managing Editor), Bashir Ahmad, Tomas Dominguez Benavides, Antonio Carbone, Yeol Je Cho, Liang-Ju Chu, Sompong Dhompongsa, Marlène Frigon, Lech Gorniewicz, Lishan Liu, Jong Seo Park, Simeon Reich, B.E. Rhoades, Biagio Ricceri, Wataru Takahashi, Peter Wong, Hong-Kun Xu, L.C. Zeng (Editors). A short description of the aims of the journal is the following: "The JP Journal of Fixed Point Theory and Applications is a fully refereed international journal, which published original research papers and survey articles in all aspects of Fixed Point Theory and their Applications. Topics in detail to be covered are new developments in fixed point theory as well as in related topological methods: ramifications to symplectic topology, dynamical systems and global analysis, significant applications in nonlinear analysis, mathematical economics and computation theory, contributions to important problems in geometry, fluid dynamics and mathematical physics and other such areas of interest."

The purpose of the monograph is to present the most important results in the field of fixed point theory. Each chapter starts with precursors, guidelines and general references of the topic. Our book is based, to a certain extent, on the authors' former book *Fixed Point Theory 1950-2000: Romanian Contributions*, House of the Book of Science, Cluj-Napoca, 2002.

The References of the book are organized in two sections. First part consists of an exhaustive bibliography of the fixed point theory of Romanian authors, while the second part is a general references list containing:

• basic references of the fixed point theory

as well as,

• papers of Romanian authors which have applied the fixed point theory.

The list of symbols, the index of terms and the author's conclude the book.

Throughout the book, the symbol **B[...]** indicates titles from the **Romanian Bibliography of the Fixed Point Theory**, while **R[...]** refers to titles from the **General References** list.

Finally, we would like to point out that, by this book, our intention was not only **to provide a tool for further research**, but also, to give, in each chapter, a guideline of the field, **to have, at a glance, the entire history of the topic.**

Cluj-Napoca, September 2008 The Authors