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DEGENERATE STIRLING NUMBERS AND A FAMILY OF BELL POLYNOMIALS

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Abstract. In this paper, we employ generating functions' techniques to obtain some identities involving degenerate Bell polynomials, multivariate Bell polynomials, and Carlitz degenerate Stirling numbers. Moreover, we obtain some formulas related to an explicit representation and recurrence relations for Lah polynomials.

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1. INTRODUCTION

We follow the notation and terminology used in [6]. For any $\lambda \in \mathbb{R}$, the degenerate exponential functions are defined by

$$(1+\lambda z)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{z^n}{n!}, \tag{1}$$

where $(x)_{n,\lambda}$ denotes the generalized factorial of x of order n and increment λ defined by $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$, for $n \geq 1$. It is clear that $\lim\limits_{\lambda\to 0}(1+\lambda z)^{\frac{x}{\lambda}} = e^{xz}$.

The (signed) Stirling numbers of the first kind $s(n,k)$ appear as the coefficients in the expansion

$$(x)_n = \sum_{k=0}^{n} s(n,k) x^k, \tag{2}$$

where $(x)_n := (x)_{n,1}$.

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As the inversion formula of (2), the Stirling numbers of the second kind $S(n,k)$ are defined as the coefficients in the expansion

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k. \tag{3}$$

The exponential generating functions are respectively

$$\frac{1}{k!}\left(\ln\left(1+z\right)\right)^k = \sum_{n=k}^{\infty} s\left(n,k\right)\frac{z^n}{n!} \tag{4}$$

and

$$\frac{1}{k!}\left(e^z-1\right)^k = \sum_{n=k}^{\infty} S\left(n,k\right)\frac{z^n}{n!}. \tag{5}$$

The Stirling number of the second kind $S(n,k)$ counts the number of ways of partitioning a set of n elements into k non-empty subsets. The (total) number of partitions of a set of n elements is the Bell number B_n, $(n \geq 0)$. Thus we note that

$$\mathrm{B}_n = \sum_{k=0}^{n} S(n,k), \quad (n \geq 0). \tag{6}$$

Further, the Bell polynomials are given by

$$\phi_n(x) = \sum_{k=0}^{n} S(n,k)x^k, \tag{7}$$

with the exponential generating function

$$\exp(x(e^z-1)) = \sum_{n=0}^{\infty} \phi_n(x)\frac{z^n}{n!}. \tag{8}$$

See [3, 4, 8] for more details.

Carlitz in [5] introduced the degenerate Stirling numbers and proved numerous properties. Recall that the degenerate Stirling numbers of first kind are defined as the coefficients in the expansion

$$(x)_n = \sum_{k=0}^{n} s_\lambda(n,k)(x)_{k,\lambda}, \tag{9}$$

with the exponential generating function

$$\frac{1}{k!}\left(\frac{(1+z)^\lambda-1}{\lambda}\right)^k = \sum_{n=k}^{\infty} s_\lambda(n,k)\frac{z^n}{n!}. \tag{10}$$

As the inversion formula of (9), the degenerate Stirling numbers of the second kind appear as the coefficients in the expansion

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_\lambda(n,k)(x)_k, \tag{11}$$

with the exponential generating function

$$\frac{1}{k!}\left((1+\lambda z)^{\frac{1}{\lambda}}-1\right)^k = \sum_{n=k}^{\infty} S_\lambda(n,k)\frac{z^n}{n!} \tag{12}$$

and given explicitly by [9]

$$S_\lambda(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(j)_{n,\lambda}. \tag{13}$$

The degenerate Bell polynomials $\phi_{n,\lambda}(x)$ are defined in [9, p. 212, Equation (4.3)] (see also [11]) by

$$\phi_{n,\lambda}(x) = \sum_{k=0}^{n} S_\lambda(n,k)x^k. \tag{14}$$

Clearly, we have

$$\exp\left(x\left((1+\lambda z)^{\frac{1}{\lambda}}-1\right)\right) = \sum_{n=0}^{\infty}\phi_{n,\lambda}(x)\frac{z^n}{n!}. \tag{15}$$

As pointed out by Carlitz [5], $S_{-1}(n,k) = s_{-1}(n,k) = L(n,k)$, where $L(n,k)$ denote the Lah numbers [16] given explicitly by

$$L(n,k) = \frac{n!}{k!}\binom{n-1}{k-1}.$$

Further, the (unsigned) Lah polynomials $L_n(x)$ are given by $L_n(x) := \phi_{n,-1}(x)$.

The (exponential) partial Bell partition polynomials $B_{n,k}(x_1,x_2,\ldots)$ in an infinite number of variables x_j, $(j \geq 1)$, were introduced by Bell [1] as a mathematical tool for representing the n-th derivative of composite functions. They are defined by their generating function

$$\sum_{n=k}^{\infty} B_{n,k}(x_1,x_2,\ldots)\frac{z^n}{n!} = \frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m\frac{z^m}{m!}\right)^k \tag{16}$$

and are given explicitly by the formula

$$B_{n,k}(x_1,x_2,\ldots,x_n) = \sum_{\pi(n,k)}\frac{n!}{k_1!\cdots k_n!}\left(\frac{x_1}{1!}\right)^{k_1}\left(\frac{x_2}{2!}\right)^{k_2}\cdots\left(\frac{x_n}{n!}\right)^{k_n}, \tag{17}$$

where

$$\pi(n,k) = \left\{(k_1,\ldots,k_n)\in\mathbb{N}^n : \sum_{i=1}^{n}k_i = k,\ \sum_{i=1}^{n}ik_i = n\right\}.$$

It is easy to obtain the following expression:

$$B_{n,k}(abx_1, ab^2x_2, \ldots, ab^nx_n) = a^kb^nB_{n,k}(x_1, x_2, \ldots, x_n). \tag{18}$$

It is well-known that for appropriate choices of the variables x_j, the (exponential) partial Bell partition polynomials can be reduced to some special combinatorial sequences. We will mention the following special cases:

$s(n,k) = B_{n,k}(0!, -1!, 2!, -3!, \ldots)$, (signed) Stirling numbers of the first kind,

$S(n,k) = B_{n,k}(1, 1, 1, \ldots)$, Stirling numbers of the second kind,

$L(n,k) = B_{n,k}(1!, 2!, 3!, \ldots)$, (unsigned) Lah numbers.

The (exponential) complete Bell partition polynomials are defined by

$$\exp\left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m!}\right) = \sum_{n=0}^{\infty} B_n(x_1, x_2, \ldots, x_n)\frac{z^n}{n!}, \tag{19}$$

from which, it follows that

$$B_n(x_1, x_2, \cdots, x_n) = \sum_{k=0}^{n} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}). \tag{20}$$

For more details, we refer the reader to [6, 7, 12, 13].

In the present paper, we study three families of sequences, the degenerate Bell polynomials, the multivariate Bell polynomials and Carlitz degenerate Stirling numbers. We employ generating functions' techniques to obtain some identities involving these sequences. Moreover, we obtain some formulas related to an explicit representation and recurrence relations for Lah polynomials.

2. DEGENERATE BELL POLYNOMIALS AND MULTIVARIATE BELL POLYNOMIALS

First, we derive some basic identities for the degenerate Stirling numbers and degenerate Bell polynomials.

LEMMA 2.1. *For $n \geq 0$, we have*

$$B_{n,k}\left((\lambda)_1, (\lambda)_2, \ldots, (\lambda)_{n-k+1}\right) = \lambda^n S_{\frac{1}{\lambda}}(n,k). \tag{21}$$

Proof. Using generating function (12), we obtain

$$\begin{aligned}\sum_{n=k}^{\infty} S_{\frac{1}{\lambda}}(n,k)\frac{z^n}{n!} &= \frac{1}{k!}\left(\left(1+\frac{1}{\lambda}z\right)^{\lambda} - 1\right)^k \\ &= \frac{1}{k!}\left(\sum_{m=1}^{\infty}\frac{1}{\lambda^m}\binom{\lambda}{m}z^m\right)^k \\ &= \frac{1}{k!}\left(\sum_{m=1}^{\infty}\frac{(\lambda)_m}{\lambda^m}\frac{z^m}{m!}\right)^k.\end{aligned}$$

By using (16), we obtain

$$\sum_{n=k}^{\infty} S_{\frac{1}{\lambda}}(n,k)\frac{z^n}{n!} = \sum_{n=k}^{\infty} B_{n,k}\left(\frac{(\lambda)_1}{\lambda}, \frac{(\lambda)_2}{\lambda^2}, \dots\right)\frac{z^n}{n!}.$$

Equating the coefficients of $\frac{z^n}{n!}$ in both sides of the last expression and using (18), we get

$$\begin{aligned} S_{\frac{1}{\lambda}}(n,k) &= B_{n,k}\left(\frac{(\lambda)_1}{\lambda}, \frac{(\lambda)_2}{\lambda^2}, \dots\right) \\ &= \frac{1}{\lambda^n} B_{n,k}\left((\lambda)_1, (\lambda)_2, \dots\right). \end{aligned}$$

This completes the proof. □

REMARK 2.2. Notice that the formula (21) simplifies the result obtained more recently by Qi et *al.* in [14].

As consequence of Lemma 2.1, we have

$$S_\lambda(n,k) = \lambda^n B_{n,k}\left(\left(\frac{1}{\lambda}\right)_1, \left(\frac{1}{\lambda}\right)_2, \dots, \left(\frac{1}{\lambda}\right)_{n-k+1}\right) \tag{22}$$

and

$$L(n,k) = (-1)^n B_{n,k}\left((-1)_1, (-1)_2, \dots, (-1)_{n-k+1}\right).$$

LEMMA 2.3. *For $n \geq 0$, we have*

$$B_n\left(\frac{x}{\lambda}(\lambda)_1, \frac{x}{\lambda}(\lambda)_2, \dots, \frac{x}{\lambda}(\lambda)_n\right) = \lambda^n \phi_{n,\frac{1}{\lambda}}\left(\frac{x}{\lambda}\right). \tag{23}$$

Proof. From (20) and (18), we obtain

$$\begin{aligned} B_n\left(\frac{x}{\lambda}(\lambda)_1, \frac{x}{\lambda}(\lambda)_2, \dots, \frac{x}{\lambda}(\lambda)_n\right) &= \sum_{k=0}^{n} B_{n,k}\left(\frac{x}{\lambda}(\lambda)_1, \frac{x}{\lambda}(\lambda)_2, \dots, \frac{x}{\lambda}(\lambda)_n\right) \\ &= \sum_{k=0}^{n} B_{n,k}\left((\lambda)_1, (\lambda)_2, \dots, (\lambda)_n\right)\left(\frac{x}{\lambda}\right)^k. \end{aligned}$$

From Lemma 2.1, we get

$$B_n\left(\frac{x}{\lambda}(\lambda)_1, \frac{x}{\lambda}(\lambda)_2, \dots, \frac{x}{\lambda}(\lambda)_n\right) = \lambda^n \sum_{k=0}^{n} S_{\frac{1}{\lambda}}(n,k)\left(\frac{x}{\lambda}\right)^k.$$

Now, using (14), we get the desired result (23) asserted by Lemma 2.3. □

THEOREM 2.4. *The following holds:*

$$B_n\left(x(x)_1, x(x)_2, \dots, x(x)_n\right) = \sum_{k=0}^{n} s(n,k)\phi_k(x)\, x^k. \tag{24}$$

Proof. By using (4) and (8), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}s\left(n,k\right)\phi_{k}\left(x\right)x^{k}\right)\frac{z^{n}}{n!}&=\sum_{k=0}^{\infty}\phi_{k}\left(x\right)x^{k}\sum_{n=k}^{\infty}s\left(n,k\right)\frac{z^{n}}{n!}\\&=\sum_{k=0}^{\infty}\phi_{k}\left(x\right)\frac{1}{k!}\left(x\ln\left(1+z\right)\right)^{k}\\&=\exp\left(x\left(\left(1+z\right)^{x}-1\right)\right).\end{aligned}$$

Thus, by applying the assertion (19), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}s\left(n,k\right)\phi_{k}\left(x\right)x^{k}\right)\frac{z^{n}}{n!}&=\exp\left(\sum_{m=1}^{\infty}x\left(x\right)_{m}\frac{z^{m}}{m!}\right)\\&=\sum_{n=0}^{\infty}B_{n}\left(x\left(x\right)_{1},x\left(x\right)_{2},\ldots,x\left(x\right)_{n}\right)\frac{z^{n}}{n!}.\end{aligned}$$

Equating the coefficients of $\frac{z^n}{n!}$ in both sides of the above equation, we get the desired result. □

COROLLARY 2.5. *For x nonzero, we have*

$$\sum_{k=0}^{n}s(n,k)\phi_{k}\left(x\right)x^{k}=x^{n}\phi_{n,\frac{1}{x}}(x). \tag{25}$$

Proof. Setting $\lambda := x$ in (21), multiplying both sides by x^k, and summing the resulting expression for $k = 0, 1, \ldots, n$, we obtain

$$\sum_{k=0}^{n}x^{k}B_{n,k}\left(\left(x\right)_{1},\left(x\right)_{2},\ldots\right)=x^{n}\sum_{k=0}^{n}S_{\frac{1}{x}}\left(n,k\right)x^{k},$$

which, by virtue of (18) and (14) yields

$$\sum_{k=0}^{n}B_{n,k}\left(x\left(x\right)_{1},x\left(x\right)_{2},\ldots\right)=x^{n}\phi_{n,\frac{1}{x}}(x).$$

Finally, by using (20) and (24), we obtain the desired formula (25). □

THEOREM 2.6. *For $n \geq 0$, the following relations hold true:*

$$S_{\lambda}(n,k)=B_{n,k}\left(\left(1\right)_{1,\lambda},\left(1\right)_{2,\lambda},\ldots,\left(1\right)_{n,\lambda}\right) \tag{26}$$

and

$$\phi_{n,\lambda}(x)=B_{n}\left(x\left(1\right)_{1,\lambda},x\left(1\right)_{2,\lambda},\ldots,x\left(1\right)_{n,\lambda}\right). \tag{27}$$

Proof. Using the relation

$$(1)_{n,\lambda}=\lambda^{n}\left(\frac{1}{\lambda}\right)_{n},$$

$B_n\left(x\,(1)_{1,\lambda}\,, x\,(1)_{2,\lambda}\,, \ldots, x\,(1)_{n,\lambda}\right)$ may be written as

$$B_n\left(x\,(1)_{1,\lambda}\,, x\,(1)_{2,\lambda}\,, \ldots, x\,(1)_{n,\lambda}\right) = \sum_{k=0}^{n} B_{n,k}\left(x\lambda\left(\frac{1}{\lambda}\right)_1, x\lambda^2\left(\frac{1}{\lambda}\right)_2, \ldots, x\lambda^{n-k+1}\left(\frac{1}{\lambda}\right)_{n-k+1}\right).$$

Thus, according to (18) and Lemma 2.1, it follows that

$$B_n\left(x\,(1)_{1,\lambda}\,, x\,(1)_{2,\lambda}\,, \ldots, x\,(1)_{n,\lambda}\right) = \sum_{k=0}^{n} S_\lambda(n,k)x^k = \phi_{n,\lambda}(x).$$

□

Thus, setting $\lambda = -1$ in (26) and (27), we get the following explicit representations for Lah numbers and Lah polynomials involving multivariate Bell polynomials.

Corollary 2.7. *We have*

$$L\,(n,k) = B_{n,k}\left(\langle 1\rangle_1\,, \langle 1\rangle_2\,, \ldots\right)$$

and

$$L_n\,(x) = B_n\left(x\,\langle 1\rangle_1\,, x\,\langle 1\rangle_2\,, \ldots\right),$$

where $\langle x\rangle_n := (x)_{n,-1}$.

From the general theory of partition polynomials [6, p. 414, Theorem 11.2], one can deduce the following recurrence relations.

Theorem 2.8. *For* $n, k \geq 0$, *the degenerate Stirling numbers satisfy the recurrence relations*

$$S_\lambda\,(n+1,k+1) = \sum_{i=0}^{n-k}\binom{n}{i}(1)_{i+1,\lambda}\,S_\lambda\,(n-i,k)$$

and

$$S_\lambda\,(n+1,k+1) = \frac{1}{k+1}\sum_{i=0}^{n-k}\binom{n+1}{i+1}(1)_{i+1,\lambda}\,S_\lambda\,(n-i,k)\,.$$

Substituting $\lambda = -1$ into Theorem 2.8, we get the following recurrence relations for Lah numbers.

Corollary 2.9. *We have*

$$L\,(n+1,k+1) = \binom{n}{i}(i+1)!L\,(n-i,k)$$

and

$$L\,(n+1,k+1) = \frac{1}{k+1}\sum_{i=0}^{n-k}\binom{n+1}{i+1}(i+1)!L\,(n-i,k)\,.$$

THEOREM 2.10. *For $n \geq 0$, the degenerate Bell polynomials $\phi_{n,\lambda}(x)$ satisfy the recurrence relation*

$$\phi_{n+1,\lambda}(x) = x \sum_{k=0}^{n} \binom{n}{k} (1)_{k+1,\lambda} \, \phi_{n-k,\lambda}(x) . \tag{28}$$

By setting $\lambda = -1$ in (28), we obtain the following corollary.

COROLLARY 2.11. *We have*

$$L_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} (k+1)! L_{n-k}(x) .$$

3. BELL POLYNOMIALS AND DEGENERATE BELL POLYNOMIALS

An explicit formula of Bell polynomials $\phi_n(x)$ involving degenerate Bell polynomials $\phi_{n,\lambda}(x)$ is given in the following theorem.

THEOREM 3.1. *For $n \geq 0$, we have*

$$\phi_n(x) = \sum_{k=0}^{n} S(n,k) \, \phi_{k,\lambda}(x) \, \lambda^{n-k}. \tag{29}$$

Proof. Generating function (8) may be expressed as

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\phi_n(x)}{\lambda^n} \frac{z^n}{n!} &= \exp\left(x\left(e^{\frac{z}{\lambda}} - 1\right)\right) \\
&= \exp\left(x\left(\left(1 + \lambda\left(\frac{e^z - 1}{\lambda}\right)\right)^{\frac{1}{\lambda}} - 1\right)\right) \\
&= \sum_{k=0}^{\infty} \phi_{k,\lambda}(x) \frac{1}{k!} \left(\frac{e^z - 1}{\lambda}\right)^k \\
&= \sum_{k=0}^{\infty} \frac{\phi_{k,\lambda}(x)}{\lambda^k} \frac{1}{k!} \left(e^z - 1\right)^k .
\end{aligned}$$

Using (5), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\phi_n(x)}{\lambda^n} \frac{z^n}{n!} &= \sum_{k=0}^{\infty} \frac{\phi_{k,\lambda}(x)}{\lambda^k} \sum_{n=k}^{\infty} S(n,k) \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} S(n,k) \frac{\phi_{k,\lambda}(x)}{\lambda^k} \right) \frac{z^n}{n!}.
\end{aligned}$$

Equating the coefficients of $\frac{z^n}{n!}$ in both sides of the above equation, we get the desired result. $\square$

Corollary 3.2. *We have*

$$\phi_n(x) = \sum_{k=0}^{n} (-1)^{n-k} S(n,k) L_k(x).$$

Remark 3.3. Note that, the above identity, can be found in [2] for the (signed) Lah polynomials.

Now, by means of the Stirling transform [15], we obtain the following theorem.

Theorem 3.4. *For $n \geq 0$, we have*

$$\phi_{n,\lambda}(x) = \sum_{k=0}^{n} s(n,k) \phi_k(x) \lambda^{n-k}.$$

Note that, Kargın and Corcino [10] obtained the recurrence relation for the generalized exponential polynomials, in particular for the degenerate Bell polynomials. However, we give here a different proof.

Corollary 3.5. *For $n \geq 0$, we have*

$$\phi_{n+1,\lambda}(x) = (x - n\lambda) \phi_{n,\lambda}(x) + x\phi'_{n,\lambda}(x).$$

Proof. From the recurrence relation for the (signed) Stirling numbers of the first kind

$$s(n+1,k) = s(n,k-1) - ns(n,k),$$

we have

$$\begin{aligned} \frac{\phi_{n+1,\lambda}(x)}{\lambda^{n+1}} &= \sum_{k=1}^{n+1} s(n+1,k) \frac{\phi_k(x)}{\lambda^k} \\ &= \sum_{k=0}^{n} s(n,k) \frac{\phi_{k+1}(x)}{\lambda^{k+1}} - n\sum_{k=0}^{n} s(n,k) \frac{\phi_k(x)}{\lambda^k}. \end{aligned}$$

Using the well-known relation

$$\phi_{n+1}(x) = x\left(1 + \frac{\mathrm{d}}{\mathrm{d}x}\right)\phi_n(x),$$

we obtain

$$\begin{aligned} \frac{\phi_{n+1,\lambda}(x)}{\lambda^{n+1}} &= x\sum_{k=0}^{n} s(n,k) \frac{1}{\lambda^{k+1}} \left(1 + \frac{\mathrm{d}}{\mathrm{d}x}\right)\phi_k(x) - n\sum_{k=0}^{n} s(n,k) \frac{\phi_k(x)}{\lambda^k} \\ &= \frac{x}{\lambda}\sum_{k=0}^{n} s(n,k) \frac{\phi_k(x)}{\lambda^k} + \frac{x}{\lambda}\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{k=0}^{n} s(n,k) \frac{\phi_k(x)}{\lambda^k}\right) \\ &- n\sum_{k=0}^{n} s(n,k) \frac{\phi_k(x)}{\lambda^k}. \end{aligned}$$

Thus

$$\frac{\phi_{n+1,\lambda}(x)}{\lambda^{n+1}} = \frac{x}{\lambda}\frac{\phi_{n,\lambda}(x)}{\lambda^n} + \frac{x}{\lambda}\frac{\phi'_{n,\lambda}(x)}{\lambda^n} - n\frac{\phi_{n,\lambda}(x)}{\lambda^n}.$$

This completes the proof of Corollary 3.5 . □

Corollary 3.6. *For $n \geq 0$, we have*

$$L_{n+1}(x) = (x+n)L_n(x) + xL'_n(x).$$

4. DEGENERATE BELL POLYNOMIALS AND STIRLING NUMBERS

In order to obtain an explicit formula of the degenerate Bell polynomials $\phi_{n,\lambda}(x)$ in terms of Stirling numbers of the second kind $S(n,k)$, we consider the following exponential generating function

$$(1+\lambda\ln(1+z))^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty}\alpha_{n,\lambda}\frac{z^n}{n!}. \tag{30}$$

From the binomial theorem, we have

$$\begin{aligned}(1+\lambda\ln(1+z))^{\frac{1}{\lambda}} &= \sum_{m=0}^{\infty}\left(\frac{1}{\lambda}\right)_m\frac{\lambda^m}{m!}(\ln(1+z))^m\\ &= \sum_{m=0}^{\infty}(1)_{m,\lambda}\sum_{n=m}^{\infty}s(n,m)\frac{z^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(1)_{m,\lambda}s(n,m)\right)\frac{z^n}{n!}.\end{aligned}$$

Then, we obtain

$$\alpha_{n,\lambda} = \sum_{m=0}^{n}(1)_{m,\lambda}s(n,m) \tag{31}$$

which can be computed by the following recurrence relation: if we construct an infinite matrix $(A_{n,m}(\lambda))_{n,m\geq 0}$ with the final sequence given by $A_{n,0}(\lambda) = (1)_{n,\lambda}$ and each entry given by

$$A_{n,m+1}(\lambda) = A_{n+1,m}(\lambda) - mA_{n,m}(\lambda),$$

then the first line of the matrix is $A_{0,m}(\lambda) = \alpha_{m,\lambda}$.

Theorem 4.1. *For $n \geq 0$, we have*

$$\phi_{n,\lambda}(x) = \sum_{k=0}^{n}S(n,k)B_k(x\alpha_{1,\lambda}, x\alpha_{2,\lambda}, \ldots, x\alpha_{k,\lambda}). \tag{32}$$

Proof. By using (4) and (15), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}s\left(n,k\right)\phi_{k,\lambda}\left(x\right)\right)\frac{z^{n}}{n!}&=\sum_{k=0}^{\infty}\phi_{k,\lambda}\left(x\right)\left(\sum_{n=k}^{\infty}s\left(n,k\right)\right)\frac{z^{n}}{n!}\\&=\sum_{k=0}^{\infty}\phi_{k,\lambda}\left(x\right)\frac{1}{k!}\left(\ln\left(1+z\right)\right)^{k}\\&=\exp\left(x\left(\left(1+\lambda\ln\left(1+z\right)\right)^{\frac{1}{\lambda}}-1\right)\right).\end{aligned}$$

We now apply (30) and (19). Henceforth, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}s\left(n,k\right)\phi_{k,\lambda}\left(x\right)\right)\frac{z^{n}}{n!}&=\exp\left(x\sum_{m=1}^{\infty}\alpha_{m,\lambda}\frac{z^{m}}{m!}\right)\\&=\sum_{n=0}^{\infty}B_{n}\left(x\alpha_{1,\lambda},x\alpha_{2,\lambda},\ldots,x\alpha_{n,\lambda}\right)\frac{z^{n}}{n!}.\end{aligned}$$

Equating the coefficients of $\frac{z^n}{n!}$ in both sides of the above expression, we get

$$\sum_{k=0}^{n}s\left(n,k\right)\phi_{k,\lambda}\left(x\right)=B_{n}\left(x\alpha_{1,\lambda},x\alpha_{2,\lambda},\ldots,x\alpha_{n,\lambda}\right).$$

Now, by means of the Stirling transform [15], we obtain the desired result. □

5. BELL POLYNOMIALS AND DEGENERATE STIRLING NUMBERS

In this section, we derive several interesting identities between Bell polynomials and degenerate Stirling numbers.

THEOREM 5.1. *For $n \geq 0$, we have*

$$\phi_{n}\left(x\right)=\sum_{k=0}^{n}s_{\lambda}\left(n,k\right)B_{k}\left(x\phi_{1,\lambda},x\phi_{2,\lambda},\ldots,x\phi_{k,\lambda}\right),\tag{33}$$

where $\phi_{n,\lambda} := \phi_{n,\lambda}(1)$ denote the degenerate Bell numbers.

Proof. By using (12) and (8), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}S_{\lambda}\left(n,k\right)\phi_{k}\left(x\right)\right)\frac{z^{n}}{n!}&=\sum_{k=0}^{\infty}\phi_{k}\left(x\right)\sum_{n=k}^{\infty}S_{\lambda}\left(n,k\right)\frac{z^{n}}{n!}\\&=\sum_{k=0}^{\infty}\phi_{k}\left(x\right)\frac{1}{k!}\left(\left(1+\lambda z\right)^{\frac{1}{\lambda}}-1\right)^{k}\\&=\exp\left(x\left(e^{\left(1+\lambda z\right)^{\frac{1}{\lambda}}-1}-1\right)\right).\end{aligned}$$

Thus, by applying the assertion (19), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}S_{\lambda}\left(n,k\right)\phi_{k}\left(x\right)\right)\frac{z^{n}}{n!}&=\exp\left(x\sum_{m=1}^{\infty}\phi_{m,\lambda}\frac{z^{m}}{m!}\right)\\&=\sum_{n=0}^{\infty}B_{n}\left(x\phi_{1,\lambda},x\phi_{2,\lambda},\ldots,x\phi_{n,\lambda}\right)\frac{z^{n}}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$, we obtain

$$\sum_{k=0}^{n}S_{\lambda}\left(n,k\right)\phi_{k}\left(x\right)=B_{n}\left(x\phi_{1,\lambda},x\phi_{2,\lambda},\ldots,x\phi_{n,\lambda}\right).$$

The inversion of the last formula gives (33). □

Corollary 5.2. *For $n\geq 0$, the following hold:*

$$\phi_{n}\left(x\right)=\sum_{k=0}^{n}L\left(n,k\right)B_{k}\left(xL_{1}\left(1\right),xL_{2}\left(1\right),\ldots\right).$$

Theorem 5.3. *For $n\geq 0$, we have*

$$\phi_{n}\left(x\right)=\sum_{k=0}^{n}S_{\lambda}\left(n,k\right)\lambda^{k}B_{k}\left(x\phi_{1,\frac{1}{\lambda}}\left(\frac{1}{\lambda}\right),x\phi_{2,\frac{1}{\lambda}}\left(\frac{1}{\lambda}\right),\ldots,x\phi_{k,\frac{1}{\lambda}}\left(\frac{1}{\lambda}\right)\right).$$

Proof. By using (10) and (8), we find

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}s_{\lambda}\left(n,k\right)\phi_{k}\left(x\right)\right)\frac{z^{n}}{n!}&=\sum_{k=0}^{\infty}\phi_{k}\left(x\right)\sum_{n=k}^{\infty}s_{\lambda}\left(n,k\right)\frac{z^{n}}{n!}\\&=\sum_{k=0}^{\infty}\phi_{k}\left(x\right)\frac{1}{k!}\left(\frac{(1+z)^{\lambda}-1}{\lambda}\right)^{k}\\&=\exp\left(x\left(e^{\frac{1}{\lambda}\left((1+z)^{\lambda}-1\right)}-1\right)\right).\end{aligned}$$

We make use of Lemma 2.3, to get

$$e^{\frac{1}{\lambda}\left((1+z)^{\lambda}-1\right)}-1=\sum_{n=1}^{\infty}\lambda^{n}\phi_{n,\frac{1}{\lambda}}\left(\frac{1}{\lambda}\right)\frac{z^{n}}{n!}.$$

Now, it easy to see that

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}s_{\lambda}\left(n,k\right)\phi_{k}\left(x\right)\right)\frac{z^{n}}{n!}=\exp\left(x\sum_{m=1}^{\infty}\lambda^{m}\phi_{m,\frac{1}{\lambda}}\left(\frac{1}{\lambda}\right)\frac{z^{m}}{m!}\right)$$
$$=\sum_{n=0}^{\infty}B_{n}\left(x\lambda\phi_{1,\frac{1}{\lambda}}\left(\frac{1}{\lambda}\right),\ldots,x\lambda^{n}\phi_{n,\frac{1}{\lambda}}\left(\frac{1}{\lambda}\right)\right)\frac{z^{n}}{n!}$$
$$=\sum_{n=0}^{\infty}\lambda^{n}B_{n}\left(x\phi_{1,\frac{1}{\lambda}}\left(\frac{1}{\lambda}\right),\ldots,x\phi_{n,\frac{1}{\lambda}}\left(\frac{1}{\lambda}\right)\right)\frac{z^{n}}{n!}.$$

Comparing the coefficients of $\frac{z^n}{n!}$, we obtain

$$\sum_{k=0}^{n}s_{\lambda}\left(n,k\right)\phi_{k}\left(x\right)=\lambda^{n}B_{n}\left(x\phi_{1,\frac{1}{\lambda}}\left(\frac{1}{\lambda}\right),x\phi_{2,\frac{1}{\lambda}}\left(\frac{1}{\lambda}\right),\ldots,x\phi_{n,\frac{1}{\lambda}}\left(\frac{1}{\lambda}\right)\right).$$

Finally, by means of the degenerate Stirling transform, we get the desired result. □

COROLLARY 5.4. *We have*

$$\phi_{n}\left(x\right)=\sum_{k=0}^{n}\left(-1\right)^{k}L\left(n,k\right)B_{k}\left(xL_{1}\left(-1\right),xL_{2}\left(-1\right),\ldots\right).$$

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