INTERLACING THEOREMS FOR THE ZEROS OF ORTHOGONAL POLYNOMIALS FROM DIFFERENT SEQUENCES

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It is well known that if $\{p_n\}_{n=1}^{\infty}$ is a sequence of orthogonal polynomials, the zeros of p_n are real and simple and the zeros $x_{1,n} < x_{2,n} < \ldots < x_{n,n}$ of p_n and $x_{1,n-1} < x_{2,n-1} < \ldots < x_{n-1,n-1}$ of p_{n-1} separate each other as follows

$$x_{1,n} < x_{1,n-1} < x_{2,n} < x_{2,n-1} < \cdots < x_{n-1,n} < x_{n-1,n-1} < x_{n,n}.$$

An important question that arises in the study of such interlacing properties is whether and when the zeros of two polynomials p_n and r_m, $m = n$ or $n-1$, where $\{p_n\}_{n=1}^{\infty}$ and $\{r_m\}_{m=1}^{\infty}$ are different sequences of orthogonal polynomials on the same interval, separate each other. In 1989 R. Askey conjectured that the zeros of Jacobi polynomials $p_n = P_n^{(\alpha,\beta)}$ and $r_n = P_n^{(\alpha+2,\beta)}$ interlace. He also posed the question whether the zeros of $p_n = P_n^{(\alpha,\beta)}$ and $r_n = P_n^{(\gamma,\beta)}$ interlace when $\alpha < \gamma \leq \alpha + 2$.

Recently [?] it was proved that the zeros of $p_n = P_n^{(\alpha,\beta)}$ and $r_n = P_n^{(\gamma,\delta)}$ interlace when $\alpha < \gamma \leq \alpha+2$ and $\beta-2 \leq \delta < \beta$. Furthermore, it was also shown that the zeros of the $p_n = P_n^{(\alpha,\beta)}$ and $r_{n-1} = P_{n-1}^{(\gamma,\delta)}$ interlace when $\alpha < \gamma \leq \alpha + 2$ and $\beta < \delta \leq \beta + 2$.

We aim to do a comprehensive study of the interlacing properties of the zeros of other classes of classical orthogonal polynomials, including Gegenbauer, Laguerre and discrete orthogonal polynomials such as Meixner, Krawtchouk and Hahn polynomials, where the parameters are shifted continuously. The same method also provides mixed recurrence relations and interlacing results for the zeros of Al-Salam-Chihara, continuous q-ultraspherical, q-Meixner-Pollaczek and q-Laguerre polynomials of the same or adjacent degree as one of the parameters is shifted by integer values or continuously within a certain range. Numerical examples are given to illustrate cases where the zeros do not separate each other.

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