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# DECAY RESULTS FOR A VISCOELASTIC WAVE EQUATION WITH DISTRIBUTED DELAY IN BOUNDARY FEEDBACK

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**Abstract.** In this work, a nonlinear viscoelastic wave equation is studied. By supposing distributed delay feedback acting on the boundary, we establish the general decay rate under suitable hypothesis.

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**Key words.** Wave equation, general decay, distributed delay term, viscoelastic term, boundary feedback.

## 1. INTRODUCTION

In the present work, we consider the following wave equation

$$(1)\quad \begin{cases} u_{tt} - \Delta u(t) + \int_0^t h(t-\varrho)\Delta u(\varrho)\mathrm{d}\varrho = 0, \text{ in } \Omega \times \mathbb{R}_+, \\ \frac{\partial u}{\partial \nu} - \int_0^t h(t-\varrho)\frac{\partial}{\partial \nu}u(\varrho)\mathrm{d}\varrho + \beta_1 g_1(u_t) + \int_{\tau_1}^{\tau_2} |\beta_2(s)| g_2(u_t(t-s))\mathrm{d}s = 0, \\ \qquad \text{on } \Gamma_1 \times \mathbb{R}_+, \\ u(x,t) = 0, \text{ on } \Gamma_0 \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \text{ in } \Omega, \\ u_t(x,-t) = f_0(x,t), \text{ in } \Gamma_1 \times (0,\tau_2), \end{cases}$$

where $\Omega \in \mathbb{R}^N (N \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_0$, $\Gamma_1$ and $\Gamma_0$ are closed and disjoint, $\nu$ is the outward normal to $\Gamma$, $\beta_1$ is positive constant. $\tau_1 < \tau_2$ are non-negative constants such that $\beta_2 : [\tau_1, \tau_2] \to \mathbb{R}$ represents distributive time delay, $h$ is a positive function representing the kernel of memory, $g_1, g_2$ are specific functions.

The time delay is an important factor in various natural and physical phenomena, as the response to the applied force is affected by the time factor, the same for the transfer of materials and information, and their condition is all subject to the time factor. Recently, dealing with the delay factor has been an active area in recent years, and many authors have been interested in this

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type of damping as it greatly affects the stability of systems and the existence of solutions. In the case ($g_1(s) = g_2(s) = s$), this term has several types: delay ($u_t(x, t-\tau)$) see ([13],[17],[22]), distributed delay ($\int_{\tau_1}^{\tau_2} |\beta(s)| u_t(t-s) \mathrm{d}s$) as in the following papers ([7]-[11],[18]) and time-varying delay ($u_t(x, t-\tau(t))$) for example see ([15],[19],[20],[21]).
Even in the general case of the functions ($g_1, g_2$), many problems have been studied, where we find the term delay in the equation or in the boundary feedback.
In [3], the authors considered the following problem:

$$u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)\mathrm{d}s + \mu_1 g_1(u_t) + \mu_2 g_2(u_t(t-\tau) = 0, \tag{2}$$

they proved the existence of global solution, and a genaral stability result.
There are also some important works, including [20], where the authors are concerned with the following problem

$$\begin{aligned} &u_{tt} - \Delta u = 0, \\ &u(x,t) = 0, \text{ on } \Gamma_0 \times \mathbb{R}_+, \\ &\frac{du}{d\nu} + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0, \text{ on } \Gamma_1 \times \mathbb{R}_+, \end{aligned} \tag{3}$$

they proved under suitable supposition ($\mu_2 < \mu_1$) that the general energy is exponentially stable. Also, in [21] the authors considered the following problem:

$$\begin{aligned} &u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)\mathrm{d}s + \mu_2 g_2(u_t(t-\tau) = 0, \\ &u(x,t) = 0, \text{ on } \Gamma_0 \times \mathbb{R}_+, \\ &\frac{du}{d\nu} + \mu_1 g_1(u_t) = 0, \text{ on } \Gamma_1 \times \mathbb{R}_+, \end{aligned} \tag{4}$$

and they established the global existence and asymptotic behavior of problem (4). The term of viscoelastic was also introduced in many papers, including ([5],[6],[4],[12]-[14],[16]).

Starting from all these works and supplementing them, we will try to study our problem (1), as we consider the distributed delay within the boudary feedback, and this makes our problem different from what was previously studied. Under suitable conditions on various functions we will prove the general decay result.

Our paper is divided into several sections: in the next section we lay down the hypotheses, concepts and lemmas we need, and in the third section we prove our main result. Finally, we give the conclusion.

## 2. PRELIMINARIES

For studying our problem, in this section we will need some materials.
Firstly, introducing the following hypothesis for $\beta_2$, $h, g_1$ and $g_2$:

**(A1)** $h : \mathbb{R}_+ \to \mathbb{R}_+$ are non-increasing $C^1$ functions satisfying

$$h(0) > 0,\ l_0 = \int_0^\infty h(\varrho)\mathrm{d}\varrho < \infty,\ 1 - l_0 = l > 0. \tag{5}$$

**(A2)** $\exists\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ a non-increasing $C^1$ function, satisfying

$$\vartheta(t)h(t) + h'(t) \le 0,\ \ \forall t \ge 0. \tag{6}$$

**(A3)**$g_1 : \mathbb{R} \to \mathbb{R}$ is a non decreasing $C^1$ function and $H : \mathbb{R}_+ \to \mathbb{R}_+$ is a convex, increasing $C^1(\mathbb{R}_+) \cap C^1(]0,\infty[)$ function satisfying

$$\begin{aligned} &H(0) = 0, \\ &H \text{ is linear on } [0,\varepsilon], \text{ or } H'(0) = 0 \text{ and } H''(t) > 0 \text{ on } ]0,\varepsilon], \end{aligned} \tag{7}$$

and

$$\begin{aligned} &c_0|s| \le g_1(s) \le c_1|s| \text{ if } |s| \ge \varepsilon, \\ &s^2 + g_1^2(s) \le H^{-1}(sg_1(s)) \text{ if } |s| \le \varepsilon, \end{aligned} \tag{8}$$

where $H^{-1}$ is the inverse function of $H$ and $\varepsilon, c_0, c_1$ are positive constants.
**(A4)** $g_2 : \mathbb{R} \to \mathbb{R}$ is a odd non-decreasing $C^1$ function, such that $\exists\alpha_1 > \frac{1}{2}$ and $c_2, \alpha_2 > 0$,

$$\begin{aligned} &|g_2'(s)| \le c_2, \\ &\alpha_1 s g_2(s) \le G(s) \le \alpha_2 s g_1(s), \end{aligned} \tag{9}$$

where $G(s) = \int_0^s g_2(\sigma)\mathrm{d}\sigma$.
**(A5)** $\beta_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$2\alpha_2 \int_{\tau_1}^{\tau_2} |\beta_2(s)|\mathrm{d}s < \beta_1. \tag{10}$$

Let us indroduce

$$(h \circ \psi)(t) := \int_\Omega \int_0^t h(t-\varrho)|\psi(t) - \psi(\varrho)|^2 \mathrm{d}\varrho \mathrm{d}x.$$

Secondly, as in [18], take the following new variables

$$y(x,\rho,s,t) = u_t(x, t - s\rho)$$

which satisfy

$$\begin{cases} sy_t(x,\rho,s,t) + y_\rho(x,\rho,s,t) = 0, \\ y(x,0,s,t) = u_t(x,t). \end{cases} \tag{11}$$

So, problem (1) can be written as

$$
(12)\qquad
\begin{aligned}
&u_{tt} - \Delta u(t) + \int_0^t h(t-\varrho)\Delta u(\varrho)\mathrm{d}\varrho = 0, \ \text{ in } \ \Omega \times \mathbb{R}_+,\\
&\frac{\partial u}{\partial \nu} - \int_0^t h(t-\varrho)\frac{\partial}{\partial \nu}u(\varrho)\mathrm{d}\varrho + \beta_1 g_1(u_t)\\
&+ \int_{\tau_1}^{\tau_2} |\beta_2(s)| g_2(y(x,1,s,t))\mathrm{d}s = 0, \ \text{ on } \Gamma_1 \times \mathbb{R}_+,\\
&s y_t(x,\rho,s,t) + y_\rho(x,\rho,s,t) = 0, \ \text{ on } \ \Gamma_1 \times \mathbb{R}_+,\\
&u(x,t) = 0, \ \text{ on } \ \Gamma_0 \times \mathbb{R}_+,\\
&u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), \ \text{ in } \ \Omega,\\
&y(x,\rho,s,0) = f_0(x,\rho s), \ \text{ in } \ \Gamma_1 \times (0,1)\times(0,\tau_2).
\end{aligned}
$$

Now, we give the energy functional.

LEMMA 2.1. *The energy functional $E$, defined by*

$$
(13)\qquad
\begin{aligned}
E(t) = {}& \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t h(\varrho)\mathrm{d}\varrho\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(h\circ\nabla u)(t)\\
&+ \int_{\Gamma_1}\int_0^1\int_{\tau_1}^{\tau_2} s|\beta_2(s)| G(y(x,\rho,s,t))\mathrm{d}s\mathrm{d}\rho\mathrm{d}\Gamma,
\end{aligned}
$$

*satisfies*

$$
(14)\qquad
\begin{aligned}
E'(t) \le {}& -\mu_1\int_{\Gamma_1} u_t g_1(u_t)\mathrm{d}\Gamma + \frac{1}{2}(h'\circ\nabla u)(t) - \frac{1}{2}h(t)\|\nabla u(t)\|_2^2\\
&- \mu_2\int_{\Gamma_1}\int_{\tau_1}^{\tau_2} |\beta_2(s)| y(x,1,s,t) g_2(y(x,1,s,t))\mathrm{d}s\mathrm{d}\Gamma \le 0,
\end{aligned}
$$

*where* $\mu_1 = \beta_1 - 2\alpha_2\int_{\tau_1}^{\tau_2}|\beta_2(s)|\mathrm{d}s > 0$ *and* $\mu_2 = 2\alpha_1 - 1 > 0$.

*Proof.* Multiplying $(12)_1$ by $u_t$, then integrating over $\Omega$, we find

$$
(15)\qquad
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|u_t(t)\|_2^2 + (1 - \int_0^t h(\varrho)\mathrm{d}\varrho)\|\nabla u(t)\|_2^2 + (h\circ\nabla u)(t)\right\}\\
&+ \beta_1\int_{\Gamma_1} u_t g_1(u_t)\mathrm{d}\Gamma - \frac{1}{2}(h'\circ\nabla u)(t) + \frac{1}{2}h(t)\|\nabla u(t)\|_2^2\\
&+ \int_{\Gamma_1}\int_{\tau_1}^{\tau_2} |\beta_2(s)| g_2(y(x,1,s,t)) u_t\mathrm{d}s\mathrm{d}\Gamma = 0.
\end{aligned}
$$

Now, multiplying the equation $(12)_3$ by $|\beta_2(s)|g_2(y(x,\rho,s,t))$, and integrating over $\Gamma_1\times(0,1)\times(\tau_1,\tau_2)$, and using $(11)_2$, we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1}\int_0^1\int_{\tau_1}^{\tau_2} s|\beta_2(s)|.G(y(x,\rho,s,t))\mathrm{d}s\mathrm{d}\rho\mathrm{d}\Gamma\\
&=-\int_{\Gamma_1}\int_0^1\int_{\tau_1}^{\tau_2}|\beta_2(s)|.g_2(y)y_\rho\mathrm{d}s\mathrm{d}\rho\mathrm{d}\Gamma\\
&=-\int_{\Gamma_1}\int_0^1\int_{\tau_1}^{\tau_2}|\beta_2(s)|\frac{d}{\mathrm{d}\rho}G(y(x,\rho,s,t))\mathrm{d}s\mathrm{d}\rho\mathrm{d}\Gamma\\
&=\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}|\beta_2(s)|\Big(G(y(x,0,s,t))-G(y(x,1,s,t))\Big)\mathrm{d}s\mathrm{d}\Gamma\\
&=\left(\int_{\tau_1}^{\tau_2}|\beta_2(s)|\mathrm{d}s\right)\int_{\Gamma_1}G(u_t)\mathrm{d}\Gamma\\
&-\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}|\beta_2(s)|.G(y(x,1,s,t))\mathrm{d}s\mathrm{d}\Gamma.
\end{aligned}\tag{16}
$$

By combining (15) and (16), we find (14) and

$$
\begin{aligned}
E'(t)=&-\beta_1\int_{\Gamma_1}u_tg_1(u_t)\mathrm{d}\Gamma+\frac{1}{2}(h'\circ\nabla u)(t)-\frac{1}{2}h(t)\|\nabla u(t)\|_2^2\\
&-\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}|\beta_2(s)|g_2(y(x,1,s,t))u_t\mathrm{d}s\mathrm{d}\Gamma\\
&+\left(\int_{\tau_1}^{\tau_2}|\beta_2(s)|\mathrm{d}s\right)\int_{\Gamma_1}G(u_t)\mathrm{d}\Gamma\\
&-\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}|\beta_2(s)|.G(y(x,1,s,t))\mathrm{d}s\mathrm{d}\Gamma.
\end{aligned}\tag{17}
$$

At this point, let use $G^*$ the conjugate function of the convex function $G$:

$$G^*(s)=\sup_{t\geq 0}(st-G(t)).$$

Then, $G^*$ is the Legendre transform of $G$, which is given by (Arnold [2], p.61-62)

$$G^*(s)=s(G')^{-1}(s)-G[(G')^{-1}(s)],\ \ \forall s\geq 0.\tag{18}$$

and $G^*$ satisfies the generalized Young inequality

$$st\leq G^*(s)+G(t),\ \ \forall s,t\geq 0.\tag{19}$$

From the definition of $G$, we find

$$G^*(s)=sg_2^{-1}(s)-G(g_2^{-1}(s)),\ \ \forall s\geq 0.\tag{20}$$

Hence, by (20) and (9) we have

$$
\begin{aligned}
G^*(g_2(y(x,1,s,t))&=y(x,1,s,t)g_2(y(x,1,s,t))-G(y(x,1,s,t))\\
&\leq(1-\alpha_1)y(x,1,s,t)g_2(y(x,1,s,t)).
\end{aligned}\tag{21}
$$

Using (19),(21) and (9), we find

$$
\begin{aligned}
E'(t) \leq & -\beta_1 \int_{\Gamma_1} u_t g_1(u_t) \mathrm{d}\Gamma + \frac{1}{2}(h' \circ \nabla u)(t) - \frac{1}{2} h(t) \|\nabla u(t)\|_2^2 \\
& + (1-\alpha_1) \int_{\Gamma_1} \int_{\tau_1}^{\tau_2} |\beta_2(s)| y(x,1,s,t) g_2(y(x,1,s,t)) \mathrm{d}s \mathrm{d}\Gamma \\
& + 2\left( \int_{\tau_1}^{\tau_2} |\beta_2(s)| \mathrm{d}s \right) \int_{\Gamma_1} G(u_t) \mathrm{d}\Gamma \\
& - \int_{\Gamma_1} \int_{\tau_1}^{\tau_2} |\beta_2(s)|.G(y(x,1,s,t)) \mathrm{d}s \mathrm{d}\Gamma, \\
\leq & -\left( \beta_1 - 2\alpha_2 \int_{\tau_1}^{\tau_2} |\beta_2(s)| \mathrm{d}s \right) \int_{\Gamma_1} u_t g_1(u_t) \mathrm{d}\Gamma \\
& + (1-2\alpha_1) \int_{\Gamma_1} \int_{\tau_1}^{\tau_2} |\beta_2(s)| y(x,1,s,t) g_2(y(x,1,s,t)) \mathrm{d}s \mathrm{d}\Gamma \\
& + \frac{1}{2}(h' \circ \nabla u)(t) - \frac{1}{2} h(t) \|\nabla u(t)\|_2^2.
\end{aligned} \tag{22}
$$

By setting $\mu_1 = \beta_1 - 2\alpha_2 \int_{\tau_1}^{\tau_2} |\beta_2(s)| \mathrm{d}s$ and $\mu_2 = 2\alpha_1 - 1$, we obtain (14). Hence, (5)-(10) give $E$ is a non-increasing function. This completes of the proof. □

Now, we give the well posedness result for the problem (12), which can be established by Faedo-Galarkin method and combination between the results ([1],[5],[10]).

THEOREM 2.2. *Suppose that (5)-(10) are satisfied. Then, for any* $u_0, u_1 \in H^1_{\Gamma_0}(\Omega) \cap L^2(\Omega)$, *and* $f_0 \in L^2(\Gamma_1, (0,1), (\tau_1, \tau_2))$, *there exists a weak solution* $(u, y)$ *of problem (12) such that*

$$
\begin{aligned}
& u, u_t \in C(]0,T[, H^1_{\Gamma_0}(\Omega)) \cap C^1(]0,T[, L^2(\Omega)), \\
& u_{tt} \in C(]0,T[, L^2(\Omega)), \\
& y \in C(]0,T[, L^2(\Gamma_1, (0,1), (\tau_1, \tau_2))).
\end{aligned}
$$

Also, to achiev our goal, we need the following Lemma

LEMMA 2.3 (Jensen's Inequality). *If* $H$ *is a convex function on* $[a,b]$, $h : \Sigma \to [a,b]$ *and* $q$ *are integrable functions on* $\Sigma$, $q(x) \geq 0$ *and* $\int_\Sigma q(x) \mathrm{d}x = Q > 0$, *then*

$$
H\left( \frac{1}{Q} \int_\Sigma h(x) q(x) \mathrm{d}x \right) \leq \frac{1}{Q} \int_\Sigma H(h(x)) q(x) \mathrm{d}x. \tag{23}
$$

## 3. GENERAL DECAY

In this section, we state and prove our general decay result of the system (12). For this goal, we set

$$\Psi(t) \; := \; \int_{\Omega} u(t)u_t(t)\mathrm{d}x, \tag{24}$$

$$\Phi(t) \; := \; -\int_{\Omega} u_t \int_0^t h(t-\varrho)(u(t)-u(\varrho))\mathrm{d}\varrho\mathrm{d}x \tag{25}$$

and

$$\Theta(t) \; := \; \int_{\Gamma_1}\int_0^1\int_{\tau_1}^{\tau_2} se^{-\rho s}|\beta_2(s)|G(y(x,\rho,s,t))\mathrm{d}s\mathrm{d}\rho\mathrm{d}\Gamma. \tag{26}$$

LEMMA 3.1. *The functional* $\Psi(t)$ *defined in* (24) *satisfies, for* $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$,

$$\begin{aligned}\Psi'(t) &\leq -(l-\varepsilon_1-(\varepsilon_2+\varepsilon_3)c_p)\|\nabla u\|_2^2 \\ &+ \|u_t\|_2^2 + c(\varepsilon_1)\int_{\Gamma_1} g_1^2(u_t)\mathrm{d}\Gamma \\ &+ c(\varepsilon_2)(h\circ\nabla u)(t) + c(\varepsilon_3)\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}|\beta_2(s)g_2^2(y(x,1,s,t))\mathrm{d}s\mathrm{d}\Gamma.\end{aligned} \tag{27}$$

*Proof.* A differentiation of (24) and using $(12)_1$, gives

$$\begin{aligned}\Psi'(t) &= \|u_t\|_2^2 + \int_{\Omega} u_{tt}u\mathrm{d}x \\ &= \|u_t\|_2^2 + \int_{\Omega} u(t)\left[\Delta u - \int_0^t h(t-\varrho)\Delta u(\varrho)\mathrm{d}\varrho\right]\mathrm{d}x \\ &= \|u_t\|_2^2 - (1-\int_0^t h(\varrho)\mathrm{d}\varrho)\|\nabla u\|_2^2 \\ &+ \underbrace{\int_{\Omega}\nabla u(t)\int_0^t h(t-\varrho)(\nabla u(t)-\nabla u(\varrho))\mathrm{d}\varrho\mathrm{d}x}_{J_1} \\ &- \underbrace{\beta_1\int_{\Gamma_1} ug_1(u_t)\mathrm{d}\Gamma}_{J_2} + \underbrace{\int_{\Gamma_1} u\int_{\tau_1}^{\tau_2}|\beta_2(s)|g_2(y(x,1,s,t))\mathrm{d}s\Big)\mathrm{d}\Gamma}_{J_3}.\end{aligned} \tag{28}$$

We estimate the last 3 terms of the RHS of (28). Applying Hölder's, Poincaré and Young's inequalities, and (5), we find for $\varepsilon_i > 0, i = 1, 2, 3$,

$$J_1 \;\leq\; \varepsilon_1\|\nabla u\|_2^2 + c(\varepsilon_1)\int_{\Gamma_1} g_1^2(u_t)\mathrm{d}\Gamma. \tag{29}$$

and

$$J_2 \;\leq\; \varepsilon_2 c_p\|\nabla u\|_2^2 + c(\varepsilon_2)(h\circ\nabla u)(t). \tag{30}$$

Similarly, we have

$$J_3 \le \varepsilon_3 c_p \|\nabla u\|_2^2 + c(\varepsilon_3)\int_{\Gamma_1}\int_{\tau_1}^{\tau_2} |\beta_2(s) g_2^2(y(x,1,s,t))\mathrm{d}s\mathrm{d}\Gamma. \tag{31}$$

Combining (29)-(31) and (28), we get

$$\begin{aligned}\Psi'(t) \le\ & \|u_t\|_2^2 - (l - \varepsilon_1 - (\varepsilon_2+\varepsilon_3)c_p)\|\nabla u\|_2^2 + c(\varepsilon_1)\int_{\Gamma_1} g_1^2(u_t)\mathrm{d}\Gamma \\ & + c(\varepsilon_2)(h\circ\nabla u)(t) + c(\varepsilon_3)\int_{\Gamma_1}\int_{\tau_1}^{\tau_2} |\beta_2(s) g_2^2(y(x,1,s,t))\mathrm{d}s\mathrm{d}\Gamma.\end{aligned}$$

□

Lemma 3.2. *The functional $\Phi(t)$ defined in (25) satisfies, for any $\delta_1 > 0$, $\delta_2 > 0$,*

$$\begin{aligned}\Phi'(t) \le & -\left(\int_0^t h(\varrho)\mathrm{d}\varrho - \delta_2\right)\|u_t\|_2^2 \\ & + \delta_1(2-l)\|\nabla u\|_2^2 - c(\delta_2)(h'\circ\nabla u)(t) \\ & + c(\delta_1)(h\circ\nabla u)(t) + c\int_{\Gamma_1} g_1^2(u_t)\mathrm{d}\Gamma \\ & + c\int_{\Gamma_1}\int_{\tau_1}^{\tau_2} |\beta_2(s) g_2^2(y(x,1,s,t))\mathrm{d}s\mathrm{d}\Gamma.\end{aligned} \tag{32}$$

*Proof.* A differentiation of (25) and using $(12)_1$, gives

$$\begin{aligned}\Phi'(t) =& -\int_\Omega u_{tt}\int_0^t h(t-\varrho)(u(t)-u(\varrho))\mathrm{d}\varrho\mathrm{d}x \\ & - \int_\Omega u_t \int_0^t h'(t-\varrho)(u(t)-u(\varrho))\mathrm{d}\varrho\mathrm{d}x - \left(\int_0^t h(\varrho)\mathrm{d}\varrho\right)\|u_t\|_2^2 \\ =& \int_\Omega \left[\Delta u - \int_0^t h(t-\varrho)\Delta u(\varrho))\mathrm{d}\varrho\right]\left[\int_0^t h(t-\varrho)(u(t)-u(\varrho))\mathrm{d}\varrho\right]\mathrm{d}x \\ & - \int_\Omega u_t \int_0^t h'(t-\varrho)(u(t)-u(\varrho))\mathrm{d}\varrho\mathrm{d}x - \left(\int_0^t h(\varrho)\mathrm{d}\varrho\right)\|u_t\|_2^2 \\ =& \underbrace{\int_\Omega \nabla u \int_0^t h(t-\varrho)(\nabla u(t) - \nabla u(\varrho))\mathrm{d}\varrho\mathrm{d}x}_{J_1} \\ & \underbrace{-\int_\Omega\left(\int_0^t h(t-\varrho)\nabla u(\varrho)\mathrm{d}\varrho\right).\left(\int_0^t h(t-\varrho)(\nabla u(t)-\nabla u(\varrho))\mathrm{d}\varrho\right)\mathrm{d}x}_{J_2} \\ & \underbrace{-\beta_1\int_{\Gamma_1} g_1(u_t)\left(\int_0^t h(t-\varrho)(u(t)-u(\varrho))\mathrm{d}\varrho\right)\mathrm{d}\Gamma}_{J_3}\end{aligned} \tag{33}$$

$$\underbrace{-\int_{\Gamma_1}\left(\int_{\tau_1}^{\tau_2}|\beta_2(s)|g_2(y(x,1,s,t))\mathrm{d}s\right)\left(\int_0^t h(t-\varrho)(u(t)-u(\varrho))\mathrm{d}\varrho\right)\mathrm{d}\Gamma}_{J_4}$$

$$\underbrace{-\int_{\Omega}u_t\int_0^t h'(t-\varrho)(u(t)-u(\varrho))\mathrm{d}\varrho\mathrm{d}x}_{J_5}-\left(\int_0^t h(\varrho)\mathrm{d}\varrho\right)\|u_t\|_2^2.$$

We estimate the terms of the RHS of (33). Applying Hölder's, Poincaré and Young's inequalities, (5), we find

$$J_1 \leq \delta_1\|\nabla u\|_2^2 + c(\delta_1)(h\circ\nabla u)(t), \tag{34}$$

and

$$\begin{aligned} J_2 &\leq \left(\int_0^t h(\varrho)\mathrm{d}\varrho\right)\int_{\Omega}\left(\nabla u(t)\int_0^t h(t-\varrho)(\nabla u(t)-\nabla u(\varrho))\mathrm{d}\varrho\right)\mathrm{d}x \\ &\quad -\int_{\Omega}\left(\int_0^t h(t-\varrho)(\nabla u(t)-\nabla u(\varrho))\mathrm{d}\varrho\right)^2\mathrm{d}x \\ &\leq \delta_1(1-l)\|\nabla u\|_2^2 + c(\delta_1)(h\circ\nabla u)(t), \end{aligned} \tag{35}$$

Similarly, we have

$$\begin{aligned} J_3 &\leq c\int_{\Gamma_1} g_1^2(u_t)\mathrm{d}\Gamma + c(h\circ\nabla u)(t), \\ J_4 &\leq c\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}|\beta_2(s)g_2^2(y(x,1,s,t))\mathrm{d}s\mathrm{d}\Gamma + c(h\circ\nabla u)(t), \\ J_5 &\leq \delta_2\|u_t\|_2^2 - c(\delta_2)(h'\circ\nabla u)(t). \end{aligned} \tag{36}$$

A substitution of (34)-(36) into (33), gives (32). □

LEMMA 3.3. *The functional $\Theta(t)$ defined in (26) satisfies*

$$\begin{aligned} \Theta'(t) &\leq -\eta_1\int_{\Gamma_1}\int_0^1\int_{\tau_1}^{\tau_2} s|\beta_2(s)|.G(y(x,\rho,s,t))\mathrm{d}s\mathrm{d}\rho\mathrm{d}\Gamma \\ &\quad + \beta_1\int_{\Gamma_1} u_t g_1(u_t)\mathrm{d}\Gamma \\ &\quad - \eta_1\alpha_1\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}|\beta_2(s)|y\,(x,1,s,t)\,g_2(y\,(x,1,s,t))\mathrm{d}s. \end{aligned} \tag{37}$$

*Proof.* By differentiating of $\Theta(t)$, and using $(12)_2$, we have

$$\begin{aligned} \Theta'(t) &= -\int_{\Gamma_1}\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}|\beta_2(s)|.y_\rho g_2(y((x,\rho,s,t))\mathrm{d}s\mathrm{d}\rho\mathrm{d}\Gamma \\ &= -\int_{\Gamma_1}\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\beta_2(s)|.G(y(x,\rho,s,t))\mathrm{d}s\mathrm{d}\rho\mathrm{d}\Gamma \end{aligned}$$

$$-\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}|\beta_2(s)|\Big[e^{-s}G(y\,(x,1,s,t))-G(y\,(x,0,s,t))\Big]\mathrm{d}s\mathrm{d}x.$$

Applying $y(x,0,s,t)=u_t(x,t)$, and $e^{-s}\le e^{-s\rho}\le 1$, for any $0<\rho<1$, and setting $\eta_1=e^{-\tau_2}$, we obtain

$$\begin{aligned}\Theta'(t) \quad \le \quad & -\eta_1\int_{\Gamma_1}\int_0^1\int_{\tau_1}^{\tau_2}s|\beta_2(s)|.G(y(x,\rho,s,t))\mathrm{d}s\mathrm{d}\rho\mathrm{d}\Gamma\\ & -\eta_1\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}|\beta_2(s)|G(y\,(x,1,s,t))\mathrm{d}s\mathrm{d}\Gamma\\ & +(\int_{\tau_1}^{\tau_2}|\beta_2(s)|\mathrm{d}s)\int_{\Gamma_1}G(u_t)\mathrm{d}\Gamma,\end{aligned}$$

by (9) and (10), we find (37).

□

Now, we introduce the functional

$$\mathcal{G}(t) \quad := \quad NE(t)+\Psi(t)+M\Phi(t)+\Theta(t). \tag{38}$$

for some positive constants $N,M$ to be determined.

Lemma 3.4. *There exist* $\gamma_i,t_0>0,\ \ i=1,...,5$ *satisfying*

$$\mathcal{G}'(t) \quad \le \quad -\gamma_1E(t)+\gamma_2\int_{\Gamma_1}g_1^2(u_t)\mathrm{d}\Gamma+\gamma_3(h\circ\nabla u)(t),\quad \forall t\ge t_0. \tag{39}$$

*and*

$$\gamma_4E(t)\le\mathcal{G}(t)\le\gamma_5E(t). \tag{40}$$

*Proof.* Since the function $h$ is positive and continuous, for all $t_0>0$, we have

$$\int_0^t h(\varrho)\mathrm{d}\varrho\ge\int_0^{t_0}h(\varrho)\mathrm{d}\varrho:=h_0,\ \ \forall t\ge t_0.$$

By differentiation of (38), using 14, the Lemmas 3.1, 3.2 and 3.3, we obtain

$$\begin{aligned}\mathcal{G}'(t)&:=NE'(t)+\Psi'(t)+M\Phi'(t)+\Theta'(t)\\ &\le-\Big(M(h_0-\delta_2)-1\Big)\|u_t\|_2^2\\ &-\Big((l-\varepsilon_1-c_p(\varepsilon_2+\varepsilon_3))-M\delta_1(1-l)\Big)\|\nabla u\|_2^2\\ &+\Big(c(\varepsilon_2)+Mc(\delta_1)\Big)(h\circ\nabla u)(t)+\Big(\frac{N}{2}-Mc(\delta_2)\Big)(h'\circ\nabla u)(t)\end{aligned} \tag{41}$$

$$
\begin{aligned}
&+\Big(c(\varepsilon_1)+cM\Big)\int_{\Gamma_1} g_1^2(u_t)\mathrm{d}\Gamma-\Big(\mu_1 N-\beta_1\Big)\int_{\Gamma_1} u_t g_1(u_t)\mathrm{d}\Gamma\\
&+\Big(c(\varepsilon_3)+Mc\Big)\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}|\beta_2(s)|g_2^2(y(x,1,s,t))\mathrm{d}s\mathrm{d}\Gamma\\
&-\eta_1\int_{\Gamma_1}\int_0^1\int_{\tau_1}^{\tau_2} s|\beta_2(s)|.G(y(x,\rho,s,t))\mathrm{d}s\mathrm{d}\rho\mathrm{d}\Gamma\\
&-\Big(\mu_2 N+\eta_1\alpha_1\Big)\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}|\beta_2(s)||y(x,1,s,t)g_2(y(x,1,s,t))\mathrm{d}s\mathrm{d}\Gamma.
\end{aligned}
$$

Now, we choose $\varepsilon_i, i=1,2,3$ so small that

$$l_1 := l-\varepsilon_1-c_p(\varepsilon_2+\varepsilon_3))>0.$$

Next, letting $\delta_2=\frac{h_0}{2}$, then we select $M$ large enough such that $\frac{h_0}{2}M-1>0$, then we pick $\delta_1=\frac{l_1}{2(1-l)M}$. Therefore, (41) becomes, for positive constants $d_i, i=1,...,5$

$$
\begin{aligned}
\mathcal{G}'(t)&\le -d_1\|u_t\|_2^2-d_2\|\nabla u\|_2^2+d_3(h\circ\nabla u)(t)\\
&+(\frac{N}{2}-c)(h'\circ\nabla u)(t)\\
&+d_4\int_{\Gamma_1} g_1^2(u_t)\mathrm{d}\Gamma-\Big(\mu_1 N-\beta_1\Big)\int_{\Gamma_1} u_t g_1(u_t)\mathrm{d}\Gamma\\
&+d_5\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}|\beta_2(s)|g_2^2(y(x,1,s,t))\mathrm{d}s\mathrm{d}\Gamma\\
&-\eta_1\int_{\Gamma_1}\int_0^1\int_{\tau_1}^{\tau_2} s|\beta_2(s)|.G(y(x,\rho,s,t))\mathrm{d}s\mathrm{d}\rho\mathrm{d}\Gamma\\
&-\Big(\mu_2 N+\eta_1\alpha_1\Big)\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}|\beta_2(s)||y(x,1,s,t)g_2(y(x,1,s,t))\mathrm{d}s\mathrm{d}\Gamma.
\end{aligned}
\tag{42}
$$

By using $(9)_1$, we get

$$
\begin{aligned}
&\mathcal{G}'(t)\le -d_1\|u_t\|_2^2-d_2\|\nabla u\|_2^2+d_3(h\circ\nabla u)(t)+(\frac{N}{2}-c)(h'\circ\nabla u)(t)\\
&+d_4\int_{\Gamma_1} g_1^2(u_t)\mathrm{d}\Gamma-\Big(\mu_1 N-\beta_1\Big)\int_{\Gamma_1} u_t g_1(u_t)\mathrm{d}\Gamma\\
&-\eta_1\int_{\Gamma_1}\int_0^1\int_{\tau_1}^{\tau_2} s|\beta_2(s)|.G(y(x,\rho,s,t))\mathrm{d}s\mathrm{d}\rho\mathrm{d}\Gamma\\
&-\Big(\mu_2 N+\eta_1\alpha_1-c_2d_5\Big)\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}|\beta_2(s)||y(x,1,s,t)g_2(y(x,1,s,t))\mathrm{d}s\mathrm{d}\Gamma.
\end{aligned}
\tag{43}
$$

On the other hand, from (24)-(26), by using Hölder, Young's and Poincaré inequalities, we get

$$
\begin{aligned}
|\mathcal{G}(t) - NE(t)| &\le \frac{1}{2}\Big(\|u_t(t)\|_2^2 + c_p\|\nabla u(t)\|_2^2\Big) \\
&+ \frac{M}{2}\Big(\|u_t(t)\|_2^2 + c_p(h\circ\nabla u)(t)\Big) \\
&+ \int_{\Gamma_1}\int_0^1\int_{\tau_1}^{\tau_2} s e^{-\rho s}|\beta_2(s)|.G(y(x,\rho,s,t))\mathrm{d}s\mathrm{d}\rho\mathrm{d}\Gamma.
\end{aligned} \tag{44}
$$

Using the fact that $e^{-\rho s} < 1$, we find

$$
\begin{aligned}
|\mathcal{G}(t) - NE(t)| \le &\frac{1}{2}\Big(\|u_t(t)\|_2^2 + c_p\|\nabla u(t)\|_2^2\Big) \\
&+ \frac{M}{2}\Big(\|u_t(t)\|_2^2 + c_p(h\circ\nabla u)(t)\Big) \\
&+ \int_{\Gamma_1}\int_0^1\int_{\tau_1}^{\tau_2} s|\beta_2(s)|.G(y(x,\rho,s,t))\mathrm{d}s\mathrm{d}\rho\mathrm{d}\Gamma \\
&\le CE(t).
\end{aligned} \tag{45}
$$

Hence

$$
(N-C)E(t)| \le \mathcal{G}(t) \le (N+C)E(t). \tag{46}
$$

Now, by choosing $N$ large enough such that

$$
N - C > 0,\ \ \mu_1 N - \beta_1 > 0,\ \ \frac{N}{2} - c > 0,\ \ \mu_2 N + \eta_1\alpha_1 - c_2 d_5 > 0,
$$

and exploiting (13), estimates (43) and (46), respectively, give (39) and (40). □

THEOREM 3.5. *Suppose that (5)-(10) are satisfied, there exist positive constants* $\lambda_1, \lambda_2, t_0$ *and* $\varepsilon_0 \in (0,\varepsilon]$ *such that the energy of (12) satisfies:*

$$
E(t) \le \lambda_1 H^{-1}\Big\{\lambda_2\Big(1 + \int_{t_0}^{t}\vartheta(\sigma)\mathrm{d}\sigma\Big)\Big\},\ \ \forall t \ge t_0, \tag{47}
$$

*where* $H(t) := \int_t^1 \frac{1}{\mathcal{R}(\varrho)}\mathrm{d}\varrho$, *and*

$$
\mathcal{R}(t) = \begin{cases} t, & \text{if } H \text{ is linear on } [0,\varepsilon], \\ tH'(\varepsilon_0 t), & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on } (0,\varepsilon]. \end{cases} \tag{48}
$$

*Proof.* Multiplying (39) by $\vartheta(t)$, using (5) and (14), we find

$$\begin{aligned}\vartheta(t)\mathcal{G}'(t) &\leq -\gamma_1\vartheta(t)E(t) + \gamma_2\vartheta(t)\int_{\Gamma_1} g_1^2(u_t)\mathrm{d}\Gamma + \gamma_3\vartheta(t)(h\circ\nabla u)(t)\\ &\leq -\gamma_1\vartheta(t)E(t) + \gamma_2\vartheta(t)\int_{\Gamma_1} g_1^2(u_t)\mathrm{d}\Gamma - \gamma_3(h'\circ\nabla u)(t)\\ &\leq -\gamma_1\vartheta(t)E(t) + \gamma_2\vartheta(t)\int_{\Gamma_1} g_1^2(u_t)\mathrm{d}\Gamma - 2\gamma_3 E'(t).\end{aligned} \tag{49}$$

Since $\vartheta(t)$ is non-increasing function, we have

$$\frac{d}{dt}\Big(\vartheta(t)\mathcal{G}(t) + 2\gamma_3E(t)\Big) \leq -\gamma_1\vartheta(t)E(t) + \gamma_2\vartheta(t)\int_{\Gamma_1} g_1^2(u_t)\mathrm{d}\Gamma. \tag{50}$$

Let

$$\mathcal{K}(t) := \vartheta(t)\mathcal{G}(t) + 2\gamma_3E(t) \sim E(t). \tag{51}$$

Hence, we obtain

$$\mathcal{K}'(t) \leq -\gamma_1\vartheta(t)E(t) + \gamma_2\vartheta(t)\int_{\Gamma_1} g_1^2(u_t)\mathrm{d}\Gamma,\ \ \forall t\geq t_0. \tag{52}$$

To arrive at our main result, it remains to estimate the last term of the inequality (52).
To this aim we consider

$$\Gamma_1^1 := \Big\{x\in\Gamma_1 :\ \ |u_t|>\varepsilon\Big\}\ \text{ and }\ \Gamma_1^2 := \Big\{x\in\Gamma_1 :\ \ |u_t|\leq\varepsilon\Big\}. \tag{53}$$

By (8) and (14), we have

$$\gamma_2\vartheta(t)\int_{\Gamma_1^1} g_1^2(u_t)\mathrm{d}\Gamma \leq \gamma_2\vartheta(0)\int_{\Gamma_1^1} g_1^2(u_t)\mathrm{d}\Gamma \leq -\lambda_3E'(t), \tag{54}$$

where $\lambda_3 = \frac{\gamma_2 c_1\vartheta(0)}{\mu_1}$.

At this stage, we have two cases to discuss:

**Case 1.** $H$ is linear on $[0,\varepsilon]$: According (8) and (14), we get

$$\gamma_2\vartheta(t)\int_{\Gamma_1^2} g_1^2(u_t)\mathrm{d}\Gamma \leq \gamma_2\vartheta(0)\int_{\Gamma_1^2} g_1^2(u_t)\mathrm{d}\Gamma \leq -\lambda_4E'(t), \tag{55}$$

where $\lambda_4 = \frac{\gamma_2 c\vartheta(0)}{\mu_1}$.
Substituting (54) and (55) into (52), we find

$$\begin{aligned}\mathcal{K}_1'(t) &\leq -\gamma_1\vartheta(t)E(t)\\ &= -\lambda_5\vartheta(t)\mathcal{R}\left(\frac{E(t)}{E(0)}\right),\ \ \forall t\geq t_0,\end{aligned} \tag{56}$$

where

$$\mathcal{K}_1(t) = (\mathcal{K}(t) + \lambda E(t)) \sim E(t), \tag{57}$$

and $\lambda = \lambda_3 + \lambda_4$, $\lambda_5 = \gamma_1 E(0)$. Integrating (56) over $(t_0, t)$ and using (57), yields (47).

**Case 2**. $H$ is nonlinear: From (8), (14) and by Jensen's inequality (23), with $\Sigma = \Gamma_1^2$, $q(x) = 1$ and $f(x) = H^{-1}(u_t(x)g_1(u_t(x)))$, we get

$$\begin{aligned}\lambda_2\vartheta(t)\int_{\Gamma_1^2} g_1^2(u_t)\mathrm{d}\Gamma &\le \lambda_2\vartheta(t)\int_{\Gamma_1^2} H^{-1}(u_t g_1(u_t))\mathrm{d}\Gamma\\ &\le \lambda_2\vartheta(t)|\Gamma_1^2|H^{-1}\left(\frac{1}{|\Gamma_1^2|}\int_{\Gamma_1^2} u_t g_1(u_t)\mathrm{d}\Gamma\right)\\ &\le \lambda_2\vartheta(t)|\Gamma_1^2|H^{-1}\left(-\frac{E'(t)}{\mu_1|\Gamma_1^2|}\right),\end{aligned} \tag{58}$$

Substituting (54) and (58) into (52), we find

$$\mathcal{K}_2'(t) \le -\gamma_1\vartheta(t)E(t) + \lambda_6\vartheta(t)H^{-1}\left(-\frac{E'(t)}{\mu_1|\Gamma_1^2|}\right), \quad \forall t \ge t_0, \tag{59}$$

where

$$\mathcal{K}_2(t) = (\mathcal{K}(t) + \lambda_3 E(t)) \sim E(t), \tag{60}$$

and $\lambda_6 = \lambda_2|\Gamma_1^2|$.
Now, for $0 < \varepsilon_0 < \varepsilon$ and $\delta_0 > 0$, according (59), (18) and (19) we have

$$\begin{aligned}&\left\{H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{K}_2(t) + \delta_0 E(t)\right\}'\\ &= \varepsilon_0\frac{E'(t)}{E(0)}H''\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{K}_2(t) + H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{K}_2'(t) + \delta_0 E'(t)\\ &\le -\gamma_1\vartheta(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)E(t) + \lambda_6\vartheta(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\\ &\times H^{-1}\left(-\frac{E'(t)}{\mu_1|\Gamma_1^2|}\right) + \delta_0 E'(t)\\ &\le -\gamma_1\vartheta(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)E(t) + \lambda_6\vartheta(t)H^*\left(H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right)\\ &- \frac{\lambda_6\vartheta(t)}{\mu_1|\Gamma_1^2|}E'(t) + \delta_0 E'(t)\\ &= -\gamma_1\vartheta(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)E(t) + \lambda_6\vartheta(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\varepsilon_0\frac{E(t)}{E(0)}\\ &- \lambda_6\vartheta(t)H\left(\varepsilon_0\frac{E(t)}{E(0)}\right) - \frac{\lambda_6\vartheta(t)}{\mu_1|\Gamma_1^2|}E'(t) + \delta_0 E'(t)\\ &\le -(\gamma_1 E(0) - \lambda_6\varepsilon_0)\vartheta(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\frac{E(t)}{E(0)} + \left(\delta_0 - \frac{\lambda_6\vartheta(0)}{\mu_1|\Gamma_1^2|}\right)E'(t).\end{aligned}$$

Now choose $\varepsilon_0$ so small that $\lambda_7 := \gamma_1 E(0) - \lambda_6\varepsilon_0 > 0$, and let $\delta_0$ be large enough such that $\delta_0 - \frac{\lambda_6\vartheta(0)}{\mu_1|\Gamma_1^2|} > 0$. Hence, we find

$$\begin{aligned}&\left\{H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{K}_2(t)+\delta_0E(t)\right\}' \le -\lambda_7\vartheta(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\frac{E(t)}{E(0)}\\&= -\lambda_7\vartheta(t)\mathcal{R}\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\ \ \forall t\ge t_0.\end{aligned}\tag{61}$$

At this point, we consider

$$\mathcal{K}_3(t)=\begin{cases}\mathcal{K}_1(t), & \text{if H is linear on } [0,\varepsilon],\\ H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{K}_2(t)+\delta_0E(t), & \text{if } H'(0)=0 \text{ and } H''>0 \text{ on } (0,\varepsilon].\end{cases}$$

Then, from (56) and (61), we get

$$\mathcal{K}_3'(t)\le -\lambda_8\vartheta(t)\mathcal{R}\left(\varepsilon_0\frac{E(t)}{E(0)}\right),\forall t\ge t_0.$$

Since $\mathcal{K}_3(t)\sim E(t)$, $\exists \xi_1,\xi_2>0$ such that

$$\xi_1\mathcal{K}_3(t)\le E(t)\le \xi_2\mathcal{K}_3(t).\tag{62}$$

Introducing the functional

$$\mathcal{B}(t):=\xi_1\frac{\mathcal{K}_3(t)}{E(0)},\tag{63}$$

we have

$$\mathcal{B}(t)\le\frac{E(t)}{E(0)}<1.\tag{64}$$

According (48), (63), (64) and since $H$ is a increasing finction, we find

$$\begin{aligned}\mathcal{B}'(t)&\le -\frac{\xi_1\lambda_8}{E(0)}\vartheta(t)\mathcal{R}\left(\frac{E(t)}{E(0)}\right)\\&\le -\lambda_9\vartheta(t)\mathcal{R}\left(\frac{E(t)}{E(0)}\right).\end{aligned}\tag{65}$$

By integrating (65) over $(t_0,t)$ and using $H'(t)=-\frac{1}{\mathcal{R}(t)}$, we have

$$H(\mathcal{B}(t))-H(\mathcal{B}(0))\ge\lambda_9\int_{t_0}^t\vartheta(\sigma)\mathrm{d}\sigma.\tag{66}$$

Since, $H^{-1}$ is decreasing functional, we obtain

$$\mathcal{B}(t)\le H^{-1}\left(H(\mathcal{B}(0))+\lambda_9\int_{t_0}^t\vartheta(\sigma)\mathrm{d}\sigma\right),\ \ \forall t\ge t_0.\tag{67}$$

Now $\mathcal{B}(t)\sim E(t)$, yields (47). The proof is complete. □

## 4. CONCLUSION

The purpose of this work was to study the general decay of solutions for a viscoelastic wave equations with distributed delay in boundary feedback. This type of problem is frequently found in some mathematical models in applied sciences. Especially in the theory of viscoelasticity. In the next work, we try to add other dampings and terms (Balakrishnan-Taylor damping, dispersion and Logarithmic terms).

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