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ON MIDPOINT AND TRAPEZOID TYPE INEQUALITIES FOR MULTIPLICATIVE INTEGRALS

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Abstract. The purpose of this paper is to establish some Hermite-Hadamard type inequalities for multiplicative convex functions. First, we obtain two equality for * differentiable functions. Then using these inequalities and multiplicative convex functions, we establish some inequalities related to the right and left hand side of Hermite-Hadamard inequality for multiplicative integrals.

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1. INTRODUCTION

Theory of convexity had played a significant role in many of mathematical and engineering sciences and also provide a general and unified framework for studying a wide classes of unrelated problems. Convexity in connection with integral inequalities is an interesting field of research. The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g., [6, 8], [15, p. 137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Over the last twenty years, the numerous studies have focused on to obtain new bound for left hand side and right and side of the inequality (1). For some examples, please refer to ([2], [4]–[7], [9]–[14], [16]–[19]) and the references therein.

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2. MULTIPLICATIVE CALCULUS

2.1. MULTIPLICATIVE DERIVATIVES AND INTEGRALS

Recall multiplicative derivative which can be found in [3].

DEFINITION 2.1. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a positive function. The multiplicative derivative of the function f is given by

$$\frac{\mathrm{d}^* f}{\mathrm{d}t}(t) = f^*(t) = \lim_{h \to 0} \left(\frac{f(t+h)}{f(t)} \right)^{\frac{1}{h}}.$$

If f has positive values and is differentiable at t, then f^* exists and the relation between f^* and ordinary derivative f' is as follows:

$$f^*(t) = \mathrm{e}^{[\ln f(t)]'} = \mathrm{e}^{\frac{f'(t)}{f(t)}}$$

Recall also that the concept of the multiplicative integral called * integral is denoted by $\int\limits_a^b (f(x))^{\mathrm{d}x}$ which introduced by Bashirov et al. in [3]. While the sum of the terms of product is used in the definition of a classical Riemann integral of f on $[a, b]$, the product of terms raised to power is used in the definition multiplicative integral of f on $[a, b]$.

The following properties of * differentiable exist:

THEOREM 2.2. *Let f and g be * differentiable functions. If c is arbitrary constant, then functions cf, fg, $f+g$, f/g and f^g are * differentiable and*

(i) $(cf)^*(t) = f^*(t)$.

(ii) $(fg)^*(t) = f^*(t) g^*(t)$.

(iii) $(f+g)^*(t) = f^*(t)^{\frac{f(t)}{f(t)+g(t)}} g^*(t)^{\frac{g(t)}{f(t)+g(t)}}$.

(iv) $\left(\frac{f}{g}\right)^*(t) = \frac{f^*(t)}{g^*(t)}$.

(v) $(f^g)^*(t) = f^*(t)^{g(t)} f(t)^{g'(t)}$.

There is the following relation between Riemann integral and multiplicative integral [3]:

PROPOSITION 2.3. *If f is Riemann integrable on $[a, b]$, then f is multiplicative integrable on $[a, b]$ and*

$$\int_a^b (f(x))^{\mathrm{d}x} = \mathrm{e}^{\int_a^b \ln(f(x))\mathrm{d}x}.$$

Moreover, Bashirov et al [3] show that multiplicative integrable has the following results and properties:

PROPOSITION 2.4. *If f is positive and Riemann integrable on $[a, b]$, then f is *integrable on $[a, b]$ and*

(i) $\int_a^b ((f(x))^p)^{dx} = \int_a^b ((f(x))^{dx})^p$,

(ii) $\int_a^b (f(x)g(x))^{dx} = \int_a^b (f(x))^{dx} \cdot \int_a^b (g(x))^{dx}$,

(iii) $\int_a^b \left(\frac{f(x)}{g(x)}\right)^{dx} = \frac{\int_a^b (f(x))^{dx}}{\int_a^b (f(x))^{dx}}$,

(iv) $\int_a^b (f(x))^{dx} = \int_a^c (f(x))^{dx} \cdot \int_c^b (f(x))^{dx}, \ \ a \leq c \leq b$,

(v) $\int_a^a (f(x))^{dx} = 1$ *and* $\int_a^b (f(x))^{dx} = \left(\int_b^a (f(x))^{dx}\right)^{-1}$.

THEOREM 2.5 (Multiplicative Integration by Parts). *Let $f : [a,b] \to \mathbb{R}$ be * differentiable, let $g : [a,b] \to \mathbb{R}$ be differentiable so the function f^g is * integrable. Then*

$$\int_a^b (f^*(x)^{g(x)})^{dx} = \frac{f(b)^{g(b)}}{f(a)^{g(a)}} \cdot \frac{1}{\int_a^b (f(x)^{g'(x)})^{dx}}.$$

2.2. HERMITE-HADAMARD INEQUALITY AND MULTIPLICATIVELY CONVEXITY

For the our main results we need the following definition.

DEFINITION 2.6. A non-empty set K is said to be convex if for every $a, b \in K$ we have

$$a + \mu(b - a) \in K, \ \forall \mu \in [0,1].$$

DEFINITION 2.7. A function f is said to be convex on a set K, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \ \forall t \in [0,1].$$

DEFINITION 2.8. A function f is said to be *log* or multiplicatively convex function on set K, if

$$f(tx + (1-t)y) \leq [f(x)]^t \cdot [f(y)]^{1-t}, \forall t \in [0,1].$$

PROPOSITION 2.9. *If f and g are log (multiplicatively) convex functions, then the functions fg and $\frac{f}{g}$ are log (multiplicatively) convex functions.*

The classical Hermite-Hadamard inequality for convex function given by the inequality (1).

Hermite-Hadamard inequality for multiplicatively convex function is proved by Ali et al. in [1] as follows:

THEOREM 2.10. *Let f be a positive and multiplicatively convex function on interval $[a,b]$,then the following inequalities hold*

$$f\left(\frac{a+b}{2}\right) \leq \left(\int_a^b (f(x))^{\mathrm{d}x}\right)^{\frac{1}{b-a}} \leq G(f(a), f(b)), \tag{2}$$

where $G(.,.)$ is a geometric mean.

The main purpose of this paper is to establish some inequalities connected with the left and right part of (2).

3. MIDPOINT TYPE INEQUALITIES FOR MULTIPLICATIVE INTEGRALS

LEMMA 3.1. *Let $f:[a,b]\to\mathbb{R}$ be * differentiable, let $g:[a,b]\to\mathbb{R}$ and $h: J\subset\mathbb{R}\to[a,b]$ be two differentiable functions. Then we have*

$$\int_a^b (f^*(h(x))^{g(x)h'(x)})^{\mathrm{d}x} = \frac{f(h(b))^{g(b)}}{f(h(a))^{g(a)}}.\frac{1}{\int_a^b (f(h(x))^{g'(x)})^{\mathrm{d}x}}.$$

Proof. The proof is obvious from the properties of multiplicative derivatives and integrals. □

Before we prove our results, we give the following lemma.

LEMMA 3.2. *Let $f: I^0 \subseteq \mathbb{R} \to \mathbb{R}^+$ be a * differentiable mapping on I^0, $a,b\in I^0$ with $a<b$. If f^* is * integrable on $[a,b]$, then we have*

$$\frac{\left(\int_a^b (f(x)^{\mathrm{d}x}\right)^{\frac{1}{b-a}}}{f\left(\frac{a+b}{2}\right)}$$
$$= \left[\int_0^{\frac{1}{2}} \left([f^*(at+(1-t)b)]^t\right)^{\mathrm{d}t} . \int_{\frac{1}{2}}^1 \left([f^*(at+(1-t)b)]^{(t-1)}\right)^{\mathrm{d}t}\right]^{b-a}.$$

Proof. By using the Lemma 3.1, we obtain

$$\left(\int_0^{\frac{1}{2}} \left([f^*(at+(1-t)b)]^t\right)^{\mathrm{d}t}\right)^{(b-a)}$$
$$\times\left(\int_{\frac{1}{2}}^1 \left([f^*(at+(1-t)b)]^{t-1}\right)^{\mathrm{d}t}\right)^{(b-a)}$$

$$
\begin{aligned}
&= \int_0^{\frac{1}{2}} \left([f^*(at+(1-t)b)]^{t(b-a)} \right)^{\mathrm{d}t} \cdot \int_{\frac{1}{2}}^{1} \left([f^*(at+(1-t)b)]^{(t-1)(b-a)} \right)^{\mathrm{d}t} \\
&= \left(f\left(\frac{a+b}{2}\right) \right)^{-\frac{1}{2}} \frac{1}{\int_0^{\frac{1}{2}} \left([f(at+(1-t)b)]^{-1} \right)^{\mathrm{d}t}} \cdot \frac{1}{\left(f\left(\frac{a+b}{2}\right)\right)^{\frac{1}{2}}} \\
&\times \frac{1}{\int_{\frac{1}{2}}^{1} \left([f(at+(1-t)b)]^{-1} \right)^{\mathrm{d}t}} \\
&= \left(f\left(\frac{a+b}{2}\right) \right)^{-1} \frac{1}{\left(\int_0^{\frac{1}{2}} (f(at+(1-t)b))^{\mathrm{d}t} \right)^{-1}} \\
&\times \frac{1}{\left(\int_{\frac{1}{2}}^{1} (f(at+(1-t)b))^{\mathrm{d}t} \right)^{-1}} \\
&= \frac{\int_0^{\frac{1}{2}} (f(at+(1-t)b))^{\mathrm{d}t} \cdot \int_{\frac{1}{2}}^{1} (f(at+(1-t)b))^{\mathrm{d}t}}{f\left(\frac{a+b}{2}\right)} \\
&= \frac{\int_0^{1} (f(at+(1-t)b))^{\mathrm{d}t}}{f\left(\frac{a+b}{2}\right)} = \frac{\left(\int_a^b (f(x))^{\mathrm{d}x} \right)^{\frac{1}{b-a}}}{f\left(\frac{a+b}{2}\right)}.
\end{aligned}
$$

This completes the proof. $\square$

Now, using Lemma 3.2, we give the following theorems.

THEOREM 3.3. *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}^+$ be a * differentiable mapping on I^0, $a, b \in I^0$ with $a < b$. If f is increasing on $[a, b]$ and f^* is multiplicatively convex on $[a, b]$, then we have*

$$
\left| \frac{\left(\int_a^b (f(x))^{\mathrm{d}x} \right)^{\frac{1}{b-a}}}{f\left(\frac{a+b}{2}\right)} \right| \le \left(f^*(a) f^*(b) \right)^{\frac{b-a}{8}}.
$$

Proof. Using Lemma 3.2 and the multiplicative convexity of f^*, we have

$$
\begin{aligned}
\left|\frac{\left(\int\limits_a^b (f(x)^{\mathrm{d}x}\right)^{\frac{1}{b-a}}}{f\left(\frac{a+b}{2}\right)}\right| &\leq \left|\int\limits_0^{\frac{1}{2}} \left([f^*(at+(1-t)b)]^t\right)^{\mathrm{d}t}\right.\\
&\quad \times \left.\int\limits_{\frac{1}{2}}^1 \left([f^*(at+(1-t)b)]^{(t-1)}\right)^{\mathrm{d}t}\right|^{b-a}\\
&\leq \left[\left|\int\limits_0^{\frac{1}{2}} \left([f^*(at+(1-t)b)]^t\right)^{\mathrm{d}t}\right|\left|\int\limits_{\frac{1}{2}}^1 \left([f^*(at+(1-t)b)]^{(t-1)}\right)^{\mathrm{d}t}\right|\right]^{b-a}\\
&\leq \exp\left((b-a)\int\limits_0^{\frac{1}{2}} |\ln f^*(at+(1-t)b)^t|\mathrm{d}t\right)\\
&\quad \times \exp\left((b-a)\int\limits_{\frac{1}{2}}^1 |\ln f^*(at+(1-t)b)^{(t-1)}|\mathrm{d}t\right)\\
&= \exp\left((b-a)\int\limits_0^{\frac{1}{2}} |t\ln f^*(at+(1-t)b)|\mathrm{d}t\right)\\
&\quad \times \exp\left((b-a)\int\limits_{\frac{1}{2}}^1 |(t-1)\ln f^*(at+(1-t)b)|\mathrm{d}t\right)\\
&= \exp\left((b-a)\int\limits_0^{\frac{1}{2}} t\ln f^*(at+(1-t)b)\mathrm{d}t\right)\\
&\quad \times \exp\left((b-a)\int\limits_{\frac{1}{2}}^1 (1-t)\ln f^*(at+(1-t)b)\mathrm{d}t\right)\\
&\leq \exp\left((b-a)\int\limits_0^{\frac{1}{2}} t\ln\left(f^*(a)^t f^*(b)^{1-t}\right)\mathrm{d}t\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \exp\left((b-a)\int_{\frac{1}{2}}^{1}(1-t)\ln\left(f^*(a)^t f^*(b)^{1-t}\right)\mathrm{d}t\right)\\
&\le \exp\left((b-a)\int_{0}^{\frac{1}{2}} t(t\ln f^*(a)+(1-t)\ln f^*(b))\mathrm{d}t\right)\\
&\quad\times \exp\left((b-a)\int_{\frac{1}{2}}^{1}(1-t)(t\ln f^*(a)+(1-t)\ln f^*(b))\mathrm{d}t\right)\\
&= (f^*(a)f^*(b))^{\frac{b-a}{8}},
\end{aligned}
$$

where we have used the facts that $\int_0^{\frac{1}{2}} t(1-t)\mathrm{d}t = \int_{\frac{1}{2}}^{1}(1-t)t\mathrm{d}t = \frac{1}{12}$, $\int_0^{\frac{1}{2}} t^2\mathrm{d}t = \frac{1}{24}$, $\int_{\frac{1}{2}}^{1}(1-t)^2\mathrm{d}t = \frac{1}{24}$. This completes the proof. □

THEOREM 3.4. *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}^+$ be a * differentiable mapping on I^0, $a, b \in I^0$ with $a < b$. If f is increasing on $[a, b]$ and $(\ln f^*)^q$, $q > 1$, is convex on $[a, b]$, then we have*

$$
\left|\frac{\left(\int_a^b (f(x)^{\mathrm{d}x}\right)^{\frac{1}{b-a}}}{f\left(\frac{a+b}{2}\right)}\right| \le (f^*(a)f^*(b))^{\frac{b-a}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}},
$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Lemma 3.2 and the Hölder's integral inequality it follows that

$$
\begin{aligned}
\left|\frac{\left(\int_a^b (f(x)^{\mathrm{d}x}\right)^{\frac{1}{b-a}}}{f\left(\frac{a+b}{2}\right)}\right| &\le \left[\left|\int_0^{\frac{1}{2}} \left([f^*(at+(1-t)b)]^t\right)^{\mathrm{d}t}\right.\right.\\
&\quad\times \left.\left.\int_{\frac{1}{2}}^{1}\left([f^*(at+(1-t)b)]^{(t-1)}\right)^{\mathrm{d}t}\right|\right]^{b-a}
\end{aligned}
\tag{3}
$$

$$\leq \left[\left| \int_0^{\frac{1}{2}} \left([f^*(at+(1-t)b)]^t \right)^{\mathrm{d}t} \right| \right.$$

$$\left. \times \left| \int_{\frac{1}{2}}^1 \left([f^*(at+(1-t)b)]^{(t-1)} \right)^{\mathrm{d}t} \right| \right]^{b-a}$$

$$\leq \left[\exp\left((b-a) \int_0^{\frac{1}{2}} |\ln f^*(at+(1-t)b)^t| \mathrm{d}t \right) \right.$$

$$\left. \times \exp\left((b-a) \int_{\frac{1}{2}}^1 |\ln f^*(at+(1-t)b)^{(t-1)}| \mathrm{d}t \right) \right]$$

$$= \exp\left((b-a) \int_0^{\frac{1}{2}} \left(|t \ln f^*(at+(1-t)b)] \right| \mathrm{d}t \right)$$

$$\times \exp\left((b-a) \int_{\frac{1}{2}}^1 \left(|(t-1) \ln f^*(at+(1-t)b)] \right| \mathrm{d}t \right)$$

$$\leq \exp\left((b-a) \left(\int_0^{\frac{1}{2}} |t|^p \mathrm{d}t \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} |\ln f^*(at+(1-t)b))|^q \mathrm{d}t \right)^{\frac{1}{q}} \right)$$

$$\times \exp\left((b-a) \left(\int_{\frac{1}{2}}^1 |t-1|^p \mathrm{d}t \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 |\ln f^*(at+(1-t)b)|^q \mathrm{d}t \right)^{\frac{1}{q}} \right),$$

where $\frac{1}{p}+\frac{1}{q}=1$. Using the convexity of $(\ln f^*)^q$, we obtain

$$\begin{aligned} & \int_0^{\frac{1}{2}} (\ln f^*(ta+(1-t)b))^q \, \mathrm{d}t \\ & \leq \int_0^{\frac{1}{2}} [t((\ln f^*(a))^q + (1-t)\,(\ln f^*(b))^q] \mathrm{d}t \\ & = \frac{(\ln f^*(a))^q + 3(\ln f^*(b))^q}{8}, \end{aligned} \tag{4}$$

$$
\begin{aligned}
&\int_{\frac{1}{2}}^{1} (\ln f^{*}(ta+(1-t)b))^{q}\, \mathrm{d}t \\
&\le \int_{\frac{1}{2}}^{1} [t(\ln f^{*}(a))^{q} + (1-t)\,(\ln f^{*}(b))^{q}]\mathrm{d}t \qquad (5)\\
&= \frac{3(\ln f^{*}(b))^{q} + (\ln f^{*}(b))^{q}}{8}.
\end{aligned}
$$

If we substitute the inequalities (4) and (5) in (3), then we have

$$
\begin{aligned}
&\left| \frac{\left(\int\limits_a^b (f(x)^{\mathrm{d}x}\right)^{\frac{1}{b-a}}}{f\left(\frac{a+b}{2}\right)} \right| \\
\le\ & \exp\left((b-a)\left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{(\ln f^{*}(a))^{q}+3(\ln f^{*}(b))^{q}}{8}\right)^{\frac{1}{q}}\right) \\
&\times \exp\left((b-a)\left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{3\,(\ln f^{*}(a))^{q}+(\ln f^{*}(b))^{q}}{8}\right)^{\frac{1}{q}}\right) \\
\le\ & \exp\left(\frac{b-a}{2^{1+\frac{1}{p}}}\left(\frac{1}{(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{(\ln f^{*}(a))^{q}+3(\ln f^{*}(b))^{q}}{8}\right)^{\frac{1}{q}}\right.\right. \\
+\ & \left.\left.\left(\frac{(3\ln f^{*}(a))^{q}+(\ln f^{*}(b))^{q}}{8}\right)^{\frac{1}{q}}\right]\right).
\end{aligned}
$$

Here, we use the facts that

$$
\int_0^{\frac{1}{2}} t^{p}\mathrm{d}t = \int_{\frac{1}{2}}^{1} |1-t|^{p}\mathrm{d}t = \int_{\frac{1}{2}}^{1} (1-t)^{p}\mathrm{d}t = \frac{1}{(p+1)2^{p+1}}.
$$

Now, let $a_1 = (\ln f^{*}(a))^{q}$, $b_1 = 3(\ln f^{*}(b))^{q}$, $a_2 = 3\,(\ln f^{*}(a))^{q}$ and $b_2 = (\ln f^{*}(b))^{q}$. Using the facts that,

$$
\sum_{k=1}^{n} (a_k+b_k)^{s} \le \sum_{k=1}^{n} a_k^{s} + \sum_{k=1}^{n} b_k^{s},\ 0 \le s < 1
$$

and $1+3^{\frac{1}{q}} \le 4$, we have

$$
\begin{aligned}
&\left(\frac{(\ln f^{*}(a))^{q}+3(\ln f^{*}(b))^{q}}{8}\right)^{\frac{1}{q}} + \left(\frac{3\,(\ln f^{*}(a))^{q}+(\ln f^{*}(b))^{q}}{8}\right)^{\frac{1}{q}} \\
\le\ & \left(\frac{1}{8}\right)^{\frac{1}{q}}\left(1+3^{\frac{1}{q}}\right)[\ln f^{*}(a)+\ln f^{*}(b)] \le 4\left(\frac{1}{8}\right)^{\frac{1}{q}} \ln\left(f^{*}(a)f^{*}(b)\right).
\end{aligned}
$$

This completes the proof. □

4. TRAPEZOID TYPE INEQUALITIES FOR MULTIPLICATIVE INTEGRALS

In this section we obtain some trapezoid type inequalities for multiplicatively convex functions. First we give the following lemma.

LEMMA 4.1. *Let $f : I^0 \subseteq \mathbb{R} \to \mathbb{R}^+$ be a * differentiable mapping on I^0, $a, b \in I^0$ with $a < b$. If f^* is * integrable on $[a, b]$, then we have*

$$\frac{\sqrt{f(a)f(b)}}{\left(\int_a^b (f(x)^{\mathrm{d}x}\right)^{\frac{1}{b-a}}} = \left[\int_0^1 \left([f^*(at + (1-t)b)]^{\frac{1}{2}-t}\right)^{\mathrm{d}t}\right]^{b-a}.$$

Proof. Using Lemma 3.1, we have

$$\begin{aligned}
&\left[\int_0^1 \left([f^*(at + (1-t)b)]^{\left(\frac{1}{2}-t\right)}\right)^{\mathrm{d}t}\right]^{b-a} \\
&= \int_0^1 \left([f^*(at + (1-t)b)]^{\left(\frac{1}{2}-t\right)(b-a)}\right)^{\mathrm{d}t} \\
&= \frac{(f(a))^{\frac{1}{2}}}{(f(b))^{\frac{-1}{2}}} \cdot \frac{1}{\int_0^1 (f(ta + (1-t)b))^{\mathrm{d}t}} \\
&= \frac{\sqrt{f(a)f(b)}}{\left(\int_a^b (f(x)^{\mathrm{d}x}\right)^{\frac{1}{b-a}}}.
\end{aligned}$$

This completes the proof. $\square$

THEOREM 4.2. *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}^+$ be a * differentiable mapping on I^0, $a, b \in I^0$ with $a < b$. If f is increasing on $[a, b]$ and f^* is multiplicatively convex on $[a, b]$, then we have*

$$\left|\frac{\sqrt{f(a)f(b)}}{\left(\int_a^b (f(x)^{\mathrm{d}x}\right)^{\frac{1}{b-a}}}\right| \leq (f^*(a)f^*(b))^{\frac{1}{8}}.$$

Proof. Using Lemma 4.1, we have

$$\begin{aligned}
&\left|\frac{\sqrt{f(a)f(b)}}{\left(\int\limits_a^b (f(x)^{\mathrm{d}x}\right)^{\frac{1}{b-a}}}\right| \leq \left[\left|\int\limits_0^1 \left([f^*(at+(1-t)b)]^{\frac{1}{2}-t}\right)^{\mathrm{d}t}\right|\right]^{b-a} \\
&\leq \exp\left((b-a)\int\limits_0^1 (|\ln f^*(at+(1-t)b)]^{\left(\frac{1}{2}-t\right)}|\mathrm{d}t\right) \\
&= \exp\left((b-a)\int\limits_0^1 \left|t-\frac{1}{2}\right| |\ln f^*(at+(1-t)b)|\,\mathrm{d}t\right).
\end{aligned} \tag{6}$$

Since f^* is multiplicatively convex, we get

$$\begin{aligned}
&\int\limits_0^1 \left|t-\frac{1}{2}\right| \ln f^*(at+(1-t)b)\mathrm{d}t \\
&\leq \int\limits_0^1 \left|t-\frac{1}{2}\right| [(1-t)\ln f^*(a)+t\ln f^*(b)]\,\mathrm{d}t \\
&= \ln f^*(a)\int\limits_0^1 \left|t-\frac{1}{2}\right|(1-t)\mathrm{d}t + \ln f^*(b)\int\limits_0^1 \left|t-\frac{1}{2}\right| t\mathrm{d}t \\
&= \frac{\ln f^*(a)+\ln f^*(b)}{8}.
\end{aligned} \tag{7}$$

If we substitute the inequality (7) in (6), we obtain

$$\begin{aligned}
\left|\frac{\sqrt{f(a)f(b)}}{\left(\int\limits_a^b (f(x)^{\mathrm{d}x}\right)^{\frac{1}{b-a}}}\right| &\leq \exp\left((b-a)\int\limits_0^1 \left|t-\frac{1}{2}\right| \ln f^*(at+(1-t)b)\mathrm{d}t\right) \\
&\leq \exp\left((b-a)\frac{\ln f^*(a)+\ln f^*(b)}{8}\right) \\
&= (f^*(a)f^*(b))^{\frac{b-a}{8}},
\end{aligned}$$

which completes the proof. $\square$

Theorem 4.3. *Let $f : I^0 \subset R \to R$ be a * differentiable mapping on I^0, $a, b \in I^0$ with $a < b$. If f is increasing on $[a, b]$ and $(\ln f^*)^q$, $q > 1$, is convex* *on* $[a,b]$*, then we have*

$$\left|\frac{\sqrt{f(a)f(b)}}{\left(\int\limits_a^b (f(x)^{\mathrm{d}x}\right)^{\frac{1}{b-a}}}\right| \leq \left(f^*(a)f^*(b)\right)^{(b-a)\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 4.1 and the Hölder's inequality it follows that

$$\left|\frac{\sqrt{f(a)f(b)}}{\left(\int\limits_a^b (f(x)^{\mathrm{d}x}\right)^{\frac{1}{b-a}}}\right| \leq \left[\left|\int\limits_0^1 \left([f^*(at+(1-t)b)]^{\frac{1}{2}-t}\right)^{\mathrm{d}t}\right|\right]^{b-a}$$

$$\leq \exp\left((b-a)\int\limits_0^1 |\ln f^*(at+(1-t)b)^{\left(\frac{1}{2}-t\right)}|\mathrm{d}t\right) \tag{8}$$

$$= \exp\left((b-a)\int\limits_0^1 \left|\left(\frac{1}{2}-t\right)\ln f^*(at+(1-t)b)\right|\mathrm{d}t\right)$$

$$\leq \exp\left((b-a)\int\limits_0^1 \left|t-\frac{1}{2}\right| |\ln f^*(at+(1-t)b)|\mathrm{d}t\right)$$

$$\begin{aligned} &\leq \exp\left((b-a)\left(\int\limits_0^1 \left|t-\frac{1}{2}\right|^p \mathrm{d}t\right)^{\frac{1}{p}}\right. \\ &\left.\times\left(\int_0^1 (\ln f^*(at+(1-t)b))^q\mathrm{d}t\right)^{\frac{1}{q}}\right). \end{aligned} \tag{9}$$

Using the convexity of $(\ln f^*)^q$, we obtain

$$\begin{aligned} &\int_0^1 (\ln f^*(at+(1-t)b))^q\mathrm{d}t \\ &\leq \int_0^1 [t(\ln f^*(a))^q+(1-t)(\ln f^*(b))^q]\mathrm{d}t \\ &= \frac{(\ln f^*(a))^q+(\ln f^*(b))^q}{2}. \end{aligned} \tag{10}$$

Further we have

$$\int_0^1 \left| t - \frac{1}{2} \right|^p \mathrm{d}t = \frac{1}{(p+1)2^p} \tag{11}$$

By combining (8)–(11), we get

$$\begin{aligned} & \left| \frac{\sqrt{f(a)f(b)}}{\left(\int\limits_a^b (f(x)^{\mathrm{d}x} \right)^{\frac{1}{b-a}}} \right| \\ \leq \ & \exp \left(\frac{b-a}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{(\ln f^*(a))^q + (\ln f^*(b))^q}{2} \right)^{\frac{1}{q}} \right) \\ \leq \ & \exp \left((b-a) \left(\frac{1}{2} \right)^{1+\frac{1}{q}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} (\ln f^*(a) f^*(b)) \right). \end{aligned}$$

Here we use the inequality $c^\lambda + d^\gamma \leq (c+d)^\lambda$ for $c, d > 0$ and $\lambda > 1$.

Thus, the proof is completed. □

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