MATHEMATICA, 65 (88), N° 1, 2023, pp. 139–149

TWO DIMENSIONAL GAUSSIAN PELL SEQUENCES

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Abstract. In this paper, a new approach is taken toward the generalization of Pell sequences into the complex plane. It is shown that the Pell numbers are generalized to two dimensions. For special entries of this new sequence, some relations with classic Pell sequence are constructed. Binet formula, generating function, explicit closed formula, sum formula for the new two dimensional Gaussian Pell sequence are investigated. The relation with classic Pell-Lucas numbers and two dimensional Gaussian Pell numbers are obtained by using the Binet formula. By matrix algebra, we obtain matrix representations of two dimensional Gaussian Pell sequences.

MSC 2020. 11B37, 11B39, 11B83, 11A07, 11A41, 11A51, 11B50, 11B65, 11B75.

Key words. Pell numbers, Pell-Lucas numbers, Gaussian Pell numbers, generating function.

1. INTRODUCTION AND PRELIMINARIES

Complex numbers $z = a + ib$, $a, b \in \mathbb{Z}$ were investigated by Gauss in 1832 so they are called Gaussian numbers. Horadam introduced complex Fibonacci number called the Gaussian Fibonacci number. Jordan studied two of the complex Fibonacci sequences and extended some relations which are known about the Fibonacci sequences. Berzsenyi [1], denoted a natural way of extension of the Fibonacci numbers into the complex plane and obtained some interesting identities for the classical Fibonacci numbers. Harman [4], Jordan [5] demonstrated an extension of Fibonacci and Lucas numbers into the complex plane. Pethe and Horadam [6] studied generalized Gaussian Fibonacci numbers. Halici and Oz [2] defined Gaussian Pell and Pell-Lucas numbers. Then they generalized numbers to Gaussian Pell polynomials [3]. Soykan [7] studied on summing formulas for generalized Fibonacci and Gaussian generalized Fibonacci numbers. For $n \in \mathbb{Z}$, the classic Pell and Pell-Lucas sequences are defined by respectively

$$\begin{aligned} P_{n+2} &= 2P_{n+1} + P_n, \quad P_0 = 0,\ P_1 = 1 \\ Q_{n+2} &= 2Q_{n+1} + Q_n,\ Q_0 = 2,\ Q_1 = 2. \end{aligned}$$

The author thanks the referee for his helpful comments and suggestions.

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DOI: 10.24193/mathcluj.2023.1.15

The elements of the classic Pell and Pell-Lucas with negative indices are demonstrated by

$$P_{-n} = (-1)^{n+1} P_n, \quad Q_{-n} = (-1)^n Q_n.$$

Halıcı and Oz defined the Gaussian Pell $\{GP_n\}_{n=0}^{\infty}$ sequence by

$$GP_{n+2} = 2GP_{n+1} + GP_n, \ \ GP_0 = \mathrm{i}, \ GP_1 = 1.$$

The Gaussian Pell-Lucas sequence $\{GQ_n\}_{n=0}^{\infty}$ is defined by

$$GQ_{n+2} = 2GQ_{n+1} + GQ_n, \quad GQ_0 = 2 - 2\mathrm{i}, \ GQ_1 = 2 + 2\mathrm{i}.$$

Also, the Gaussian Pell and Pell-Lucas sequences satisfy the following identities:

$$GP_n = P_n + \mathrm{i}P_{n-1},$$

$$GQ_n = Q_n + \mathrm{i}Q_{n-1}.$$

The Gaussian Jacobsthal-Lucas sequence $\{GC_n\}_{n=0}^{\infty}$ is defined by the following recurrence relation

$$GC_n = C_n + \mathrm{i}C_{n-1}.$$

2. TWO DIMENSIONAL GAUSSIAN PELL SEQUENCE

Definition 2.1. Let n, m be any integers. The two dimensional Gaussian Pell sequence is defined by

$$\begin{aligned} G(n+2,m) &= 2G(n+1,m) + G(n,m), \\ G(n,m+2) &= 2G(n,m+1) + G(n,m), \end{aligned}$$

$G(0,0) = 0, \ G(1,0) = 1, \ G(0,1) = \mathrm{i}, \ G(1,1) = 2(1+\mathrm{i}).$

If we use Definition 2.1 for $m = 0$ and $n = 0$ respectively, we get

$$\begin{aligned} G(n+2,0) &= 2G(n+1,0) + G(n,0), \\ G(0,m+2) &= 2G(0,m+1) + G(0,m). \end{aligned}$$

By the initial conditons and the induction method, it is easily obtained that

$$G(n,0) = P_n,$$

$$G(0,m) = \mathrm{i}P_m.$$

Theorem 2.2. *Let n be any integer. Then*

$$G(n,1) = P_n G(1,1) + P_{n-1} G(0,1),$$

$$G(n,1) = 2P_n + \mathrm{i}P_{n+1}.$$

Proof. The induction method is used to prove this theorem. For $m = 1$,

$$G(1,1) = 2P_1(1+\mathrm{i}) + \mathrm{i}P_0.$$

Assume that the claim is true for $m \leq k$. We show that the assertion is satisfied for $m = k+1$

$$\begin{aligned} G(k+1,1) &= 2G(k,1) + G(k-1,1) \\ &= 2[P_kG(1,1) + P_{k-1}G(0,1)] + [P_{k-1}G(1,1) + P_{k-2}G(0,1)] \\ &= G(0,1)\,[2P_{k-1} + P_{k-2}] + G(1,1)\,[P_{k-1} + 2P_k] \\ &= P_kG(0,1) + P_{k+1}G(1,1). \end{aligned}$$

□

THEOREM 2.3. *Let n be any integer. Then*

$$G(1,m) = P_mG(1,1) + P_{m-1}G(1,0),$$

$$G(1,m) = P_{m+1} + 2\mathrm{i}P_m.$$

Proof. We use the induction method to prove this theorem. For $m = 1$,

$$G(1,1) = P_1(1+\mathrm{i}) + P_0.$$

Suppose that the claim is true for $m \leq k$, so

$$G(1,k) = P_kG(1,1) + P_{k-1}G(1,0).$$

We show that the claim is true for $m = k+1$.

$$\begin{aligned} G(1,k+1) &= 2G(1,k) + G(1,k-1) \\ &= 2[P_k2(1+\mathrm{i}) + P_{k-1}] + [P_{k-1}2(1+\mathrm{i}) + P_{k-2}] \\ &= 2(1+\mathrm{i})P_{k+1} + P_k. \end{aligned}$$

□

THEOREM 2.4. *Let n, m be any integers. Then*

$$G(n,m) = P_mG(n,1) + P_{m-1}G(n,0).$$

Proof. We use the induction method to prove this theorem. For $m = 1$,

$$G(n,1) = P_1G(n,1) + P_0G(n,0).$$

Suppose that the claim is true for $k \leq m$, so

$$G(n,k) = P_kG(n,1) + P_{k-1}G(n,0).$$

It is showed that the claim is satisfied for $m = k+1$.

$$\begin{aligned} G(n,k+1) &= 2G(n,k) + G(n,k-1) \\ &= 2[P_kG(n,1) + P_{k-1}G(n,0)] + [P_{k-1}G(n,1) + P_{k-2}G(n,0)] \\ &= P_{k+1}G(n,1) + P_kG(n,0). \end{aligned}$$

□

Corollary 2.5. *Let n, m be any integers. Then*

$$G(n,m) = P_{m+1}P_n + \mathrm{i}P_mP_{n+1}. \tag{1}$$

Proof. By Theorems 2.3, 2.7, 2.2, we have

$$\begin{aligned} G(n,m) &= P_mG(n,1) + P_{m-1}G(n,0) \\ &= P_m(2P_n + \mathrm{i}P_{n+1}) + P_{m-1}P_n \\ &= P_{m+1}P_n + \mathrm{i}P_mP_{n+1}. \end{aligned}$$

□

Corollary 2.6.

$$G(n,m) + G(m,n) = \frac{2Q_{m+n+1} - 2(-1)^mQ_{n-m}}{8}(1+\mathrm{i}).$$

Proof. By the property of Pell numbers, we can write the folowing identity

$$P_{m+1}P_n + P_mP_{n+1} = \frac{2Q_{m+n+1} - 2(-1)^mQ_{n-m}}{8}.$$

By adding both sides of the following equalities, the proof is easily obtained:

$$\begin{aligned} G(n,m) &= P_{m+1}P_n + \mathrm{i}P_mP_{n+1}, \\ G(m,n) &= P_{n+1}P_m + \mathrm{i}P_nP_{m+1}. \end{aligned}$$

It is seen that the commutative property is not satisfied by $G(n,m)$. □

Theorem 2.7. *The elements of the two dimensional Gaussian Pell sequence with negative indices satisfy the following equality*

$$G(-n,-m) = (-1)^{n+m+1}\, G(m-1, n-1).$$

Proof. By the equality (1),

$$\begin{aligned} G(-n,-m) &= P_{-m+1}P_{-n} + \mathrm{i}P_{-m}P_{-n+1} \\ &= (-1)^{n+1}\, P_n\, (-1)^m\, P_{m-1} + \mathrm{i}\, (-1)^n\, P_{n-1}\, (-1)^{m+1}\, P_m \\ &= (-1)^{m+n+1}\, (P_nP_{m-1} + \mathrm{i}P_{n-1}P_m). \end{aligned}$$

□

Corollary 2.8.

$$\begin{aligned} G(n+1,m+1) &= P_{n+1}P_{m+1}2(1+\mathrm{i}) + G(m,n), \\ G(n+2,m+2) &= (1+\mathrm{i})(3P_{n+1}P_{m+1} + 4P_nP_m) + 6G(n,m) + 2G(m,n). \end{aligned}$$

Proof. By the equality (1),

$$\begin{aligned} G(n+1,m+1) &= P_{n+1}P_{m+2} + \mathrm{i}P_{n+2}P_{m+1} \\ &= P_{n+1}(2P_{m+1} + P_m) + \mathrm{i}P_{m+1}(2P_{n+1} + P_n) \\ &= 2P_{n+1}P_{m+1} + P_{n+1}P_m \\ &\quad +\mathrm{i}(2P_{n+1}P_{m+1} + P_nP_{m+1}), \end{aligned}$$

$$\begin{aligned} G(n+2,m+2) &= P_{n+2}P_{m+3}+\mathrm{i}P_{n+3}P_{m+2} \\ &= (2P_{n+1}+P_n)(2P_{m+2}+P_{m+1}) \\ &\quad +\mathrm{i}(2P_{m+1}+P_m)(2P_{n+2}+P_{n+1}) \\ &= (2P_{n+1}+P_n)(5P_{m+1}+P_m) \\ &\quad +\mathrm{i}(2P_{m+1}+P_m)(5P_{n+1}+P_n) \\ &= 10(1+\mathrm{i}P_{m+1}P_{n+1}+2(1+i)P_nP_m \\ &\quad +5G(n,m)+4G(m,n). \end{aligned}$$

□

Corollary 2.9. *By Definition 2.1, it is obtained that*

$$\begin{aligned} G(n,m) &= 2G(n-1,m)+G(n-2,m) \\ &= 4G(n-1,m-1)+2G(n-1,m-2) \\ &\quad +2G(n-2,m-1)+G(n-2,m-2). \end{aligned}$$

Theorem 2.10 (Generating Function). *The generating function for the two dimensional Gaussian Pell sequence is*

$$G(x,y)=\sum_{m,n=0}^{\infty}G(n,m)x^ny^m=\frac{x+iy+(2+2\mathrm{i})xy-\frac{2\mathrm{i}xy^3}{1-2y-y^2}-\frac{2yx^3}{1-2x-x^2}}{1-4xy-2x^2y-2xy^2-x^2y^2}.$$

Proof. By Corollary 2.8 and $\sum_{n=0}^{\infty}P_nx^n=\frac{x}{1-2x-x^2}$, we have

$$G(x,y)=G(0,0)+G(1,0)x+G(0,1)y+G(1,1)xy+\sum_{m,n=2}^{\infty}G(n,m)x^ny^m$$

$$-4xyG(x,y)=-4xyG(0,0)-4\sum_{m,n=2}^{\infty}G(n-1,m-1)x^ny^m$$

$$-x^2y^2G(x,y)=-\sum_{m,n=2}^{\infty}G(n-2,m-2)x^ny^m,$$

$$\begin{aligned} -2x^2yG(x,y) &= -2\sum_{n=2}^{\infty}G(n-2,0)x^ny-2\sum_{m,n=2}^{\infty}G(n-2,m-1)x^ny^m \\ &= -2y\sum_{n=2}^{\infty}P_{n-2}x^n-2\sum_{m,n=2}^{\infty}G(n-2,m-1)x^ny^m \\ &= \frac{-2yx^3}{1-2x-x^2}-2\sum_{m,n=2}^{\infty}G(n-2,m-1)x^ny^m, \end{aligned}$$

and

$$\begin{aligned}
-2xy^2G(x,y) &= -2\sum_{m=2}^{\infty}G(0,m-2)xy^m - 2\sum_{m,n=2}^{\infty}G(n-1,m-2)x^ny^m \\
&= -2\mathrm{i}x\sum_{m=2}^{\infty}P_{m-2}y^m - \sum_{m,n=2}^{\infty}G(n-1,m-2)x^ny^m \\
&= -\frac{2\mathrm{i}xy^3}{1-2y-y^2} - \sum_{m,n=2}^{\infty}G(n-1,m-2)x^ny^m
\end{aligned}$$

$$\begin{aligned}
&\left[1-4xy-2x^2y-2xy^2-x^2y^2\right]G(x,y) \\
= \quad & x+\mathrm{i}y+(2+2\mathrm{i}xy+\frac{2ixy^3}{1-y-2y^2}+\frac{2yx^3}{1-x-2x^2}.
\end{aligned}$$

□

THEOREM 2.11 (Binet Formula). *The Binet Formula for the two dimensional Gaussian Pell sequence is*

$$\begin{aligned}
G(n,m) = \Big(&\frac{(1+\sqrt{2})^{m+n+1}+(1-\sqrt{2})^{m+n+1}}{8} \\
&-\frac{(-1)^n((1+\sqrt{2})^{m-n+1}+(1-\sqrt{2})^{m-n+1})}{8}\Big) \\
&\mathrm{i}\Big(\frac{(1+\sqrt{2})^{m+n+1}+(1-\sqrt{2})^{m+n+1}}{8} \\
&-\frac{(-1)^m((1+\sqrt{2})^{n-m+1}+(1-\sqrt{2})^{n-m+1})}{8}.
\end{aligned}$$

Proof. By the equality (1) and the Binet formula for the Pell sequence, it is obtained that

$$\alpha = 1+\sqrt{2}, \quad \beta = 1-\sqrt{2}, \quad P_n = \frac{\alpha^n-\beta^n}{\alpha-\beta},$$

$$\begin{aligned}
G(n,m) &= P_{m+1}P_n + iP_mP_{n+1} \\
&= \frac{\alpha^{m+1}-\beta^{m+1}}{2\sqrt{2}}\frac{\alpha^n-\beta^n}{2\sqrt{2}} + \mathrm{i}\frac{\alpha^m-\beta^m}{2\sqrt{2}}\frac{\alpha^{n+1}-\beta^{n+1}}{2\sqrt{2}}.
\end{aligned}$$

□

COROLLARY 2.12. *By the Binet Formula for the two dimensional Gaussian Pell sequence, the following relation between Pell-Lucas numbers and two dimensional Gaussian Pell numbers is obtained:*

$$G(n,m) = \frac{Q_{m+n+1}-(-1)^nQ_{m-n+1}}{8} + \mathrm{i}\frac{Q_{m+n+1}-(-1)^mQ_{n-m+1}}{8}.$$

Theorem 2.13 (Explicit Closed Formula). *The explicit closed formula for two dimensional Gaussian Pell sequence is*

$$\begin{aligned} G(n,m) &= \frac{1}{8}\sum_{k=0}^{\lfloor \frac{m+n+1}{2} \rfloor} \frac{m+n+1}{m+n+1-k}\binom{m+n+1-k}{k}2^{m+n+1-k}(1+\mathrm{i}) \\ &\quad -\frac{(-1)^n}{8}\sum_{k=0}^{\lfloor \frac{m-n+1}{2} \rfloor} \frac{m-n+1}{m-n+1-k}\binom{m-n+1-k}{k}2^{m-n+1-k}. \end{aligned}$$

Proof. The explicit closed formula for Pell-Lucas numbers is

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k}\binom{n-k}{k}2^{n-k}.$$

Then, by Corollary 2.11, we get

$$G(n,m) = -\frac{(-1)^m}{8}\sum_{k=0}^{\lfloor \frac{n-m+1}{2} \rfloor} \frac{n-m+1}{n-m+1-k}\binom{n-m+1-k}{k}2^{n-m+1-k}.$$

□

Theorem 2.14 (Sum Formula). *The sum formula for two dimensional Gaussian Pell sequence is*

$$\begin{aligned} \sum_{k=0}^{n}\sum_{j=0}^{m} G(k,j) &= [\frac{Q_{n+m+1}+2Q_{n+m+2}+Q_{n+m+3}}{32} - \frac{Q_{m+2}+2Q_{m+1}}{32} \\ &\quad -\frac{Q_n+2Q_{n+1}+Q_{n+2}-4}{32} -](1+\mathrm{i}) \\ &\quad +\mathrm{i}\frac{Q_{n+m-1}+2Q_{n+m}+Q_{n+m+1}}{32} - \mathrm{i}\frac{Q_{m-1}-6}{32} \\ &\quad +\mathrm{i}\frac{Q_n+2Q_{n-1}+Q_{n-2}}{32}. \end{aligned}$$

Proof. By Corallary 2.11 and the sum formula for the Pell-Lucas sequence $\sum_{k=0}^{n} Q_k = \frac{Q_{n+1}+Q_n}{2}$, we get

$$\begin{aligned} \sum_{k=0}^{n}\sum_{j=0}^{m} G(k,j) &= \sum_{k=0}^{n}\sum_{j=0}^{m} \frac{Q_{k+j+1}-(-1)^k Q_{j-k+1}}{8} + \mathrm{i}\frac{Q_{k+j+1}-(-1)^j Q_{k-j+1}}{8} \\ &= \sum_{k=0}^{n}\sum_{j=0}^{m} \frac{Q_{k+j+1}}{8} - \sum_{k=0}^{n}\sum_{j=0}^{m} \frac{(-1)^k Q_{j-k+1}}{8} \end{aligned}$$

$$+\sum_{k=0}^{n}\sum_{j=0}^{m}\mathrm{i}\frac{Q_{k+j+1}}{8}-\sum_{k=0}^{n}\sum_{j=0}^{m}\mathrm{i}\frac{(-1)^{j}Q_{k-j+1}}{8}.$$

Then

$$\begin{aligned}\sum_{k=0}^{n}\sum_{j=0}^{m}G(k,j) &= \sum_{k=0}^{n}\left(\sum_{j=0}^{m+k+1}\frac{Q_j}{8}-\sum_{j=0}^{k}\frac{Q_j}{8}\right)\\ &+\sum_{k=0}^{n}(-1)^{k+1}\left(\sum_{j=0}^{m-k+1}\frac{Q_j}{8}+\sum_{j=1-k}^{0}\frac{Q_j}{8}\right)\\ &+\sum_{k=0}^{n}\mathrm{i}\left(\sum_{j=0}^{m+k+1}\frac{Q_j}{8}-\sum_{j=0}^{k}\frac{Q_j}{8}\right)\\ &+\sum_{j=0}^{m}(-1)^{j+1}\mathrm{i}\left(\sum_{k=0}^{n-j+1}\frac{Q_k}{8}+\sum_{k=1-j}^{0}\frac{Q_k}{8}\right).\end{aligned}$$

By the sum formula of Pell-Lucas sequence

$$\begin{aligned}&\sum_{k=0}^{n}\sum_{j=0}^{m}G(k,j)=\sum_{k=0}^{n}\left(\frac{Q_{m+k+2}+Q_{m+k+1}}{16}-\frac{Q_{k+1}+Q_k}{16}\right)\\ &+\sum_{k=0}^{n}(-1)^{k+1}\left(\frac{-Q_{m-k+1}+Q_{m-k+2}}{16}-\frac{Q_{-k+1}+Q_{-k+2}}{16}\right)\\ &+\mathrm{i}\sum_{k=0}^{n}\left(\frac{Q_{m+k+2}+Q_{m+k+1}}{16}-\frac{Q_{k+1}+Q_k}{16}\right)\\ &+\sum_{j=0}^{m}(-1)^{j+1}\mathrm{i}\left(\frac{Q_{n-j+1}+Q_{n-j+2}}{16}-\frac{Q_{1-j}-Q_{2-j}}{16}\right),\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n}\sum_{j=0}^{m}G(k,j) &= \sum_{k=m+2}^{n+m+2}\frac{Q_k}{16}+\sum_{k=m+1}^{n+m+1}\frac{Q_k}{16}-\sum_{k=0}^{n+1}\frac{Q_k-2}{16}\\ &-\sum_{k=0}^{n}\frac{Q_k}{16}+\sum_{k=0}^{n}(-1)^{k+1}\frac{Q_{m-k+1}}{16}\\ &+\sum_{k=0}^{n}(-1)^{k+1}\frac{Q_{m-k+2}}{16}\\ &-\sum_{k=0}^{n}(-1)^{k+1}\frac{Q_{-k+1}}{16}-\sum_{k=0}^{n}(-1)^{k+1}\frac{Q_{-k+2}}{16}\end{aligned}$$

$$+\mathrm{i}\left[\sum_{k=m+2}^{n+m+2}\frac{Q_k}{16}+\sum_{k=m+1}^{n+m+1}\frac{Q_k}{16}-\sum_{k=0}^{n+1}\frac{Q_k-2}{16}-\sum_{k=0}^{n}\frac{Q_k}{16}\right]$$
$$+\mathrm{i}\left[\sum_{j=0}^{m}(-1)^{j+1}\frac{Q_{n-j+1}}{16}+\sum_{j=0}^{m}(-1)^{j+1}\frac{Q_{n-j+2}}{16}\right]$$
$$-\mathrm{i}\left[\sum_{j=0}^{m}(-1)^{j+1}\frac{Q_{1-j}}{16}-\sum_{j=0}^{m}(-1)^{j+1}\frac{Q_{2-j}}{16}\right],$$

$$\begin{aligned}\sum_{k=0}^{n}\sum_{j=0}^{m}G(k,j) \quad = \quad & \frac{Q_{m+n+3}+Q_{m+n+2}}{32}-\frac{Q_{m+3}+Q_{m+2}}{32} \\ & +\frac{Q_{m+n+2}+Q_{m+n+1}}{32} \\ & -\frac{Q_{m+2}+Q_{m+1}}{32}-\frac{Q_{n+2}+Q_{n+1}-4}{32}-\frac{Q_{n+1}+Q_n}{32} \\ & +(-1)^m\left[\frac{Q_{m+n+2}+Q_{m+n+1}}{32}-\frac{Q_{m+2}+Q_{m+1}}{32}\right] \\ & +(-1)^{m+1}\left[\frac{Q_{m+n+3}+Q_{m+n+2}}{32}-\frac{Q_{m+3}+Q_{m+2}}{32}\right] \\ & -\frac{Q_{n-1}+Q_n+4}{32}-\frac{Q_{n-2}+Q_{n-1}+8}{32} \\ & +\mathrm{i}\left[\frac{Q_{m+n+3}+Q_{m+n+2}}{32}-\frac{Q_{m+3}+Q_{m+2}}{32}\right] \\ & +\mathrm{i}\left[\frac{Q_{m+n+2}+Q_{m+n+1}}{32}-\frac{Q_{m+2}+Q_{m+1}}{32}\right] \\ & -\mathrm{i}\left[\frac{Q_{n+2}+Q_{n+1}-4}{32}-\frac{Q_{n+1}+Q_n}{32}\right] \\ & +\mathrm{i}(-1)^n\left[-\frac{Q_{m+n+1}+Q_{m+n+2}}{32}+\frac{Q_{n+1}+Q_{n+2}}{32}\right] \\ & +\mathrm{i}(-1)^{n+1}\left[-\frac{Q_{m+n+2}+Q_{m+n+3}}{32}+\frac{Q_{n+2}+Q_{n+3}}{32}\right] \\ & -\mathrm{i}\left[\frac{Q_{m-1}+Q_m+4}{32}+\frac{Q_{m-2}+Q_{m-1}+8}{32}\right].\end{aligned}$$

After some operations we have

$$\sum_{k=0}^{n}\sum_{j=0}^{m}G(k,j) \quad = \quad \frac{1}{16}\left[Q_{m+n+3}-Q_{m+3}-Q_{n+3}+2\right](1+\mathrm{i})$$

$$
\begin{aligned}
&-\frac{Q_n + \mathrm{i}Q_m + 6(1+\mathrm{i})}{16} \\
&+\frac{(-1)^m}{16}\left[Q_{m+n+2} - Q_{m+2}\right] + \frac{\mathrm{i}(-1)^n}{16}\left[Q_{m+n+2} - Q_{n+2}\right].
\end{aligned}
$$

□

THEOREM 2.15. *The following matrix equalities hold for the two dimensional Gaussian Pell sequence:*

$$
\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 2(1+\mathrm{i}) & 1 \\ i & 0 \end{bmatrix} = \begin{bmatrix} G(n+1,1) & G(n+1,0) \\ G(n,1) & G(n,0) \end{bmatrix}
$$

$$
\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1+\mathrm{i} & \mathrm{i} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} G(1,n+1) & G(0,n+1) \\ G(1,n) & G(0,n) \end{bmatrix}.
$$

Proof. The mathematical induction method is used for the proof. □

THEOREM 2.16. *The following matrix equalities are satisfied for the two dimensional Gaussian Pell sequence:*

$$
\begin{bmatrix} G(n+1,m+1) \\ G(n,m+1) \\ G(n+1,m) \\ G(n,m) \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^n \begin{bmatrix} 2(1+\mathrm{i}) \\ \mathrm{i} \\ 1 \\ 0 \end{bmatrix}
$$

Proof. The mathematical induction method is used for the proof. The assertion is true for $n = 1$. Now assume that it is true for $k \leq n$. For $k = n + 1$,

$$
\begin{aligned}
&\begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{n+1} \begin{bmatrix} 2(1+\mathrm{i}) \\ \mathrm{i} \\ 1 \\ 0 \end{bmatrix} \\
= &\begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G(n+1,m+1) \\ G(n,m+1) \\ G(n+1,m) \\ G(n,m) \end{bmatrix} = \begin{bmatrix} G(n+2,m+2) \\ G(n+1,m+2) \\ G(n+2,m+1) \\ G(n+1,m+1) \end{bmatrix}
\end{aligned}
$$

Thus, the proof is completed. □

3. CONCLUSION

In this study firstly we carried out the Pell sequence to the complex plane, then we defined the sequence into two dimensions. We called this generalized sequence two dimensional Gaussian Pell sequence. We investigated Binet formula, generating function, sum formula, explicit closed formula, and some relations between Pell sequence. Also, we get matrix equality for obtaining elements of the two-dimensional Gaussian Pell sequence.

REFERENCES

[1] G. Berzsenyi, *Gaussian Fibonacci numbers*, Fibonacci Quart., **15** (1977), 233–236

[2] S. Halici and S. Öz, *On some Gaussian Pell and Pell-Lucas numbers*, Ordu University Journal of Science and Technology, **6** (2016), 8–18.

[3] S. Halici and S. Öz, *On Gaussian Pell polynomials and their some properties*, Palest. J. Math., **7** (2018), 251–256.

[4] C. J. Harman, *Complex Fibonacci numbers,* Fibonacci Quart., **19** (1981), 82–86.

[5] J. H. Jordan, *Gaussian Fibonacci and Lucas numbers*, Fibonacci Quart., **3** (1965), 315–318 .

[6] S. Pethe and A. F. Horadam, *Generalized Gaussian Fibonacci numbers*, Bull. Aust. Math. Soc., **33** (1986), 37–48.

[7] Y. Soykan, *On summing formulas for generalized Fibonacci and Gaussian generalized Fibonacci numbers*, Advances in Research, **20** (2019), 1–15.

Received September 30, 2021
Accepted February 28, 2022

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