MATHEMATICA, 64 (87), N° 2, 2022, pp. 255–264

AN EXAMPLE OF NON-COFORMAL CLASSIFYING SPACE WITH RATIONAL $H(2)$-STRUCTURE

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Abstract. Let $Baut_1X$ and $Baut_1p$ be the Dold-Lashof classifying spaces of a space X and a fibration $p : X \to Y$, respectively. In this paper, we give an example that there exists a fibration $\xi : S^7 \times S^{11} \times S^{15} \times S^{19} \to X \stackrel{p}{\to} S^5$ such that $Baut_1X$ and $Baut_1p$ are not coformal and are rational $H(2)$-spaces.

MSC 2010. 55P62, 55R15.

Key words. Rational homotopy theory, Sullivan (minimal) model, derivation, classifying space for fibration, formal, coformal, $H(n)$-space.

1. INTRODUCTION

Let X be a connected and simply connected finite CW complex having $\dim \pi_*(X)_{\mathbb{Q}} < \infty$ ($G_{\mathbb{Q}} = G \otimes \mathbb{Q}$) and let $Baut_1X$ be the Dold-Lashof classifying space of orientable fibrations [2].

Here $aut_1X = map(X, X; id_X)$ is the identity component of the space $autX$ of self-equivalences of X. Then any orientable fibration ξ with fibre X over a base space B is the pull-back of a universal fibration by a map from B to $Baut_1X$ [2].

The Sullivan minimal model $M(X)$ ([18]) determines the rational homotopy type of X, the homotopy type of the rationalization X_0 [9] of X. The differential graded Lie algebra (DGL) $DerM(X)$, the negative derivations of $M(X)$ (see Section 2), gives rise to the DGL model for $Baut_1X$ due to Sullivan [18] (cf. [7, 19]), i.e., the spatial realization $||DerM(X)||$ is $(Baut_1X)_0$. Furthermore, for a fibration $\xi : F \to X \stackrel{p}{\to} Y$ with fiber F and base Y finite, let $aut_1p = \{f \in aut_1X \mid p \circ f = p\}$ be the monoid of fibrewise self-equivalences homotopic to the identity. Then the DGL model of the Dold-Lashof classifying space $Baut_1p$ is given by [1, Theorem 1]. See Theorem 2.2 in Section 2.

A simply connected CW complex Z of finite type is said to be *formal* if there is a DGA-quasi-isomorphism $M(Z) \to (H^*(Z; \mathbb{Q}), 0)$. Let $L(Z)$ be the Quillen DGL-model of Z [4].

The authors thank the referee for his helpful comments and suggestions.

DOI: 10.24193/mathcluj.2022.2.10

Definition 1.1 (cf. [19, II.7.(6)]). A space Z is said to be *coformal* if there is a DGL-quasi-isomorphism $L(Z) \to (\pi_*(\Omega Z)_{\mathbb{Q}}, 0)$.

This is equivalent to the fact that the differential d of a Sullivan minimal model $M(Z) = (\Lambda V, d)$ is quadratic.

For example, rational H-spaces and one-point unions of spheres are coformal [19]. Notice that $Baut_1 X$ is not formal even when $X = S^3 \times S^5$. In this case, we obtain that $M(Baut_1 X) = (\Lambda(x, y, z), d)$ with $|x| = 3$, $|y| = 4$, $|z| = 6$, $d(x) = d(y) = 0$ and $d(z) = xy$ from Theorem 2.1 of Section 2. Then the non-formality is induced by the element $[xz] \in H^9(Baut_1 X; \mathbb{Q})$. ($H^*(Baut_1 X; \mathbb{Q})$ is infinitely generated as a $\mathbb{Q}$-algebra.) But it is coformal since the differential d is quadratic. See [14, Theorem 4.1] for some non-coformal examples of the classifying space $Baut_1 X$.

A space X is said to be *pure* if $dM(X)^{even} = 0$ and $dM(X)^{odd} \subset M(X)^{even}$. A pure space is said to be an *F_0-space* (or *positively elliptic*) if $\dim \pi_{\text{even}}(X) \otimes \mathbb{Q} = \dim \pi_{\text{odd}}(X) \otimes \mathbb{Q}$ and $H^{\text{odd}}(X; \mathbb{Q}) = 0$.

In 1976, S. Halperin [8] conjectured that the Serre spectral sequences of all fibrations $X \to E \to B$ of simply connected CW complexes collapse at the E_2-terms for any F_0-space X [4]. For compact connected Lie groups G and H where H is a subgroup of G, when rank G = rank H, the homogeneous space G/H satisfies the Halperin conjecture [16]. Also, the conjecture is true when $n \leq 3$ [10]. Due to [12], the Halperin conjecture is relaxed to the form: *is $Baut_1 X$ a rational H-space if X is an F_0-space?*. Of course, even if X is not an F_0-space, $Baut_1 X$ can be a rational H-space. For example, $X = S^3$, $(Baut_1 X)_0 \simeq K(\mathbb{Q}, 4)$ [13]. When is $Baut_1 X$ or $Baut_1 p$ a rational H-space? ([1, 3.2]) Also how near is it to a rational H-space?

Definition 1.2 ([6]). A simply connected CW complex Z of finite type is an *$H(n)$-space* if there exists a map $\mu_n : G_n(Z \times Z) \to Z$ such that $\mu_n \circ i_n^l = \mu_n \circ i_n^r = p_n : G_n(Z) \to Z$. Here $p_n : G_n(Z) \to Z$ is the n-th Ganea fibration and i_n^l, $i_n^r : G_n(Z) \to G_n(Z \times Z)$ are the canonical maps induced by the standard injections of Z in $Z \times Z$.

Notice that Z is a rational $H(n)$-space (Z_0 is an $H(n)$-space) if and only if the word of length k of the differential $d = d_k + d_{k+1} + \cdots$ of $M(Z) = (\Lambda V, d)$ is bigger than n [6, Proposition 8]. Here $d_k : V \to \Lambda^k V = V \cdot \ldots \cdot V$ (k-times). Thus all spaces are rational $H(1)$-spaces and H-spaces are rational $H(\infty)$-spaces. Remark that a rational $H(m)$-space is a rational $H(n)$-space when $m > n$ and that coformal spaces are not rational $H(2)$-spaces. Recall the following problem stated in the Oberwolfach Workshop in 2009:

Problem 1.3 ([3, Problem 23]). Is $Baut_1 X$ a rational H-space if it is a rational $H(2)$-space ?

However, in this paper, we show that there exists a counter example, given by the following theorem.

THEOREM 1.4. *There exists a space X such that $Baut_1X$ is not coformal and is a rational $H(2)$-space. Furthermore, it is the total space of a fibration $\xi : S^7 \times S^{11} \times S^{15} \times S^{19} \to X \xrightarrow{p} S^5$ such that $Baut_1p$ is not coformal and is a rational $H(2)$-space, too.*

The Sullivan minimal models are given by

$$M(Baut_1X) = (\Lambda(v_2, v_3, v_9, v_{13}, v_{20}), d)$$
$$M(Baut_1p) = (\Lambda(v_2, v_3, v_5, v_9, v_{13}, v_{20}), d)$$

where $|v_n| = n$, $d(v_{13}) = v_2v_3v_9$ and $d(v_i) = 0$ for the other i. Remark that $M(\mathbb{C}P^2) = (\Lambda(v_2, v_5), d)$ where $d(v_2) = 0$ and $d(v_5) = v_2^3$. So $\mathbb{C}P^2$ is not coformal and is a rational $H(2)$-space, too. But it can not be realized as $(Baut_1X)_0$ [11, Theorem 2].

REMARK 1.5. In this case, $Baut_1X$ and $Baut_1p$ are not formal and the rational cohomologies are finitely generated as $\mathbb{Q}$-algebras:

$$H^*(Baut_1X; \mathbb{Q}) \cong \wedge(v_3, v_9) \otimes \mathbb{Q}[v_2, v_{20}, w_{16}, w_{22}]/\ I \quad \text{and}$$
$$H^*(Baut_1p; \mathbb{Q}) \cong \wedge(v_3, v_5, v_9) \otimes \mathbb{Q}[v_2, v_{20}, w_{16}, w_{22}]/\ I$$

where $w_{16} = [v_3v_{13}]$, $w_{22} = [v_9v_{13}]$ and I is the ideal generated by

$$\{v_2v_3v_9, v_3w_{16}, v_3w_{22} + v_9w_{16}, v_9w_{22}, w_{16}^2, w_{16}w_{22}, w_{22}^2\}.$$

2. MODELS

Let $M(Z) = (\Lambda V, d)$ be the Sullivan minimal model of simply connected CW complex Z of finite type [18]. It is a free $\mathbb{Q}$-commutative **d**ifferential **g**raded **a**lgebra (DGA) with a $\mathbb{Q}$-graded vector space $V = \bigoplus_{i \geq 1} V^i$ where $\dim V^i < \infty$ and a decomposable differential, i.e., $d(V^i) \subset (\Lambda^+V \cdot \Lambda^+V)^{i+1}$ and $d \circ d = 0$. Here Λ^+V is the ideal of ΛV generated by elements of positive degree. The degree of a homogeneous element x of a graded algebra is denoted as $|x|$. Then $xy = (-1)^{|x||y|}yx$ and $d(xy) = d(x)y + (-1)^{|x|}xd(y)$. Note that $M(X)$ determines the rational homotopy type of X, namely the spatial realization is given as $||M(Z)|| \simeq Z_0$. In particular,

$$V^n \cong \operatorname{Hom}(\pi_n(Z), \mathbb{Q}) \quad \text{and} \quad H^*(\Lambda V, d) \cong H^*(Z; \mathbb{Q}).$$

Here the second is an isomorphism of graded algebras. Refer to [4] for details.

Let $Der_iM(X)$ be the set of $\mathbb{Q}$-derivations of $M(X)$ decreasing the degree by i with $\sigma(xy) = \sigma(x)y + (-1)^{i \cdot |x|}x\sigma(y)$ for $x, y \in M(X)$. The boundary operator $\partial : Der_iM(X) \to Der_{i-1}M(X)$ is defined by

$$\partial(\sigma) = d \circ \sigma - (-1)^i \sigma \circ d$$

for $\sigma \in Der_iM(X)$. We denote $\oplus_{i>0} Der_iM(X)$ by $DerM(X)$ in which $Der_1M(X)$ is ∂-cycles. Then $DerM(X)$ is a DGL by the Lie bracket

$$[\sigma, \tau] := \sigma \circ \tau - (-1)^{|\sigma||\tau|}\tau \circ \sigma.$$

Furthermore, recall the definition of D. Tanré [19, p.25]: Let (L, ∂) be a DGL of finite type with positive degree. Then $C^*(L,\partial) := (\Lambda s^{-1}\sharp L, D = d_1 + d_2)$ with

$$\langle d_1 s^{-1}z; sx\rangle = -\langle z; \partial x\rangle \quad \text{and} \quad \langle d_2 s^{-1}z; sx_1, sx_2\rangle = (-1)^{|x_1|}\langle z; [x_1, x_2]\rangle,$$

where $\langle s^{-1}z; sx\rangle = (-1)^{|z|}\langle z; x\rangle$ and $\sharp L$ is the dual space of L.

Theorem 2.1 ([18, §11],[7, 19]). *For a Sullivan model $M(X) = (\Lambda V, d)$ of X, $Der(\Lambda V)$ is a DGL-model of $Baut_1 X$. Thus $C^*(Der(\Lambda V))$ is a free DGA-model of $Baut_1 X$.*

Consider the simply connected fibration $\xi : F \to X \xrightarrow{p} Y$ of finite type given by the relative model (Koszul-Sullivan extension)

$$M(Y) = (\Lambda V, d) \overset{i}{\hookrightarrow} (\Lambda V \otimes \Lambda W, D) \to (\Lambda W, \overline{D}) = M(F) \tag{1}$$

for a certain differential D with $D\,|_{\Lambda V} = d$. There is a quasi-isomorphism $M(X) \simeq (\Lambda V \otimes \Lambda W, D)$ [4]. Let $Der_{\Lambda V}(\Lambda V \otimes \Lambda W)$ be the sub-DGL in $Der(\Lambda V \otimes \Lambda W)$ of elements σ with $\sigma(v) = 0$ for $v \in V$.

Theorem 2.2 ([1, Theorem 1], [5]). *For a fibration $\xi : F \to X \xrightarrow{p} Y$ given by model (1) with F and Y finite, $Der_{\Lambda V}(\Lambda V \otimes \Lambda W)$ is a DGL-model of $Baut_1 p$. Thus $C^*(Der_{\Lambda V}(\Lambda V \otimes \Lambda W))$ is a free DGA-model of $Baut_1 p$.*

For a fibration ξ, there is a map $Baut_1 p \to Baut_1 X$ induced by the monoid inclusion $aut_1 p \hookrightarrow aut_1 X$. The DGL-map between DGL-models is given by the natural inclusion $Der_{\Lambda V}(\Lambda V \otimes \Lambda W) \hookrightarrow Der(\Lambda V \otimes \Lambda W)$.

3. THE PROOF

Convention 3.1 ([18]). For a free DGA-model $(\Lambda V, d)$, the symbol (v, f) means the *elementary derivation* that takes a generator v of V to an element f of ΛV and the other generators to 0. Note that $|(v, f)| = |v| - |f|$.

The proof of Theorem 1.4. Let the relative model of a fibration $S^7 \times S^{11} \times S^{15} \times S^{19} \to X \xrightarrow{p} S^5$ be given by

$$(\Lambda(x), 0) \to M(X) = (\Lambda(x, s_7, s_{11}, s_{15}, s_{19}), D) \to (\Lambda(s_7, s_{11}, s_{15}, s_{19}), 0)$$

where $|x| = 5$, $|s_n| = n$, $Dx = 0$, $Ds_7 = 0$, $Ds_{11} = xs_7$, $Ds_{15} = xs_{11}$, $Ds_{19} = xs_{15}$.

Let us calculate the DGA-model of $Baut_1 p$ by using Theorem 2.2. The basis of $Der_{\Lambda(x)} M(X)$ is given by the following 18-elements

$$\begin{gathered}(s_7,1),(s_7,x),(s_{11},1),(s_{11},x),(s_{11},s_7),(s_{15},1),(s_{15},x),(s_{15},s_7)\\ (s_{15},s_{11}),(s_{15},xs_7),(s_{19},1),(s_{19},x),(s_{19},s_7),(s_{19},s_{11})\\ (s_{19},s_{15}),(s_{19},xs_7),(s_{19},xs_{11}),(s_{19},s_7s_{11})\end{gathered}$$

and the differential ∂ is given by

$$\begin{aligned}
&\partial(s_{19},1)=0,\ \partial(s_{19},x)=0,\ \partial(s_{19},s_7)=0\\
&\partial(s_{19},s_{11})=(s_{19},xs_7),\ \partial(s_{19},s_{15})=(s_{19},xs_{11})\\
&\partial(s_{19},xs_7)=0,\ \partial(s_{19},xs_{11})=0,\ \partial(s_{19},s_7s_{11})=0\\
&\partial(s_{15},1)=-(s_{19},x),\ \partial(s_{15},x)=0,\ \partial(s_{15},s_7)=-(s_{19},xs_7)\\
&\partial(s_{15},s_{11})=(s_{15},xs_7)-(s_{19},xs_{11}),\ \partial(s_{15},xs_7)=0\\
&\partial(s_{11},1)=-(s_{15},x),\ \partial(s_{11},x)=0,\ \partial(s_{11},s_7)=-(s_{15},xs_7)\\
&\partial(s_7,1)=-(s_{11},x),\ \partial(s_7,x)=0.
\end{aligned}$$

Note that there are six non-exact ∂-cycles:

$$\begin{aligned}
&(s_{19},1),(s_{19},s_7),(s_{19},s_7s_{11}),(s_7,x)\\
&\sigma=(s_{19},s_{11})+(s_{15},s_7)\\
&\tau=(s_{19},s_{15})+(s_{15},s_{11})+(s_{11},s_7).
\end{aligned}$$

Let $v_{s_a,f}$ be the dual element of the derivation (s_a,f) for some $s_a\in W$ and $f\in\Lambda V\otimes\Lambda W$ with degree $+1$. Then d_1 of Section 2 is given by

$$\begin{aligned}
&d_1(v_{s_{19},1})=0,\ d_1(v_{s_{19},x})=-v_{s_{15},1},\ d_1(v_{s_{19},s_7})=0,\ d_1(v_{s_{19},s_{11}})=0\\
&d_1(v_{s_{19},s_{15}})=0,\ d_1(v_{s_{19},xs_7})=-v_{s_{19},s_{11}}+v_{s_{15},s_7}\\
&d_1(v_{s_{19},xs_{11}})=-v_{s_{19},s_{15}}+v_{s_{15},s_{11}},\ d_1(v_{s_{19},s_7s_{11}})=0\\
&d_1(v_{s_{15},1})=0,\ d_1(v_{s_{15},x})=-v_{s_{11},1},\ d_1(v_{s_{15},s_7})=0,\ d_1(v_{s_{15},s_{11}})=0\\
&d_1(v_{s_{15},xs_7})=-v_{s_{15},s_{11}}+v_{s_{11},s_7},\ d_1(v_{s_{11},1})=0\\
&d_1(v_{s_{11},x})=-v_{s_7,1},\ d_1(v_{s_{11},s_7})=0,\ d_1(v_{s_7,1})=0,\ d_1(v_{s_7,x})=0.
\end{aligned}$$

The Lie bracket of $Der_{\Lambda(x)}M(X)$ is given by

$$\begin{aligned}
&[(s_{15},1),(s_{19},s_{15})]=(s_{19},1),\ [(s_{11},1),(s_{19},s_{11})]=(s_{19},1),\\
&[(s_7,1),(s_{19},s_7)]=(s_{19},1),\ [(s_{15},x),(s_{19},s_{15})]=(s_{19},x),\\
&[(s_{11},x),(s_{19},s_{11})]=(s_{19},x),\ [(s_7,x),(s_{19},s_7)]=(s_{19},x),\\
&[(s_{11},1),(s_{19},xs_{11})]=-(s_{19},x),\ [(s_7,1),(s_{19},xs_7)]=-(s_{19},x),\\
&[(s_{15},s_7),(s_{19},s_{15})]=(s_{19},s_7),\ [(s_{11},s_7),(s_{19},s_{11})]=(s_{19},s_7),\\
&[(s_{11},1),(s_{19},s_7s_{11})]=-(s_{19},s_7),\ [(s_{15},s_{11}),(s_{19},s_{15})]=(s_{19},s_{11}),\\
&[(s_7,1),(s_{19},s_7s_{11})]=(s_{19},s_{11}),\ [(s_{11},s_7),(s_{19},xs_{11})]=(s_{19},xs_7),\\
&[(s_{11},x),(s_{19},s_7s_{11})]=-(s_{19},xs_7),\ [(s_{15},xs_7),(s_{19},s_{15})]=(s_{19},xs_7),\\
&[(s_7,x),(s_{19},s_7s_{11})]=(s_{19},xs_{11}),\ [(s_{11},1),(s_{15},s_{11})]=(s_{15},1)\\
&[(s_7,1),(s_{15},s_7)]=(s_{15},1),\ [(s_{11},x),(s_{15},s_{11})]=(s_{15},x),\\
&[(s_7,x),(s_{15},s_7)]=(s_{15},x),\ [(s_7,1),(s_{15},xs_7)]=-(s_{15},x),\\
&[(s_{11},s_7),(s_{15},s_{11})]=(s_{15},s_7),\ [(s_7,1),(s_{11},s_7)]=(s_{11},1),\\
&[(s_7,x),(s_{11},s_7)]=(s_{11},x).
\end{aligned}$$

Then d_2 of Section 2 is given by

$$
\begin{aligned}
d_2(v_{s_{19},1}) =& v_{s_7,1}v_{s_{19},s_7} + v_{s_{11},1}v_{s_{19},s_{11}} + v_{s_{15},1}v_{s_{19},s_{15}} \\
d_2(v_{s_{19},x}) =& - v_{s_7,x}v_{s_{19},s_7} - v_{s_{11},x}v_{s_{19},s_{11}} - v_{s_{15},x}v_{s_{19},s_{15}} \\
&+ v_{s_7,1}v_{s_{19},xs_7} + v_{s_{11},1}v_{s_{19},xs_{11}} \\
d_2(v_{s_{19},s_7}) =& - v_{s_{11},s_7}v_{s_{19},s_{11}} - v_{s_{15},s_7}v_{s_{19},s_{15}} + v_{s_{11},1}v_{s_{19},s_7s_{11}} \\
d_2(v_{s_{19},s_{11}}) =& - v_{s_{15},s_{11}}v_{s_{19},s_{15}} - v_{s_7,1}v_{s_{19},s_7s_{11}} \\
d_2(v_{s_{19},s_{15}}) =& 0 \\
d_2(v_{s_{19},xs_7}) =& v_{s_{15},xs_7}v_{s_{19},s_{15}} - v_{s_{11},s_7}v_{s_{19},xs_{11}} + v_{s_{11},x}v_{s_{19},s_7s_{11}} \\
d_2(v_{s_{19},xs_{11}}) =& - v_{s_7,x}v_{s_{19},s_7s_{11}} \\
d_2(v_{s_{19},s_7s_{11}}) =& 0 \\
d_2(v_{s_{15},1}) =& v_{s_7,1}v_{s_{15},s_7} + v_{s_{11},1}v_{s_{15},s_{11}} \\
d_2(v_{s_{15},x}) =& - v_{s_7,x}v_{s_{15},s_7} - v_{s_{11},x}v_{s_{15},s_{11}} + v_{s_7,1}v_{s_{15},xs_7} \\
d_2(v_{s_{15},s_7}) =& - v_{s_{11},s_7}v_{s_{15},s_{11}} \\
d_2(v_{s_{15},s_{11}}) =& 0 \\
d_2(v_{s_{15},xs_7}) =& 0 \\
d_2(v_{s_{11},1}) =& v_{s_7,1}v_{s_{11},s_7} \\
d_2(v_{s_{11},x}) =& - v_{s_7,x}v_{s_{11},s_7} \\
d_2(v_{s_{11},s_7}) =& 0 \\
d_2(v_{s_7,1}) =& 0 \\
d_2(v_{s_7,x}) =& 0.
\end{aligned}
$$

Let

$$
\begin{aligned}
v_{w_1} &:= -v_{s_{19},s_{15}} + v_{s_{15},s_{11}}, \\
v_{w_2} &:= -v_{s_{15},s_{11}} + v_{s_{11},s_7}, \\
v_{\alpha} &:= -v_{s_{19},s_{11}} + v_{s_{15},s_7}.
\end{aligned}
$$

Then the differential $\hat{D} = d_1 + d_2$ of $C^*(Der_{\Lambda(x)}M(X))$ is given by

$$
\begin{aligned}
\hat{D}(v_{s_{19},1}) =& + v_{s_7,1}v_{s_{19},s_7} + v_{s_{11},1}v_{s_{19},s_{11}} + v_{s_{15},1}v_{s_{19},s_{15}} \\
\hat{D}(v_{s_{19},x}) =& - v_{s_{15},1} - v_{s_7,x}v_{s_{19},s_7} - v_{s_{11},x}v_{s_{19},s_{11}} - v_{s_{15},x}v_{s_{19},s_{15}} \\
&+ v_{s_7,1}v_{s_{19},xs_7} + v_{s_{11},1}v_{s_{19},xs_{11}}
\end{aligned}
$$

$$
\begin{aligned}
\hat{D}(v_{s_{19},s_7}) &= -(v_{w_2}+v_{w_1})v_{s_{19},s_{11}} - v_\alpha v_{s_{19},s_{15}} + v_{s_{11},1}v_{s_{19},s_7s_{11}}\\
\hat{D}(v_{s_{19},s_{11}}) &= -v_{w_1}v_{s_{19},s_{15}} - v_{s_7,1}v_{s_{19},s_7s_{11}}\\
\hat{D}(v_{s_{19},s_{15}}) &= 0\\
\hat{D}(v_{s_{19},xs_7}) &= v_\alpha + v_{s_{15},xs_7}v_{s_{19},s_{15}} - (v_{w_2}+v_{w_1}+v_{s_{19},s_{15}})v_{s_{19},xs_{11}}\\
&\quad + v_{s_{11},x}v_{s_{19},s_7s_{11}}\\
\hat{D}(v_{s_{19},xs_{11}}) &= v_{w_1} - v_{s_7,x}v_{s_{19},s_7s_{11}}\\
\hat{D}(v_{s_{19},s_7s_{11}}) &= 0\\
\hat{D}(v_{s_{15},1}) &= v_{s_7,1}(v_\alpha + v_{s_{19},s_{11}}) + v_{s_{11},1}(v_{w_1}+v_{s_{19},s_{15}})\\
\hat{D}(v_{s_{15},x}) &= -v_{s_{11},1} - v_{s_7,x}(v_\alpha + v_{s_{19},s_{11}}) - v_{s_{11},x}(v_{w_1}+v_{s_{19},s_{15}})\\
&\quad + v_{s_7,1}v_{s_{15},xs_7}\\
\hat{D}(v_\alpha) &= +v_{w_1}v_{s_{19},s_{15}} + v_{w_1}v_{w_2} - v_{w_2}v_{s_{19},s_{15}} + v_{s_7,1}v_{s_{19},s_7s_{11}}\\
\hat{D}(v_{w_1}) &= 0\\
\hat{D}(v_{s_{15},xs_7}) &= v_{w_2}\\
\hat{D}(v_{s_{11},1}) &= v_{s_7,1}(v_{w_2}+v_{w_1}+v_{s_{19},s_{15}})\\
\hat{D}(v_{s_{11},x}) &= -v_{s_7,1} - v_{s_7,x}(v_{w_2}+v_{w_1}+v_{s_{19},s_{15}})\\
\hat{D}(v_{w_2}) &= 0\\
\hat{D}(v_{s_7,1}) &= 0\\
\hat{D}(v_{s_7,x}) &= 0.
\end{aligned}
$$

Thus the minimal model $M(Baut_1p)$ of the free DGA

$$m(Baut_1p) := C^*(Der_{\Lambda(x)}M(X))$$

is given by

$M(Baut_1p) = (\Lambda(U_{s_{19},1}, U_{s_{19},s_7}, U_{s_{19},s_7s_{11}}, U_{s_7,x}, U_\sigma, U_\tau), d)$
$|U_{s_{19},1}| = 20 \quad |U_{s_{19},s_7}| = 13 \quad |U_\sigma| = 9 \quad |U_\tau| = 5 \quad |U_{s_7,x}| = 3 \quad |U_{s_{19},s_7s_{11}}| = 2$
$d(U_{s_{19},1}) = d(U_\sigma) = d(U_\tau) = d(U_{s_7,x}) = d(U_{s_{19},s_7s_{11}}) = 0$
$d(U_{s_{19},s_7}) = -2U_{s_7,x}U_\sigma U_{s_{19},s_7,s_{11}}$.

Here the quasi-isomorphic DGA-map $\varphi : M(Baut_1p) \to m(Baut_1p)$ is given by

$$
\begin{aligned}
\varphi(U_{s_{19},s_7s_{11}}) =& v_{s_{19},s_7s_{11}} \\
\varphi(U_{s_7,x}) =& v_{s_7,x} \\
\varphi(U_\tau) =& v_{s_{19},s_{15}} \\
\varphi(U_\sigma) =& 2v_{s_{19},s_{11}} + v_\alpha + v_{s_{19},xs_{11}}v_{s_{19},s_{15}} + v_{s_{15},xs_7}v_{s_{19},s_{15}} - v_{s_{11},x}v_{s_{19},s_7s_{11}} \\
&+ v_{s_{15},xs_7}v_{w_1} + v_{s_7,x}v_{s_{19},xs_{11}}v_{s_{19},s_7s_{11}} + v_{s_7,x}v_{s_{15},xs_7}v_{s_{19},s_7s_{11}}
\end{aligned}
$$

$$
\begin{aligned}
\varphi(U_{s_{19},s_7}) =& 2v_{s_{19},s_7} + 2v_{s_{19},s_{15}}v_{s_{19},xs_7} + 2v_{s_{15},xs_7}v_{s_{19},s_{11}} + 2v_{s_{19},xs_{11}}v_{s_{19},s_{11}} \\
&+ 2v_{s_{15},x}v_{s_{19},s_7s_{11}} + 2v_{s_{19},s_{15}}v_{s_{15},xs_7}v_{s_{19},xs_{11}} + 2v^2_{s_{19},xs_{11}}v_{s_{19},s_{15}} \\
&- 2v_{s_{11},x}v_{s_{19},xs_{11}}v_{s_{19},s_7s_{11}} + 2v_{s_7,x}v_{s_{15},xs_7}v_{s_{19},xs_{11}}v_{s_{19},s_7s_{11}} \\
&+ v_{s_7,x}v^2_{s_{19},xs_{11}}v_{s_{19},s_7s_{11}} \\
\varphi(U_{s_{19},1}) =& v_{s_{19},1} + v_{w_2}v_{s_{19},x} - v_{s_{19},s_{15}}v_{s_{19},x} + v_{s_{11},x}v_{s_{19},s_7} + v_{s_{11},1}v_{s_{19},xs_7} \\
&- v_{s_{15},xs_7}v_{s_{15},1} - v_\alpha v_{s_{15},x} - v_{s_{19},s_{11}}v_{s_{15},x} - v_{w_1}v_{s_{19},xs_7}v_{s_{11},x} \\
&- 2v_{s_{19},s_{15}}v_{s_{19},xs_7}v_{s_{11},x} + v_{w_2}v_{s_{15},x}v_{s_{19},xs_{11}} - v_{w_1}v_{s_{15},x}v_{s_{19},xs_{11}} \\
&+ v_{s_{11},1}v^2_{s_{19},xs_{11}} + v_{s_{11},1}v_{s_{15},xs_7}v_{s_{19},xs_{11}} - 2v_{s_7,x}v_{s_{19},s_7}v_{s_{15},xs_7} \\
&- v_{s_7,x}v_{s_{19},s_7}v_{s_{19},xs_{11}} - v_{s_7,x}v_{s_{19},xs_{11}}v_{s_{19},s_{15}}v_{s_{19},xs_7} \\
&- 2v_{s_7,x}v_{s_{15},xs_7}v_{s_{19},s_{15}}v_{s_{19},xs_7} - 2v_{s_7,x}v_{s_{19},s_{11}}v_{s_{15},xs_7}v_{s_{19},xs_{11}} \\
&- v_{s_{19},s_{15}}v_{s_{11},x}v^2_{s_{19},xs_{11}} - v_{s_{19},s_{15}}v_{s_{11},x}v_{s_{19},xs_{11}}v_{s_{15},xs_7} \\
&+ v_{s_{15},x}v_{s_{15},xs_7}v_{s_7,x}v_{s_{19},s_7s_{11}} - v_{s_{15},x}v_{s_{19},xs_{11}}v_{s_7,x}v_{s_{19},s_7s_{11}} \\
&- v_{s_7,x}v_{s_{19},s_{11}}v^2_{s_{15},xs_7} - v_{s_7,1}v_{s_{15},xs_7}v^2_{s_{19},xs_{11}} \\
&+ v_{s_7,x}v_{s_{19},xs_7}v_{w_2}v_{s_{19},xs_{11}} - v_{s_7,x}v_{s_{19},xs_7}v_{w_1}v_{s_{19},xs_{11}} \\
&+ v_{s_{11},x}v^2_{s_{19},xs_{11}}v_{w_1} - 3v_{s_7,x}v^2_{s_{19},xs_{11}}v_{s_{19},s_{15}}v_{s_{15},xs_7} \\
&- v_{s_7,x}v^2_{s_{15},xs_7}v_{s_{19},s_{15}}v_{s_{19},xs_{11}} \\
&- v_{s_{11},x}v_{s_{19},xs_{11}}v_{s_{15},xs_7}v_{s_7,x}v_{s_{19},s_7s_{11}} \\
&- 3v_{s_7,x}v_{w_1}v_{s_{15},xs_7}v^2_{s_{19},xs_{11}}
\end{aligned}
$$

By the similar arguments, we obtain the minimal model of $Baut_1X$:

$$M(Baut_1X) = M(C^*(DerM(X))) = (\Lambda(U_{s_{19},1}, U_{s_{19},s_7}, U_{s_{19},s_7s_{11}}, U_{s_7,x}, U_\sigma), d)$$

as a sub-DGA of $M(Baut_1p)$. In this case, the element U_τ from $\partial(x,1) = (s_{19},s_{15}) + (s_{15},s_{11}) + (s_{11},s_7) = \tau$, does not exist. □

REMARK 3.2. In general, let a fibration $\xi : S^a \times S^b \times S^c \times S^d \to X \to S^e$ with $a,\ b,\ c,\ d,\ e$ odd and $e < a < 2e-1$ be given by the relative model

$$(\Lambda(x), 0) \to M(X) = (\Lambda(x, s_a, s_b, s_c, s_d), D) \to (\Lambda(s_a, s_b, s_c, s_d), 0)$$

where $|x| = e$, $|s_n| = n$, $Dx = 0$, $Ds_a = 0$, $Ds_b = xs_a$, $Ds_c = xs_b$, $Ds_d = xs_c$. Then we obtain the same result as Theorem 1.4 from a similar proof.

EXAMPLE 3.3. For any fibration $\xi : S^5 \times S^9 \times S^{13} \times S^{17} \to X \to S^5$, we can check that $Baut_1X$ and $Baut_1p$ are coformal. Especially, when ξ is given by

$$(\Lambda(x), 0) \to (\Lambda(x, s_5, s_9, s_{13}, s_{17}), D) \to (\Lambda(s_5, s_9, s_{13}, s_{17}), 0)$$

with $Dx = 0$, $Ds_5 = 0$, $Ds_9 = xs_5$, $Ds_{13} = xs_9$, $Ds_{17} = xs_{13}$, then $Baut_1X$ and $Baut_1p$ are rational H-spaces.

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Received May 18, 2021
Accepted October 26, 2021

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