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MULTIPLE SOLUTIONS TO p-KIRCHHOFF TYPE PROBLEMS INVOLVING CRITICAL SOBOLEV EXPONENT IN $\mathbb{R}^N$

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Abstract. In this paper, we use variational methods to study the existence and multiplicity of non negative solutions for a p-Kirchhoff equation involving critical Sobolev exponent.

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1. INTRODUCTION

This paper deals with the existence and multiplicity of nontrivial solutions of the following Kirchhoff problem

$$(\mathcal{P}_\lambda)\begin{cases} -\left(\alpha \left\|u\right\|^p+\beta\right)\Delta_p u=u^{p^*-1}+\lambda f u^{q-1} & \text{in } \mathbb{R}^N \\ u\geq 0,\ u\in W^{1,p}\left(\mathbb{R}^N\right), \end{cases}$$

where α and β are two positive constants, $N \geq 3$, $1 < p < N$, $1 < q < p$, λ is a positive parameter, $f \not\equiv 0$, Δ_p is the p-Laplacian operator, defined by

$$\Delta_p u=\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right),\quad 1<p<N,$$

$p^* = pN/\left(N-p\right)$ is the critical exponent and $\|\cdot\|$ is the usual norm in the space $W^{1,p}\left(\mathbb{R}^N\right)$, given by

$$\|u\|^p=\int_{\mathbb{R}^N}|\nabla u|^p\mathrm{d}x.$$

Kirchhoff type problems are often referred to as being nonlocal because of the presence of the term $\int_{\mathbb{R}^N}|\nabla u|^p\mathrm{d}x$ which implies that the equation in $(\mathcal{P}_\lambda)$ is no longer a pointwise identity. It is analogous to the stationary case of equations that arise in the study of string or membrane vibrations, namely,

$$u_{tt}-\left(a\int_{\Omega}|\nabla u|^2\mathrm{d}x+b\right)\Delta u=g\left(x,u\right),$$

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where $\Omega \subset \mathbb{R}^N$, u denotes the displacement, $g(x,u)$ is the external force and b is the initial tension while a is related to the intrinsic properties of the string (such as Young's modulus). Equations of this type were first proposed by Kirchhoff in 1883 to describe the transversal oscillations of a stretched string.

In recent years, Kirchhoff type problems in bounded or unbounded domain have been studied in many papers by using variational methods. Some interesting studies can be found in [2, 5, 6, 7, 8, 10]. This problems in the whole space $\mathbb{R}^N$ considered in general without the critical Sobolev exponent, when the difficulty is due to the lack of compactness embedding from $W^{1,p}(\mathbb{R}^N)$ into the space $L^r\left(\mathbb{R}^N\right)$. In this subcritical case, many authors considering the following equation

$$(\mathcal{P}_V)\left\{\ -\left(\alpha\left\|u\right\|^p+\beta\right)\Delta_p u+V\left(x\right)u=h(x,u)\quad\text{in }\mathbb{R}^N,\right.$$

where $N\geq 3$, $1<p<N$, $V\in C\left(\mathbb{R}^N,\mathbb{R}\right)$ and $h\in C\left(\mathbb{R}^N\times\mathbb{R},\ \mathbb{R}\right)$ is subcritical. In such problems, some conditions are imposed on the weight function $V\left(x\right)$ which are key points for recovering the compactness of Sobolev embedding. See for example [5],[11] and [7].

On the other hand, the problem $(\mathcal{P}_\lambda)$ without the nonlocal term $\alpha\left\|u\right\|^p$ is treated by Alves [1], he proves the existence of two nonnegative solutions for $(\mathcal{P}_\lambda)$ where $\alpha=0$, $\beta=1$ and f is a nonnegative function.

A natural and interesting question is whether results concerning the solutions of problem $(\mathcal{P}_\lambda)$ with $\alpha=0$ remain valid for $\alpha\neq 0$. Our answer is affirmative, but the adaptation to the procedure to our problem is not trivial at all, since the appearance of nonlocal term. In this context, we need more delicate estimates. We are concerned in finding conditions on N, p, f and λ for which problem $(\mathcal{P}_\lambda)$ possesses multiple nontrivial solutions via the variational methods. To the best of our knowledge, there is no result on the multiple nontrivial solutions to the critical problem $(\mathcal{P}_\lambda)$ in $\mathbb{R}^N$.

Before stating our results, recall that, the best Sobolev constant

$$S=\inf_{u\in W^{1,p}(\mathbb{R}^N)}\frac{\left\|u\right\|^p}{\left(\int_{\mathbb{R}^N}\left|u\right|^{p^*}\mathrm{d}x\right)^{p/p^*}},$$

is attained in $\mathbb{R}^N$ by the function

$$U\left(x\right)=\frac{\left[N\left(\left(N-p\right)/\left(p-1\right)\right)^{p-1}\right]^{(N-p)/p^2}}{\left[1+\left|x\right|^{p/(p-1)}\right]^{(N-p)/p}},$$

see [10]. We introduce the following conditions on N, p and f:

(H_0) $N/3=p/2=m,$ with $m\in\mathbb{N}^*$,
(H_1) $f^+\not\equiv 0$,
(H_2) $f\in L^{q_0}\left(\mathbb{R}^N\right)$ with $q_0=pN/\left[\left(p-q\right)N+qp\right]$,
(H_3) $\int_{\mathbb{R}^N}fU^q\mathrm{d}x>0$.

Our main results are the following.

THEOREM 1.1. *Assume that* $\alpha > 0$, $\beta > 0$, N, p *satisfy* (H_0), $1 < q < p$ *and* f *satisfies* $(H_1) - (H_2)$. *Then there exists* $\Lambda_1 > 0$ *such that problem* $(\mathcal{P}_\lambda)$ *has at least one nontrivial solution for any* $\lambda \in (0,\ \Lambda_1)$.

THEOREM 1.2. *In addition to the assumption of Theorem 1, we assume that* f *satisfies* (H_3). *Then there exists* $\Lambda_2 > 0$ *such that problem* $(\mathcal{P}_\lambda)$ *has at least two nontrivial solutions for any* $\lambda \in (0,\ \Lambda_2)$.

This paper is organized as follows. In Section 2 we give some technical results which allow us to give a variational approach of our main results that we prove in Section 3.

2. AUXILIARY RESULTS

In this paper we use the following notation: $\|.\|_r$ stands for $\|.\|_{L^r(\mathbb{R}^N)}$, B_ρ is the ball centred at 0 and of radius ρ, $\to$ (resp.$\rightharpoonup$) denotes strong (resp. weak) convergence, $u^{\pm} = \max(\pm u,\ 0)$ and $\circ_n(1)$ denotes $\circ_n(1) \to 0$ as $n \to \infty$.

Since our approach is variational, we define the functional I_λ by

$$I_\lambda(u) = \frac{\alpha}{2p}||u||^{2p} + \frac{\beta}{p}||u||^p - \frac{1}{p^*}\int_{\mathbb{R}^N} \left(u^+\right)^{p^*} \mathrm{d}x - \frac{\lambda}{q}\int_{\mathbb{R}^N} f\left(u^+\right)^q \mathrm{d}x,$$

for all $u \in W^{1,p}\left(\mathbb{R}^N\right)$. Using (H_2), it is clear that I_λ is well defined in $W^{1,p}\left(\mathbb{R}^N\right)$ and belongs to $C^1\left(W^{1,p}\left(\mathbb{R}^N\right),\ \mathbb{R}\right)$. $u \in W^{1,p}\left(\mathbb{R}^N\right) \setminus \{0\}$ is said to be a weak solution of problem $(\mathcal{P}_\lambda)$ if it satisfies $u \geq 0$ and

$$\begin{aligned}(\alpha||u||^p + \beta)\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\varphi \,\mathrm{d}x - \int_{\mathbb{R}^N} \left(u^+\right)^{p^*-1}\varphi \,\mathrm{d}x \\ - \lambda\int_{\mathbb{R}^N} f\left(u^+\right)^{q-1}\varphi \,\mathrm{d}x = 0,\end{aligned}$$

for all $\varphi \in W^{1,p}\left(\mathbb{R}^N\right)$.

To prove our main results, we need following lemmas.

LEMMA 2.1. *Assume that* $\alpha > 0$, $\beta > 0$, $N \geq 3$, $1 < p < N$, $1 < q < p$ *and* f *satisfies* (H_2). *Then there exist positive numbers* Λ_1, ρ_1 *and* δ_1 *such that for all* $\lambda \in (0,\ \Lambda_1)$ *we have*

(i) $I_\lambda(u) \geq \delta_1 > 0$, *for all* $u \in W^{1,p}\left(\mathbb{R}^N\right)$ *with* $\|u\| = \rho_1$

(ii) $I_\lambda(u) \geq -\frac{p-q}{p}\left[\left(\frac{2q}{\beta}\right)^{\frac{q}{p}} \frac{\|f\|_{q_0}}{qS^{q/p}}\right]^{p/(p-q)} \lambda^{p/(p-q)}$ *for all* $u \in B_{\rho_1}$.

Proof. Let $u \in W^{1,p}\left(\mathbb{R}^N\right) \setminus \{0\}$, then by Sobolev and Hölder inequalities, we have

$$I_\lambda(u) \geq \frac{\alpha}{2p}\|u\|^{2p} + \frac{\beta}{p}\|u\|^p - \frac{S^{-p^*/p}}{p^*}\|u\|^{p^*} - \frac{\lambda}{q}S^{-q/p}\|f\|_{q_0}\|u\|^q.$$

Let $\eta > 0$, $\rho = \|u\|$ and

$$h(\rho) = \frac{\alpha}{2p}\rho^{2p} + \frac{\beta}{p}\rho^{p} - \frac{S^{-p^*/p}}{p^*}\rho^{p^*} - \frac{\lambda}{q}S^{-q/p}\|f\|_{q_0}\rho^{q}.$$

Then

$$\begin{aligned}\frac{\lambda}{q}S^{-q/p}\|f\|_{q_0}\rho^{q} &= \left[\left(\frac{\eta p}{q}\right)^{\frac{q}{p}}\rho^{q}\right]\left[\left(\frac{\eta p}{q}\right)^{-\frac{q}{p}}\frac{\lambda}{q}S^{-q/p}\|f\|_{q_0}\right]\\ &\le \eta\rho^{p} + \frac{p-q}{p}\left[\left(\frac{q}{p\eta}\right)^{\frac{q}{p}}\frac{S^{-q/p}}{q}\|f\|_{q_0}\right]^{p/(p-q)}\lambda^{p/(p-q)}.\end{aligned}$$

Therefore,

$$h(\rho) \ge \left(\frac{\beta}{p} - \eta\right)\rho^{p} - \frac{S^{-\frac{p^*}{p}}}{p^*}\rho^{p^*} - \frac{p-q}{p}\left[\left(\frac{q}{p\eta}\right)^{\frac{q}{p}}\frac{S^{-\frac{q}{p}}}{q}\|f\|_{q_0}\right]^{\frac{p}{p-q}}\lambda^{\frac{p}{p-q}},$$

Choosing $\eta = \beta/2p$, we get

$$h(\rho) \ge \frac{\beta}{2p}\rho^{p} - \frac{S^{-\frac{p^*}{p}}}{p^*}\rho^{p^*} - \frac{p-q}{p}\left[\left(\frac{2q}{b}\right)^{\frac{q}{p}}\frac{S^{-\frac{q}{p}}}{q}\|f\|_{q_0}\right]^{\frac{p}{p-q}}\lambda^{\frac{p}{p-q}}.$$

Easy computations show that

$$\begin{aligned}\max_{\rho\ge 0} h(\rho) &= h\left(\left[\frac{\beta}{2}S^{\frac{p^*}{p}}\right]^{1/(p^*-p)}\right)\\ &= \frac{1}{N}\left(\frac{\beta}{2}S\right)^{\frac{N}{p}} - \frac{p-q}{p}\left[\left(\frac{2q}{b}\right)^{\frac{q}{p}}\frac{S^{-\frac{q}{p}}}{q}\|f\|_{q_0}\right]^{\frac{p}{p-q}}\lambda^{\frac{p}{p-q}}\end{aligned}$$

Taking

$$\rho_1 = \left[\frac{\beta}{2}S^{\frac{p^*}{p}}\right]^{1/(p^*-p)},\ \delta_1 = \frac{1}{2N}\left(\frac{\beta}{2}S\right)^{\frac{N}{p}},$$

and

$$\Lambda_1 = \frac{1}{q\|f\|_{q_0}}\left(\frac{\beta}{2q}S\right)^{\frac{q}{p}}\left(\frac{p}{2N(p-q)}\left(\frac{\beta}{2}S\right)^{\frac{N}{p}}\right)^{\frac{p-q}{p}}.$$

Then the conclusion holds. □

In the sequel of this paper we need the condition (H_0). So, we replace p by $2m$ and N by $3m$ with $m \in \mathbb{N}^*$.

Lemma 2.2. *Assume that $\alpha > 0$, $\beta > 0$, N, p satisfy (H_0), $1 < q < p$ and f satisfies (H_2). If (u_n) is a $(PS)_c$ sequence of I_λ, then $u_n \rightharpoonup u$ in $W^{1,2m}(\mathbb{R}^{3m})$ for some u with $I'_\lambda(u) = 0$.*

Proof. We have

$$c+\circ_n(1)=I_\lambda(u_n) \text{ and } \circ_n(1)=\left\langle I'_\lambda(u_n),u_n\right\rangle, \tag{1}$$

this implies that

$$\begin{aligned} c+\circ_n(1) &= I_\lambda(u_n)-\frac{1}{6m}\left\langle I'_\lambda(u_n),u_n\right\rangle \\ &\geq \frac{\alpha}{12m}\|u_n\|^{4m}+\frac{\beta}{3m}\|u_n\|^{2m}-\lambda\left(\frac{1}{q}-\frac{1}{6m}\right)S^{-q/p}\|f\|_{q_0}\|u_n\|^q. \end{aligned}$$

Then (u_n) is bounded in $W^{1,2m}(\mathbb{R}^{3m})$. Up to a subsequence if necessary, we obtain

$$u_n\rightharpoonup u \text{ in } W^{1,2m}(\mathbb{R}^{3m}),L^{6m}(\mathbb{R}^{3m}),\ u_n\to u \text{ a.e.}, \nabla u_n\to\nabla u \text{ a.e. in } \mathbb{R}^{3m}.$$

Then $\langle I'_\lambda(u_n),\varphi\rangle=0$ for all $\varphi\in C_0^\infty(\mathbb{R}^{3m})$, which means that $I'_\lambda(u)=0$. □

Next, we prove an important lemma which ensures the local compactness of the (PS) sequence for I_λ.

LEMMA 2.3. *Assume that $\alpha>0$, $\beta>0$, N, p satisfy (H_0), $1<q<p$ and f satisfies (H_2), and let $(u_n)\subset W^{1,2m}(\mathbb{R}^{3m})$ be a $(PS)_c$ sequence for I_λ for some $c\in\mathbb{R}$ such that $u_n\rightharpoonup u$ in $W^{1,2m}(\mathbb{R}^{3m})$. Then*

$$\text{either } u_n\to u \text{ or } c\geq I_\lambda(u)+C_{\alpha,\beta,S}\left(\frac{\alpha}{48m}C_{\alpha,\beta,S}+\frac{\beta}{6m}\right),$$

where $C_{\alpha,\beta,S}=\alpha S^3+\left(\alpha^2S^6+4\beta S^3\right)^{1/2}$.

Proof. By the proof of Lemma 2 we have (u_n) is a bounded sequence in $W^{1,2m}(\mathbb{R}^{3m})$. Then by (H_2) we get

$$\int_{\mathbb{R}^{3m}}f\left(u_n^+\right)^q\mathrm{d}x\to\int_{\mathbb{R}^{3m}}f\left(u^+\right)^q\mathrm{d}x. \tag{2}$$

Furthermore, if we write $v_n=u_n-u$; we derive $v_n\rightharpoonup 0$ in $W^{1,2m}(\mathbb{R}^{3m})$, and by using Brezis-Lieb Lemma [4] we have

$$\|u_n\|^{2m}=\|v_n\|^{2m}+\|u\|^{2m}+o_n(1) \text{ and } \|u_n\|_{6m}^{6m}=\|v_n\|_{6m}^{6m}+\|u\|_{6m}^{6m}+o_n(1). \tag{3}$$

Putting together (2.2) and (2.3), we get

$$c+o_n(1)=I_\lambda(u)+\frac{\alpha}{4m}\|v_n\|^{4m}+\frac{\beta}{2m}\|v_n\|^{2m}+\frac{\alpha}{2m}\|v_n\|^{2m}\|u\|^{2m}-\frac{1}{6m}\|v_n\|_{6m}^{6m},$$

and

$$o_n(1)=\alpha\|v_n\|^{4m}+\beta\|v_n\|^{2m}+2\alpha\|v_n\|^{2m}\|u\|^{2m}-\|v_n\|_{6m}^{6m}. \tag{4}$$

Therefore,

$$c+o_n(1)=I_\lambda(u)+\frac{\alpha}{12m}\|v_n\|^{4m}+\frac{\beta}{3m}\|v_n\|^{2m}+\frac{\alpha}{3m}\|v_n\|^{2m}\|u\|^{2m}. \tag{5}$$

Assume that $\|v_n\| \to l > 0$, then by (2.4) and the Sobolev inequality we obtain

$$S^{-3}l^{6m} \geq \alpha l^{4m} + \beta l^{2m},$$

this implies that

$$\begin{aligned} l^{2m} &\geq \frac{\alpha}{2}S^3 + \frac{S^3}{2}\left(\alpha^2 + 4\beta S^{-3}\right)^{1/2} \\ &= \frac{1}{2}\left(\alpha S^3 + \left(\alpha^2 S^6 + 4\beta S^3\right)^{1/2}\right) \\ &= \frac{1}{2}C_{\alpha,\beta,S}. \end{aligned}$$

From the above inequality and (2.5) we conclude

$$\begin{aligned} c &\geq I_\lambda(u) + \frac{\alpha}{12m}l^{4m} + \frac{\beta}{3m}l^{2m} \\ &\geq I_\lambda(u) + \frac{\alpha}{48m}C^2_{\alpha,\beta,S} + \frac{\beta}{6m}C_{\alpha,\beta,S} \\ &\geq I_\lambda(u) + C_{\alpha,\beta,S}\left(\frac{\alpha}{48m}C_{\alpha,\beta,S} + \frac{\beta}{6m}\right); \end{aligned}$$

This finishes the proof of lemma 3. □

3. PROOF OF THEOREM 1

Now, we proof the existence of a local minimizer.

By Lemma 1, we define

$$c_1 = \inf\left\{I_\lambda(u);\ u \in \bar{B}_{\rho_1}\right\}.$$

Using (H_1) we can choose $v \in W^{1,2m}\left(\mathbb{R}^{3m}\right)$ such that $\int_{\mathbb{R}^{3m}} f\,(v^+)^q\,\mathrm{d}x > 0$. Then there exists $t_0 > 0$ small enough such that $\|t_0 v\| < \rho_1$ and

$$\begin{aligned} I_\lambda(t_0 v) = &\frac{\alpha}{4m}t_0^{4m}\|v\|^{4m} + \frac{\beta}{2m}t_0^{2m}\|v\|^{2m} - \frac{t_0^{6m}}{6m}\int_{\mathbb{R}^{3m}}\left(v^+\right)^{6m}\mathrm{d}x \\ &- \frac{\lambda}{q}t_0^q\int_{\mathbb{R}^{3m}} f\left(v^+\right)^q\mathrm{d}x < 0, \end{aligned}$$

which implies that $c_1 < 0$. Using the Ekeland's variational principle [8], for the complete metric space $\bar{B}_{\rho_1}$ with respect to the norm of $W^{1,2m}\left(\mathbb{R}^{3m}\right)$, we obtain by Lemma 2, the existence of a $(\mathrm{PS})_{c_1}$ sequence $(u_n) \subset \bar{B}_{\rho_1}$ such that $u_n \rightharpoonup u_1$ in $W^{1,2m}\left(\mathbb{R}^{3m}\right)$ for some u_1 with $\|u_1\| \leq \rho_1$. After a direct calculation, we derive $\left\|u_1^-\right\| = \left\langle I'_\lambda(u_1), u_1^-\right\rangle = 0$, which implies $u_1 \geq 0$. As $I_\lambda(0) = 0 > c_1$ then $u_1 \neq 0$. Assume $u_n \nrightarrow u_1$ in $W^{1,2m}\left(\mathbb{R}^{3m}\right)$, then it follows from Lemma 3 that

$$c_1 \geq I_\lambda(u) + C_{\alpha,\beta,S}\left(\frac{\alpha}{48m}C_{\alpha,\beta,S} + \frac{\beta}{6m}\right) > c_1,$$

which is a contradiction. Thus u_1 is a nontrivial solution of $(\mathcal{P}_\lambda)$ with negative energy.

4. PROOF OF THEOREM 2

Now, we proof the existence of Mountain Pass type solution, here we need the condition (H_3).

LEMMA 4.1. *Assume that $\alpha > 0$, $\beta > 0$, $(H_0) - (H_3)$ hold, and let $\Lambda_2 > 0$ such that*

$$-\frac{p-q}{p}\left[\left(\frac{2q}{\beta}\right)^{\frac{q}{p}} \frac{\|f\|_{q_0}}{qS^{q/p}}\right]^{p/(p-q)} \lambda^{p/(p-q)} + C_{\alpha,\beta,S}\left(\frac{\alpha}{48m}C_{\alpha,\beta,S} + \frac{\beta}{6m}\right) > 0,$$

for all $\lambda \in (0,\Lambda_2)$. Then there exists $0 < \Lambda_* \leq \Lambda_2$ such that

$$\sup_{t\geq 0} I_\lambda(tU) < c_1 + C_{\alpha,\beta,S}\left(\frac{\alpha}{48m}C_{\alpha,\beta,S} + \frac{\beta}{6m}\right), \quad \textit{for all } \lambda \in (0,\Lambda_*).$$

Proof. We consider functions

$$\Phi_1(t) = \frac{\alpha t^{4m}}{4m}\|U\|^{4m} + \frac{\beta t^{2m}}{2m}\|U\|^{2m} - \frac{t^{6m}}{6m}\|U\|_{6m}^{6m},$$

and

$$\Phi_2(t) = \Phi_1(t) - \lambda t \int_{\mathbb{R}^{3m}} fU^q \, \mathrm{d}x.$$

So, for all $\lambda \in (0,\Lambda_2)$ we have

$$\Phi_2(0) = 0 < -\frac{p-q}{p}\left[\left(\frac{2q}{\beta}\right)^{\frac{q}{p}} \frac{\|f\|_{q_0}}{qS^{q/p}}\right]^{p/(p-q)} \lambda^{p/(p-q)} + C_{\alpha,\beta,S}\left(\frac{\alpha}{48m}C_{\alpha,\beta,S} + \frac{\beta}{6m}\right).$$

Hence, by the continuity of $\Phi_2(t)$, there exists $t_1 > 0$ small enough such that

$$\Phi_2(t) < -\frac{p-q}{p}\left[\left(\frac{2q}{\beta}\right)^{\frac{q}{p}} \frac{\|f\|_{q_0}}{qS^{q/p}}\right]^{p/(p-q)} \lambda^{p/(p-q)} + C_{\alpha,\beta,S}\left(\frac{\alpha}{48m}C_{\alpha,\beta,S} + \frac{\beta}{6m}\right),$$

for all $t \in (0,t_1)$. On the other hand, the function $\Phi_1(t)$ attains its maximum at t_2 such that

$$t_2^{2m} = \frac{\alpha\|U\|^{4m} + \left(\alpha^2\|U\|^{8m} + 4\beta\|U\|^{2m}\|U\|_{6m}^{6m}\right)^{1/2}}{2\|U\|_{6m}^{6m}}.$$

From the definition of S we have

$$\frac{\alpha t_2^{4m}}{4m}\|U\|^{4m} = \frac{\alpha}{4m}\|U\|^{4m}$$

$$\times\left[\frac{\alpha\|U\|^{4m}+\left(\alpha^{2}\|U\|^{8m}+4\beta\|U\|^{2m}\|U\|_{6m}^{6m}\right)^{1/2}}{2\|U\|_{6m}^{6m}}\right]^{2}$$

$$=\frac{\alpha}{16m}\left[\frac{\alpha\|U\|^{6m}}{\|U\|_{6m}^{6m}}+\left[\frac{\alpha^{2}\|U\|^{12m}+4\beta\|U\|^{6m}\|U\|_{6m}^{6m}}{\|U\|_{6m}^{12m}}\right]^{1/2}\right]^{2}$$

$$=\frac{\alpha}{16m}C_{\alpha,\beta,S}^{2}.$$

Similarly, we obtain

$$\frac{\beta t_{2}^{2m}}{2m}\|U\|^{2m}=\frac{\beta}{4m}C_{\alpha,\beta,S},$$

and

$$\frac{t_{2}^{6m}}{6m}\|U\|_{6m}^{6m}=\frac{S^{-3}}{48m}C_{\alpha,\beta,S}^{3}.$$

By the above estimates, we obtain

$$\begin{aligned}\Phi_{1}(t_{2}) &= \frac{\alpha t_{2}^{4m}}{4m}\|U\|^{4m}+\frac{\beta t_{2}^{2m}}{2m}\|U\|^{2m}-\frac{t_{2}^{6m}}{6m}\|U\|_{6m}^{6m}\\ &= \frac{\alpha}{16m}C_{\alpha,\beta,S}^{2}+\frac{\beta}{4m}C_{\alpha,\beta,S}-\frac{S^{-3}}{48m}C_{\alpha,\beta,S}^{3}\\ &= C_{\alpha,\beta,S}\left(\frac{\alpha}{48m}C_{\alpha,\beta,S}+\frac{\beta}{6m}\right).\end{aligned}$$

Thus we deduce that

$$\Phi_{1}(t)\leq C_{\alpha,\beta,S}\left(\frac{\alpha}{48m}C_{\alpha,\beta,S}+\frac{\beta}{6m}\right).$$

On the other hand, using Lemma 1 we see that

$$c_{1}\geq-\frac{p-q}{p}\left[\left(\frac{2q}{\beta}\right)^{\frac{q}{p}}\frac{\|f\|_{q_{0}}}{qS^{q/p}}\right]^{p/(p-q)}\lambda^{p/(p-q)}\quad\text{for all}\quad\lambda\in(0,\Lambda_{1}),$$

furthermore, if

$$\lambda<\left[\left(\frac{2q}{\beta}\right)^{\frac{q}{p}}\frac{\|f\|_{q_{0}}}{qS^{q/p}}\right]^{-p/q}\left[\frac{p}{q\,(p-q)}\int_{\mathbb{R}^{3m}}fU^{q}\mathrm{d}x\right]^{(p-q)/q},$$

we get

$$c_{1}>-\lambda\frac{t_{1}}{q}\int_{\mathbb{R}^{3m}}fU^{q}\,\mathrm{d}x.$$

Taking

$$\Lambda_{*}=\min\left\{\Lambda_{1},\ \Lambda_{2},\ \left[\left(\frac{2q}{\beta}\right)^{\frac{q}{p}}\frac{\|f\|_{q_{0}}}{qS^{q/p}}\right]^{-\frac{p}{q}}\left[\frac{p}{q\,(p-q)}\int_{\mathbb{R}^{3m}}fU^{q}\,\mathrm{d}x\right]^{\frac{p-q}{q}}\right\},$$

then we deduce that

$$\sup_{t\geq 0} I_{\lambda}(tU) < c_1 + C_{\alpha,\beta,S}\left(\frac{\alpha}{48m}C_{\alpha,\beta,S} + \frac{\beta}{6m}\right) \text{ for all } \lambda \in (0, \Lambda_*) .$$

□

Note that $I_\lambda(0) = 0$ and $I_\lambda(t_3U) < 0$ for t_3 large enough, also from Lemma 1, we know that

$$I_\lambda(u)|_{\partial B_{\rho_1}} \geq \delta_1 > 0 \text{ for all } \lambda \in (0, \Lambda_1).$$

Then, by the Mountain Pass theorem [3], there exists a $(\mathrm{PS})_{c_2}$ sequence, where

$$c_2 = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

with

$$\Gamma = \left\{\gamma \in C([0,\ 1],\ W^{1,2m}(\mathbb{R}^{3m})),\ \gamma(0) = 0 \text{ and } \gamma(1) = t_3U\right\}.$$

Using Lemma 2 we have (u_n) has a subsequence, still denoted by (u_n), such that $u_n \rightharpoonup u_2$ in $W^{1,2m}(\mathbb{R}^{3m})$, for some u_2. As $\|u_2^-\| = \langle I'_\lambda(u_2), u_2^-\rangle = 0$, we conclude that $u_2 \geq 0$. Furthermore, we know by Lemma 4 that

$$\sup_{t\geq 0} I_\lambda(tU) < c_1 + C_{\alpha,\beta,m,S}, \text{ for all } \lambda \in (0, \Lambda_*),$$

then from Lemma 3 we deduce that $u_n \to u_2$ in $W^{1,2m}(\mathbb{R}^{3m})$. Thus we obtain a critical point u_2 of I_λ satisfying $I_\lambda(u_2) > 0$, and we conclude that u_2 is a nontrivial solution of $(\mathcal{P}_\lambda)$ with positive energy.

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