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TOUCH POINTS IN IDEAL ČECH CLOSURE SPACES

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Abstract. Let $(X, f, \mathcal{I})$ be a Čech closure space with an ideal $\mathcal{I}$. For a subset A of X, the set $\widetilde{f}(A)$ of so-called a Čech touch points is defined as follows: $\widetilde{f}(A) = \{x \in X : A \cap N \notin \mathcal{I}$ for every $N \in \mathcal{N}(x)\}$. We investigate the properties of touch points and construct a topology on X from the touch points. Moreover, in an ideal Čech closure space $(X, f, \mathcal{I})$, we define f-compatibility with the ideal $\mathcal{I}$ and obtain several characterizations of the compatibility.

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Key words. Čech closure operator, ideal Čech closure space, f-compatible with an ideal.

1. INTRODUCTION

An ideal $\mathcal{I}$ on a space X is a non-empty collection of subsets of X which satisfies the following properties:

(1) $A \in \mathcal{I}$ and $B \subseteq A$ implies that $B \in \mathcal{I}$.
(2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

An ideal topological space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X and is denoted by $(X, \tau, \mathcal{I})$ (see [9, 10]). Several characterizations of the ideal structure were provided in [1-6].

First we recall several definitions. An operator $f : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ defined on the power set, $\mathcal{P}(X)$, of a set X satisfying the axioms:

(C_1)**:** $f(\emptyset) = \emptyset$;
(C_2)**:** $A \subseteq f(A)$ for every $A \subseteq X$;
(C_3)**:** $f(A \cup B) = f(A) \cup f(B)$ for all $A, B \in \mathcal{P}(X)$.

is called a Čech closure operator ([7, 8]) and the pair (X, f) is a Čech closure space. A subset A of X is said to be closed in (X, f) if $f(A) = A$ holds, and it is said to be open if its complement is closed.

The interior operator $f^* : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ is defined by means of the closure operator in the usual way: $f^*(A) = X - f(X - A)$.

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Let $(X, f, \mathcal{I})$ be a Čech closure space with an ideal $\mathcal{I}$. For a subset A of X, the set $\widetilde{f}(A)$ of so-called touch points is defined as follows:

$$\widetilde{f}(A) = \{x \in X : A \cap N \notin \mathcal{I} \text{ for every } N \in \mathcal{N}(x)\}.$$

We investigate the properties of touch points and construct a topology on X from the touch points. Moreover, in an ideal Čech closure space $(X, f, \mathcal{I})$, we define f-compatibility with the ideal $\mathcal{I}$ and obtain several characterizations of the compatibility.

2. PRELIMINARIES

REMARK 2.1. Let (X, f) be a Čech closure space.

(1) $f^*(\emptyset) = \emptyset$.
(2) $f^*(X) = X$.
(3) $f^*(A) \subseteq A$ for every $A \subseteq X$.
(4) $f^*(A \cap B) = f^*(A) \cap f^*(B)$ for all A, $B \in \mathcal{P}(X)$.

A subset N is a neighborhood of a point x (respectively, subset A) in X if $x \in f^*(N)$ (respectively, $A \subseteq f^*(N)$) holds. The collection of all neighborhoods of x will be denoted by $\mathcal{N}_x$ or $\mathcal{N}(x)$.

In (X, f), a point $x \in f(A)$ if and only if for each neighborhood N of x, $N \cap A \neq \emptyset$ holds.

DEFINITION 2.2 ([11]). Let f and f^* be the be closure function and its dual function on X. Then the neighborhood function $\mathcal{N} : X \to \mathcal{P}(\mathcal{P}(X))$ and the convergent function $\mathcal{N}^* : X \to \mathcal{P}(\mathcal{P}(X))$ assign to each $x \in X$ the collections

$$\mathcal{N}(x) = \{N \in \mathcal{P}(X) : x \in f^*(N)\}$$

$$\mathcal{N}^*(x) = \{Q \in \mathcal{P}(X) : x \in f(Q)\}$$

of its neighborhoods and convergents, respectively.

LEMMA 2.3. *Let (X, f) be a Čech closure space. Then the following properties hold*

(1) $Q \in \mathcal{N}^*(x)$ *if and only if* $X - Q \notin \mathcal{N}(x)$.
(2) $x \in f(A)$ *if and only if* $X - A \notin \mathcal{N}(x)$.
(3) $x \in f^*(A)$ *if and only if* $X - A \notin \mathcal{N}^*(x)$.

LEMMA 2.4. *Let (X, f) be a Čech closure space, then*

(1) $X \in \mathcal{N}(x)$ *for every* $x \in X$.
(2) $\emptyset \notin \mathcal{N}(x)$ *for every* $x \in X$.
(3) *If* $N \in \mathcal{N}(x)$, *then* $x \in f^*(N) \subseteq N$.
(4) *If* $N, M \in \mathcal{N}(x)$, *then we have* $N \cap M \in \mathcal{N}(x)$.
(5) *If* $N \cup M \in \mathcal{N}^*(x)$, *then we have* $N \in \mathcal{N}^*(x)$ *or* $M \in \mathcal{N}^*(x)$.

Proof. (1): For every $x \in X$, $x \in X = f^*(X)$ and $X \in \mathcal{N}(x)$ for every $x \in X$.

(2): Suppose that $\emptyset \in \mathcal{N}(x)$ for some $x \in X$, then $x \in f^*(\emptyset)$. This is contrary to $f^*(\emptyset) = \emptyset$. Hence $\emptyset \notin \mathcal{N}(x)$.

(3): Let $N \in \mathcal{N}(x)$, then by Remark 2.1 (3) $x \in f^*(N) \subset N$ and $x \in N$.

(4): If $M, N \in \mathcal{N}(x)$, then $x \in f^*(M)$ and $x \in f^*(N)$ and hence $x \in f^*(M) \cap f^*(N) = f^*(M \cap N)$. Therefore we have $N \cap M \in \mathcal{N}(x)$.

(5): If $M \cup N \in \mathcal{N}^*(x)$, then $x \in f(M \cup N) = f(M) \cup f(N)$. Hence $x \in f(M)$ or $x \in f(N)$ and $M \in \mathcal{N}^*(x)$ or $N \in \mathcal{N}^*(x)$. □

3. TOUCH POINTS

In Sections 3 and 4, an ideal Čech closure space is briefly called an ideal Čech space.

Definition 3.1. Let $(X, f, \mathcal{I})$ be an ideal Čech space. For a subset A of X, we define the following set:

$$\widetilde{f}(A) = \{x \in X : A \cap N \notin \mathcal{I} \text{ for every } N \in \mathcal{N}(x)\}.$$

$\widetilde{f}(A)$ is called the set of touch points of A with respect to $\mathcal{I}$ and f.

Lemma 3.2. *Let $(X, f, \mathcal{I})$ and $(X, g, \mathcal{J})$ be ideal Čech spaces, where $\mathcal{I}$ and $\mathcal{J}$ are ideals on X, and let A and B be subsets of X. Then the following properties hold:*

(1) *If $A \subseteq B$, then $\widetilde{f}(A) \subseteq \widetilde{f}(B)$.*
(2) *If $\mathcal{I} \subseteq \mathcal{J}$, then $\widetilde{f}(A) \supseteq \widetilde{g}(A)$.*
(3) *$\widetilde{f}(A) = f(\widetilde{f}(A)) \subseteq f(A)$.*
(4) *If $A \subseteq \widetilde{f}(A)$, then $\widetilde{f}(A) = f(\widetilde{f}(A)) = f(A)$.*
(5) *If $A \in \mathcal{I}$, then $\widetilde{f}(A) = \emptyset$.*

Proof. (1) Suppose that $x \notin \widetilde{f}(B)$. Then there exists $N \in \mathcal{N}(x)$ such that $N \cap B \in \mathcal{I}$. Since $N \cap A \subseteq N \cap B$, $N \cap A \in \mathcal{I}$. Hence $x \notin \widetilde{f}(A)$. Thus $X \setminus \widetilde{f}(B) \subseteq X \setminus \widetilde{f}(A)$ or $\widetilde{f}(A) \subseteq \widetilde{f}(B)$.

(2) Suppose that $x \in \widetilde{g}(A)$. For every $N \in \mathcal{N}(x)$, we have $N \cap A \notin \mathcal{J}$. Since $\mathcal{I} \subseteq \mathcal{J}$, $N \cap A \notin \mathcal{I}$ and $x \in \widetilde{f}(A)$. Therefore, $\widetilde{f}(A) \supseteq \widetilde{g}(A)$.

(3) We have $\widetilde{f}(A) \subseteq f(\widetilde{f}(A))$ in general. Let $x \in f(\widetilde{f}(A))$. Then $\widetilde{f}(A) \cap N \neq \emptyset$ for every $N \in \mathcal{N}(x)$. Therefore, there exists some $y \in \widetilde{f}(A) \cap N$ and $N \in \mathcal{N}(y)$. Since $y \in \widetilde{f}(A)$, $A \cap N \notin \mathcal{I}$ and hence $x \in \widetilde{f}(A)$. Hence we have $f(\widetilde{f}(A)) \subseteq \widetilde{f}(A)$ and hence $\widetilde{f}(A) = f(\widetilde{f}(A))$.

Again, let $x \in \widetilde{f}(A) = f(\widetilde{f}(A))$, then $N \cap A \notin \mathcal{I}$ for every $N \in \mathcal{N}(x)$. This implies $N \cap A \neq \emptyset$ for every $N \in \mathcal{N}(x)$. Therefore, $x \in f(A)$. This shows that $\widetilde{f}(A) = f(\widetilde{f}(A)) \subseteq f(A)$.

(4) For any subset A of X, by (3) we have $\widetilde{f}(A) = f(\widetilde{f}(A)) \subseteq f(A)$. Since $A \subseteq \widetilde{f}(A)$, $f(A) \subseteq f(\widetilde{f}(A))$ and hence $\widetilde{f}(A) = f(\widetilde{f}(A)) = f(A)$.

(5) Suppose that $x \in \widetilde{f}(A)$. Then for any $N \in \mathcal{N}(x)$, $N \cap A \notin \mathcal{I}$. But since $A \in \mathcal{I}$, $N \cap A \in \mathcal{I}$ for any $N \in \mathcal{N}(x)$. This is a contradiction. Hence $\widetilde{f}(A) = \emptyset$. □

Lemma 3.3. *Let $(X, f, \mathcal{I})$ be an ideal Čech space and $x \in X$. If $N \in \mathcal{N}(x)$, then $N \cap \widetilde{f}(A) = N \cap \widetilde{f}(N \cap A) \subseteq \widetilde{f}(N \cap A)$ for any subset A of X.*

Proof. Suppose that $N \in \mathcal{N}(x)$ and $x \in N \cap \widetilde{f}(A)$. Then $x \in N$ and $x \in \widetilde{f}(A)$. Let $V \in \mathcal{N}(x)$. Then by Lemma 2.4, $V \cap N \in \mathcal{N}(x)$ and $V \cap (N \cap A) = (V \cap N) \cap A \notin \mathcal{I}$. This shows that $x \in \widetilde{f}(N \cap A)$ and hence we obtain $N \cap \widetilde{f}(A) \subseteq \widetilde{f}(N \cap A)$. Moreover, $N \cap \widetilde{f}(A) \subseteq N \cap \widetilde{f}(N \cap A)$ and by Lemma 3.2, $\widetilde{f}(N \cap A) \subseteq \widetilde{f}(A)$ and $N \cap \widetilde{f}(N \cap A) \subseteq N \cap \widetilde{f}(A)$. Therefore, $N \cap \widetilde{f}(A) = N \cap \widetilde{f}(N \cap A)$. □

Theorem 3.4. *Let $(X, f, \mathcal{I})$ be an ideal Čech space and A, B any subsets of X. Then the following properties hold:*
(1) $\widetilde{f}(\emptyset) = \emptyset$.
(2) $\widetilde{f}(\widetilde{f}(A)) \subseteq \widetilde{f}(A)$.
(3) $\widetilde{f}(A) \cup \widetilde{f}(B) = \widetilde{f}(A \cup B)$.

Proof. (1) The proof is obvious.

(2) Let $x \in \widetilde{f}(\widetilde{f}(A))$. Then for every $N \in \mathcal{N}(x)$, $N \cap \widetilde{f}(A) \notin \mathcal{I}$ and hence $N \cap \widetilde{f}(A) \neq \emptyset$. Let $y \in N \cap \widetilde{f}(A)$. Then $N \in \mathcal{N}(y)$ and $y \in \widetilde{f}(A)$. Hence we have $N \cap A \notin \mathcal{I}$ and $x \in \widetilde{f}(A)$. This shows that $\widetilde{f}(\widetilde{f}(A)) \subseteq \widetilde{f}(A)$.

(3) It follows from Lemma 3.2, that $\widetilde{f}(A \cup B) \supseteq \widetilde{f}(A) \cup \widetilde{f}(B)$. To prove the reverse inclusion, let $x \in \widetilde{f}(A \cup B)$, then for every $U \in \mathcal{N}(x)$, $U \cap (A \cup B) = (U \cap A) \cup (U \cap B) \notin \mathcal{I}$. Hence $U \cap A \notin \mathcal{I}$ or $U \cap B \notin \mathcal{I}$ and hence $x \in \widetilde{f}(A)$ or $x \in \widetilde{f}(B)$. Therefore, $x \in \widetilde{f}(A) \cup \widetilde{f}(B)$ and $\widetilde{f}(A \cup B) \subseteq \widetilde{f}(A) \cup \widetilde{f}(B)$. Hence we obtain $\widetilde{f}(A) \cup \widetilde{f}(B) = \widetilde{f}(A \cup B)$. □

Theorem 3.5. *Let $(X, f, \mathcal{I})$ be an ideal Čech space, let A, B be subsets of X and let $cl(A) = \widetilde{f}(A) \cup A$. Then*
(1) $cl(\emptyset) = \emptyset$.
(2) $A \subseteq cl(A)$.
(3) $cl(A \cup B) = cl(A) \cup cl(B)$.
(4) $cl(A) = cl(cl(A))$.

Proof. By Theorem 3.4, we obtain

(1) $cl(\emptyset) = \widetilde{f}(\emptyset) \cup \emptyset = \emptyset$.

(2) $A \subseteq A \cup \widetilde{f}(A) = cl(A)$.

(3) $cl(A\cup B) = \widetilde{f}(A\cup B)\cup(A\cup B) = (\widetilde{f}(A)\cup\widetilde{f}(B))\cup(A\cup B) = cl(A)\cup Cl(B)$.

(4) $cl(cl(A)) = cl[\widetilde{f}(A) \cup A] = \widetilde{f}(\widetilde{f}(A) \cup A) \cup (\widetilde{f}(A) \cup A) = (\widetilde{f}(\widetilde{f}(A))) \cup \widetilde{f}(A) \cup (\widetilde{f}(A) \cup A) = \widetilde{f}(A) \cup A = cl(A)$. □

By Theorem 3.5, we obtain that $cl(A) = A \cup \widetilde{f}(A)$ is a Kuratowski closure operator. We will denote by τ_{cl} the topology generated by cl, that is,

$$\tau_{cl} = \{U \subseteq X : cl(X - U) = X - U\}.$$

THEOREM 3.6. *Let $(X, f, \mathcal{I})$ be an ideal Čech space. Then $\beta(f, \mathcal{I}) = \{V - I : V \in \mathcal{N}(x), I \in \mathcal{I}, x \in X\}$ is a basis for τ_{cl}.*

Proof. Let $(X, f, \mathcal{I})$ be an ideal Čech space. It is obvious that A is τ_{cl}-closed if and only if $\widetilde{f}(A) \subseteq A$. Now we have $U \in \tau_{cl}$ if and only if $\widetilde{f}(X - U) \subseteq X - U$ if and only if $U \subseteq X - \widetilde{f}(X - U)$.

At first, we shall show that every member of $\beta(f, \mathcal{I})$ is an open set in τ_{cl}. Let $V - I \in \beta(f, \mathcal{I})$, where $V \in \mathcal{N}(x)$ and $I \in \mathcal{I}$. For each $x \in V - I$, $x \in V$ and $V \cap (X - (V - I)) \subseteq I \in \mathcal{I}$. Hence, $x \notin \widetilde{f}(X - (V - I))$, and hence $V - I \subseteq X - \widetilde{f}(X - (V - I))$. Therefore, by the fact above, $V - I \in \tau_{cl}$.

Now, let $U \in \tau_{cl}$. By the fact above $x \in U$ implies that $x \notin \widetilde{f}(X - U)$. This implies that there exists $V \in \mathcal{N}(x)$ such that $V \cap (X - U) \in \mathcal{I}$. Now let $I = V \cap (X - U)$ and we have $x \in V - I \subseteq U$, $I \in \mathcal{I}$.

Now we need only show that β is closed under finite intersection. Let $A, B \in \beta$, then $A = H - I$ and $B = K - J$, where H, K are in $\mathcal{N}(x)$ and $I, J \in \mathcal{I}$. Now, we have

$$\begin{aligned}(H - I) \cap (K - J) &= (H \cap (X - I)) \cap (K \cap (X - J))\\ &= (H \cap K) \cap ((X - I) \cap (X - J))\\ &= (H \cap K) \cap (X - (I \cup J))\\ &= (H \cap K) - (I \cup J).\end{aligned}$$

Since $I \cup J \in \mathcal{I}$ and $H \cap K \in \mathcal{N}(x)$, $A \cap B \in \beta$. Therefore β is closed under finite intersection. Thus $\beta = \{V - I : V \in \mathcal{N}(x), I \in \mathcal{I}\}$ is a basis for τ_{cl}. □

We set $i(A) = \cup\{U : U \subseteq A, U \in \mathcal{N}(x)$ for every $x \in X\}$.

THEOREM 3.7. *Let $(X, f, \mathcal{I})$ be an ideal Čech space and $x \in X$, then the following properties are equivalent:*

(1) *$\mathcal{N}(x) \cap \mathcal{I} = \emptyset$;*
(2) *If $I \in \mathcal{I}$, then $i(I) = \emptyset$;*
(3) *For every $N \in \mathcal{N}(x)$, $N \subseteq \widetilde{f}(N)$;*
(4) *$X = \widetilde{f}(X)$.*

Proof. (1) ⇒(2): Let $\mathcal{N}(x) \cap \mathcal{I} = \{\emptyset\}$ and $I \in \mathcal{I}$. Suppose that $x \in i(I)$. Then there exists $N \in \mathcal{N}(x)$ such that $x \in N \subseteq I$. Since $I \in \mathcal{I}$ we have $\emptyset \neq \{x\} \subseteq N \in \mathcal{N}(x) \cap \mathcal{I}$. This contradicts $\mathcal{N}(x) \cap \mathcal{I} = \{\emptyset\}$. Therefore, $i(I) = \{\emptyset\}$.

(2)⇒(3): Let $x \in N$. Assume $x \notin \widetilde{f}(N)$. Then there exists $M \in \mathcal{N}(x)$ such that $N \cap M \in \mathcal{I}$. By (2), $x \in N \cap M = i(N \cap M) = \{\emptyset\}$. Hence $x \in \widetilde{f}(N)$ and $N \subseteq \widetilde{f}(N)$.

(3)⇒(4): Since $X \in \mathcal{N}(x)$, then $X = \widetilde{f}(X)$.

(4)⇒(1): $X = \widetilde{f}(X) = \{x \in X : N \cap X = N \notin \mathcal{I}$ for each $N \in \mathcal{N}(x)\}$. Hence $\mathcal{N}(x) \cap \mathcal{I} = \{\emptyset\}$. □

Lemma 3.8. *Let $(X, f, \mathcal{I})$ be an ideal Čech space and A, B be subsets of X. Then $\widetilde{f}(A) - \widetilde{f}(B) = \widetilde{f}(A - B) - \widetilde{f}(B)$.*

Proof. We have by Theorem 3.4 $\widetilde{f}(A) = \widetilde{f}[(A-B) \cup (A \cap B)] = \widetilde{f}(A-B) \cup \widetilde{f}(A \cap B) \subseteq \widetilde{f}(A - B) \cup \widetilde{f}(B)$. Thus $\widetilde{f}(A) - \widetilde{f}(B) \subseteq \widetilde{f}(A - B) - \widetilde{f}(B)$. By Lemma 3.2, $\widetilde{f}(A - B) \subseteq \widetilde{f}(A)$ and hence $\widetilde{f}(A - B) - \widetilde{f}(B) \subseteq \widetilde{f}(A) - \widetilde{f}(B)$. Hence $\widetilde{f}(A) - \widetilde{f}(B) = \widetilde{f}(A - B) - \widetilde{f}(B)$. □

Corollary 3.9. *Let $(X, f, \mathcal{I})$ be an ideal Čech space and let A, B be subsets of X with $B \in \mathcal{I}$. Then $\widetilde{f}(A \cup B) = \widetilde{f}(A) = \widetilde{f}(A - B)$.*

Proof. Since $B \in \mathcal{I}$, by Lemma 3.2 $\widetilde{f}(B) = \emptyset$. By Lemma 3.8, $\widetilde{f}(A) = \widetilde{f}(A - B)$ and by Theorem 3.4, $\widetilde{f}(A \cup B) = \widetilde{f}(A) \cup \widetilde{f}(B) = \widetilde{f}(A)$ □

4. f-COMPATIBLE IN IDEAL ČECH SPACE

Definition 4.1. Let $(X, f, \mathcal{I})$ be an ideal Čech space. We say f is f-compatible with the ideal $\mathcal{I}$, denoted $f \cong \mathcal{I}$, if the following holds for every $A \subseteq X$: For every $x \in A$ and $U \in \mathcal{N}(x)$, if $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

Theorem 4.2. *Let $(X, f, \mathcal{I})$ be an ideal Čech space and $x \in X$, the following properties are equivalent:*

1. *$f \cong \mathcal{I}$;*
2. *If a subset A of X has a cover of $U \in \mathcal{N}(x)$ each of whose intersection with A is in $\mathcal{I}$, then $A \in \mathcal{I}$;*
3. *For every $A \subseteq X$, $A \cap \widetilde{f}(A) = \emptyset$ implies that $A \in \mathcal{I}$;*
4. *For every $A \subseteq X$, $A - \widetilde{f}(A) \in \mathcal{I}$;*
5. *For every $A \subseteq X$, if A contains no nonempty subset B with $B \subseteq \widetilde{f}(B)$, then $A \in \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (2): The proof is obvious.

(2) $\Rightarrow$ (3): Let $A \subseteq X$ and $x \in A$. Then $x \notin \widetilde{f}(A)$ and there exists $V_x \in \mathcal{N}(x)$ such that $V_x \cap A \in \mathcal{I}$. Therefore, we have $A \subseteq \cup\{V_x : x \in A\}$ and by (2) $A \in \mathcal{I}$.

(3) $\Rightarrow$ (4): For any $A \subseteq X$, $A - \widetilde{f}(A) \subseteq A$ and $(A - \widetilde{f}(A)) \cap \widetilde{f}(A - \widetilde{f}(A)) \subseteq (A - \widetilde{f}(A)) \cap \widetilde{f}(A) = \emptyset$. By (3), $A - \widetilde{f}(A) \in \mathcal{I}$.

(4) $\Rightarrow$ (5): By (4), for every $A \subseteq X$, $A - \widetilde{f}(A) \in \mathcal{I}$. Let $A - \widetilde{f}(A) = J \in \mathcal{I}$, then $A = J \cup (A \cap \widetilde{f}(A))$ and by Theorem 3.4 (3) and Lemma 3.2, $\widetilde{f}(A) = \widetilde{f}(J) \cup \widetilde{f}(A \cap \widetilde{f}(A)) = \widetilde{f}(A \cap \widetilde{f}(A))$. Therefore, we have $A \cap \widetilde{f}(A) = A \cap \widetilde{f}(A \cap \widetilde{f}(A)) \subseteq \widetilde{f}(A \cap \widetilde{f}(A))$ and $A \cap \widetilde{f}(A) \subseteq A$. By the assumption $A \cap \widetilde{f}(A) = \emptyset$ and hence $A = A - \widetilde{f}(A) \in \mathcal{I}$.

(5) $\Rightarrow$ (1): Let $A \subseteq X$ and assume that for every $x \in A$, there exists $U \in \mathcal{N}(x)$ such that $U \cap A \in \mathcal{I}$. Then $A \cap \widetilde{f}(A) = \emptyset$. Since

$$(A - \widetilde{f}(A)) \cap \widetilde{f}(A - \widetilde{f}(A)) \subseteq (A - \widetilde{f}(A)) \cap \widetilde{f}(A) = \emptyset,$$

$A - \widetilde{f}(A)$ contains no nonempty subset B with $B \subseteq \widetilde{f}(B)$. By (5), $A - \widetilde{f}(A) \in \mathcal{I}$ and hence $A = A \cap (X - \widetilde{f}(A)) = A - \widetilde{f}(A) \in \mathcal{I}$. □

THEOREM 4.3. *Let $(X, f, \mathcal{I})$ be an ideal Čech space., then the following properties are equivalent:*

(1) *$f \cong \mathcal{I}$;*
(2) *For every τ_{cl}-closed subset A, $A - \widetilde{f}(A) \in \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (2): It is clear by Theorem 4.2.

(2) $\Rightarrow$ (1): Let $A \subseteq X$ and assume that for every $x \in A$, there exists $U \in \mathcal{N}(x)$, such that $U \cap A \in \mathcal{I}$. Then $A \cap \widetilde{f}(A) = \emptyset$. Since $cl(A) = A \cup \widetilde{f}(A)$ is τ_{cl}-closed, we have $(A \cup \widetilde{f}(A)) - \widetilde{f}(A \cup \widetilde{f}(A)) \in \mathcal{I}$.
Moreover,

$$\begin{aligned}(A \cup \widetilde{f}(A)) - \widetilde{f}(A \cup \widetilde{f}(A)) &= (A \cup \widetilde{f}(A)) - (\widetilde{f}(A) \cup \widetilde{f}(\widetilde{f}(A))) \\ &= (A \cup \widetilde{f}(A)) - \widetilde{f}(A) \\ &= A.\end{aligned}$$

Therefore $A \in \mathcal{I}$. □

THEOREM 4.4. *Let $(X, f, \mathcal{I})$ be an ideal Čech space. If f is f-compatible with $\mathcal{I}$, then the following equivalent properties hold:*

(1) *For every $A \subseteq X$, $A \cap \widetilde{f}(A) = \emptyset$ implies that $\widetilde{f}(A) = \emptyset$;*
(2) *For every $A \subseteq X$, $\widetilde{f}(A - \widetilde{f}(A)) = \emptyset$;*
(3) *For every $A \subseteq X$, $\widetilde{f}(A \cap \widetilde{f}(A)) = \widetilde{f}(A)$.*

Proof. First, we show that (1) holds if f is f-compatible with $\mathcal{I}$. Let A be any subset of X and $A \cap \widetilde{f}(A) = \emptyset$. By Theorem 4.2, $A \in \mathcal{I}$ and by Lemma 3.2 (5), $\widetilde{f}(A) = \emptyset$.

(1) ⇒ (2): Assume that for every $A \subseteq X$, $A \cap \widetilde{f}(A) = \emptyset$ implies that $\widetilde{f}(A) = \emptyset$. Let $B = A - \widetilde{f}(A)$, then

$$\begin{aligned} B \cap \widetilde{f}(B) &= (A - \widetilde{f}(A)) \cap \widetilde{f}(A - \widetilde{f}(A)) \\ &= (A \cap (X - \widetilde{f}(A))) \cap \widetilde{f}(A \cap (X - \widetilde{f}(A))) \\ &\subseteq \Big(A \cap (X - \widetilde{f}(A))\Big) \cap \Big(\widetilde{f}(A) \cap \widetilde{f}(X - \widetilde{f}(A))\Big) = \emptyset. \end{aligned}$$

By (1), we have $\widetilde{f}(B) = \emptyset$. Hence $\widetilde{f}(A - \widetilde{f}(A)) = \emptyset$.

(2) ⇒ (3): Assume for every $A \subseteq X$, $\widetilde{f}(A - \widetilde{f}(A)) = \emptyset$.

$$\begin{aligned} A &= (A - \widetilde{f}(A)) \cup (A \cap \widetilde{f}(A)) \\ \widetilde{f}(A) &= \widetilde{f}\Big((A - \widetilde{f}(A)) \cup (A \cap \widetilde{f}(A))\Big) \\ &= \widetilde{f}(A - \widetilde{f}(A)) \cup \widetilde{f}(A \cap \widetilde{f}(A)) \\ &= \widetilde{f}(A \cap \widetilde{f}(A)). \end{aligned}$$

(3) ⇒ (1): Let $A \subseteq X$ such that $A \cap \widetilde{f}(A) = \emptyset$. Then by the assumption, $\widetilde{f}(A \cap \widetilde{f}(A)) = \widetilde{f}(A)$. This implies that $\emptyset = \widetilde{f}(\emptyset) = \widetilde{f}(A \cap \widetilde{f}(A)) = \widetilde{f}(A)$. □

COROLLARY 4.5. *Let $(X, f, \mathcal{I})$ be an ideal Čech space. If f is f-compatible with $\mathcal{I}$, then $\widetilde{f}()$ is an idempotent operator i.e. $\widetilde{f}(A) = \widetilde{f}(\widetilde{f}(A))$ for any subset A of X.*

Proof. By Theorems 4.4 and 3.4, we obtain

$$\widetilde{f}(A) = \widetilde{f}(A \cap \widetilde{f}(A)) \subseteq \widetilde{f}(A) \cap \widetilde{f}(\widetilde{f}(A)) = \widetilde{f}(\widetilde{f}(A))$$

and by Theorem 3.4, we have $\widetilde{f}(A) = \widetilde{f}(\widetilde{f}(A))$ for any subset A of X. □

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