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ON THE SOLVABILITY OF A SYSTEM OF CAPUTO-HADAMARD FRACTIONAL HYBRID DIFFERENTIAL EQUATIONS SUBJECT TO SOME HYBRID BOUNDARY CONDITIONS

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Abstract. In this work, we give some existence and regularity results for a system of a new class of hybrid Caputo-Hadamard fractional differential equations under hybrid boundary conditions. The technique of investigation is essentially based on the use of a well known hybrid fixed point theorem.

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Key words. Hybrid fractional differential equation, Caputo-Hadamard derivative, hybrid boundary value problem, fixed point theorems.

1. INTRODUCTION

The theory of fractional differential equations has received much attention over the past years and has become an important field of investigation due to its extensive applications in numerous branches of physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology, etc. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials. As a consequence, the subject of fractional differential equations is gaining much importance and attention. The study of hybrid fractional differential equations is an attractive topic. In fact, this kind of problems are often encountered in several branches of engineering and physics, for more information, we refer the reader to [19] and the references therein. We recall that this class of equations involves the fractional derivative of an unknown hybrid function with the nonlinearity depending on it. Some recent results on hybrid differential equations can be found in (see [1, 3, 4, 5, 6, 7, 8, 9, 13, 14, 23, 24, 26, 27]).

In this work, we are concerned with the existence and uniqueness of solutions for the following system of the fractional differential equations

$$(1)\qquad \mathrm{D}^{r}_{1^+}\left[\frac{x(\tau)-\sum_{i=1}^{m}\mathrm{I}^{q_i}_{1^+}f_i(\tau,x(\tau))}{g(\tau,x(\tau))}\right]=h(\tau,x(\tau)),\qquad \tau\in \mathrm{J}.$$

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Here $\mathrm{J} := [1, T]$ and $1 < r \leq 2$ while $\mathrm{D}^{r}_{1^+}$ and $\mathrm{I}^{q_i}_{1^+}$ denote the Caputo-Hadamard fractional derivatives of orders ϵ,$\epsilon \in \{r, p\}$, $0 < p \leq 1$ and Hadamard integral of order q_i, respectively. We assume also that $g \in \mathrm{C}(\mathrm{J} \times \mathbb{R}, \mathbb{R} - \{0\})$ and $f, h \in \mathrm{C}(\mathrm{J} \times \mathbb{R}, \mathbb{R})$. The boundary conditions imposed to our problem are given by

$$\tag{2} \begin{cases} \alpha_1 \left[\frac{x(\tau)-\sum_{i=1}^{m} \mathrm{I}^{q_i} f_i(\tau,x(\tau))}{g(\tau,x(\tau))}\right]_{\tau=1} \\ +\beta_1 \mathrm{D}^{p}_{1^+} \left[\frac{x(\tau)-\sum_{i=1}^{m} \mathrm{I}^{q_i} f_i(\tau,x(\tau))}{g(\tau,x(\tau))}\right]_{\tau=1} = \gamma_1, \\ \alpha_2 \left[\frac{x(\tau)-\sum_{i=1}^{m} \mathrm{I}^{q_i} f_i(\tau,x(\tau))}{g(\tau,x(\tau))}\right]_{\tau=T} \\ +\beta_2 \mathrm{D}^{p}_{1^+} \left[\frac{x(\tau)-\sum_{i=1}^{m} \mathrm{I}^{q_i} f_i(\tau,x(\tau))}{g(\tau,x(\tau))}\right]_{\tau=T} = \gamma_2, \end{cases}$$

where $\alpha_i, \beta_i, \gamma_i, i = 1, 2$, are real constants.

We mention here that the study of Hybrid fractional problems was the subject of several works. For instance, we find in [2] some regularity results concerning this type of problems, these authors opted for the use of Krasnoselskii's fixed point theorem to investigate the scalar case with respect to the classical Caputo fractional derivatives. In the same direction, in [20], the Mönch's fixed point theorem combined with the technique of measures of weak noncompactness was successfully used to obtain some interesting regularity results.

This paper is organized as follows. In Section 2, we recall briefly some basic definitions and preliminary facts which will be used throughout subsequent sections. Section 3, contains the existence of solutions for the boundary value problem (1)–(2) which is obtained by means of an hybrid fixed point theorem for three operators in a Banach algebra due to Dhage [10]. Finally, we illustrate the obtained results by an example.

2. PRELIMINARIES

At first, we recall some basic concepts on fractional calculus and present some additional properties that will be used later. For more details, we refer to [15, 16, 18, 21, 25] and the references therein.

Definition 2.1 ([18]). The Hadamard fractional integral of order $\alpha > 0$ for a function $\omega : \mathrm{J} \to \mathbb{R}$ is defined as

$$\tag{3} \mathrm{I}^{\alpha}_{a^+}\omega(\tau) = \frac{1}{\Gamma(\alpha)} \int_a^{\tau} \left(\log \frac{\tau}{s}\right)^{\alpha-1} \omega(s) \frac{\mathrm{d}s}{s},$$

where Γ is the Gamma function.

Definition 2.2 ([18]). For a function ω given on the interval J, and $n-1 < r < n$, the Hadamard derivative of order $\alpha > 0$ is defined by

$$\tag{4} \begin{aligned} {}^{H}\mathrm{D}^{\alpha}_{a^+}\omega(\tau) &= \frac{1}{\Gamma(n-\alpha)} \left(\tau \frac{\mathrm{d}}{\mathrm{d}t}\right)^{n} \int_a^{\tau} \left(\log \frac{\tau}{s}\right)^{n-\alpha-1} \omega(s) \frac{\mathrm{d}s}{s} \\ &= \delta^n \mathrm{I}^{n-\alpha}_{a^+}\omega(\tau), \end{aligned}$$

where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of the real number α and $\delta = \tau \frac{\mathrm{d}}{\mathrm{d}t}$.

There is a recent generalization introduced in [16], where the authors define a generalization of the Hadamard fractional derivatives and present properties of such derivatives. This new generalization is now known as the Caputo-Hadamard fractional derivatives and is given by the following definition:

Definition 2.3 ([16, Caputo-Hadamard fractional derivative]). Let $\alpha \geq 0$, and $n = [\alpha] + 1$, and $\omega(\tau) \in AC_\delta^n(\mathrm{J})$, where

$$AC_\delta^n(\mathrm{J}) = \left\{\omega : \mathrm{J} \to \mathbb{R} : \delta^{n-1}\omega(\tau) \in AC(\mathrm{J})\right\},$$

and $\delta = \tau \frac{\mathrm{d}}{\mathrm{d}t}$ is the Hadamard derivative.

The Caputo-type modification of Hadamard fractional derivatives of order α is given by

$$\mathrm{D}_{a^+}^\alpha \omega(\tau) =^H \mathrm{D}_{a^+}^\alpha \left(\omega(\tau) - \sum_{k=0}^{n-1} \frac{\delta^k \omega(a)}{k!} \left(\log \frac{\tau}{s}\right)^k\right). \tag{5}$$

Theorem 2.4 ([16]). *Let $\alpha \geq 0$, and $n = [\alpha] + 1$. If $\omega(\tau) \in AC_\delta^n(\mathrm{J})$, where $0 < a < b < \infty$, then $\mathrm{D}_{a^+}^\alpha \omega(\tau)$ exists everywhere on* J *and*
(i) *if $\alpha \notin \mathbb{N} - \{0\}$, $\mathrm{D}_{a^+}^\alpha f(\tau)$ can be represented by*

$$\begin{aligned} \mathrm{D}_{a^+}^\alpha \omega(\tau) &= \frac{1}{\Gamma(n-\alpha)} \int_a^\tau \left(\log \frac{\tau}{s}\right)^{n-\alpha-1} \delta^n \omega(s) \frac{\mathrm{d}s}{s}, \\ &= \mathrm{I}_{a^+}^{n-\alpha} \delta^n \omega(\tau). \end{aligned} \tag{6}$$

(ii) *if $\alpha \in \mathbb{N} - \{0\}$, then*

$$\mathrm{D}_{a^+}^\alpha \omega(\tau) = \delta^n \omega(\tau). \tag{7}$$

In particular,

$$\mathrm{D}_{a^+}^0 \omega(\tau) = \omega(\tau). \tag{8}$$

Caputo-Hadamard fractional derivatives can also be defined on the positive half axis $\mathbb{R}^+$ by replacing a by 0 in formula (6) provided that $\omega(\tau) \in AC_\delta^n(\mathbb{R}^+)$. Thus one has

$${}^c\mathrm{D}_{a^+}^\alpha \omega(\tau) = \frac{1}{\Gamma(n-\alpha)} \int_a^\tau \left(\log \frac{\tau}{s}\right)^{n-\alpha-1} \delta^n \omega(s) \frac{\mathrm{d}s}{s}. \tag{9}$$

Proposition 2.5 ([18]). *Let $\alpha > 0, \beta > 0, n = [\alpha] + 1$, and $a > 0$, then*

$$\begin{aligned} &\mathrm{D}_{a^+}^\alpha \left(\log \frac{\tau}{a}\right)^{\beta-1} (\omega) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\log \frac{\omega}{a}\right)^{\beta-\alpha-1}, \beta > n, \\ &\mathrm{D}_{a^+}^\alpha \left(\log \frac{\tau}{a}\right)^k = 0, k = 0, 1, ..., n-1. \end{aligned} \tag{10}$$

THEOREM 2.6 ([15]). *Let* $\omega(\tau) \in AC_\delta^n(\mathrm{J})$*, and* $\alpha \geq 0, \beta \geq 0$*. Then*

$$\begin{aligned} \mathrm{I}_{a^+}^{\alpha}\mathrm{I}_{a^+}^{\beta}\omega(\tau) &= \mathrm{I}_{a^+}^{\beta+\alpha}\omega(\tau), \\ \mathrm{D}_{a^+}^{\alpha}\mathrm{I}_{a^+}^{\alpha}\omega(\tau) &= \omega(\tau), \\ \mathrm{D}_{a^+}^{\alpha}\mathrm{I}_{a^+}^{\beta}\omega(\tau) &= \mathrm{I}_{a^+}^{\beta-\alpha}\omega(\tau), \\ \mathrm{D}_{a^+}^{\alpha}\mathrm{D}_{a^+}^{\beta}\omega(\tau) &= \mathrm{D}_{a^+}^{\alpha+\beta}\omega(\tau). \end{aligned} \tag{11}$$

LEMMA 2.7 ([16]). *Let* $\alpha \geq 0$*, and* $n = [\alpha] + 1$*. If* $\omega(\tau) \in AC_\delta^n(\mathrm{J})$*, then the Caputo-Hadamard fractional differential equation*

$$\mathrm{D}_{a^+}^{\alpha}\omega(\tau) = 0, \tag{12}$$

has a solution:

$$\omega(\tau) = \sum_{k=0}^{n-1} c_k \left(\log\frac{\tau}{a}\right)^k, \tag{13}$$

and the following formula holds:

$$\mathrm{I}_{a^+}^{\alpha}\left(\mathrm{D}_{a^+}^{\alpha}\omega\right)(\tau) = \omega(\tau) + \sum_{k=0}^{n-1} c_k \left(\log\frac{\tau}{a}\right)^k, \tag{14}$$

where $c_k \in \mathbb{R}, k = 1, 2, ..., n-1$*.*

Now, let us consider the usual Banach space $\mathrm{E} = \mathrm{C}([1,T], \mathbb{R})$ endowed with the classical supremum norm

$$\|\omega\| = \sup_{\tau\in\mathrm{J}} |\omega(\tau)|\,.$$

At this level, it is necessary to recall that E is a Banach algebra.

To prove the existence result for the nonlocal boundary value problem (1)–(2), we will use the following hybrid fixed point theorem for three operators in a Banach algebra E due to Dhage, see [10].

LEMMA 2.8. *Let* S *be a closed convex bounded nonempty subset of a Banach algebra* E*, and let* $\mathrm{A}, \mathrm{C} : \mathrm{E} \to \mathrm{E}$ *and* $\mathrm{B} : \mathrm{S} \to \mathrm{E}$ *be three operators such that:*

(a) A *and* C *are Lipschitzian with a Lipschitz constants* δ *and* ρ*, respectively,*
(b) B *is compact and continuous,*
(c) $x = \mathrm{A}x\mathrm{B}\omega + \mathrm{C}x \Rightarrow x$ *in* S *for all* ω *in* S*,*
(d) $\delta M + \rho < 1$*, where* $M = \|\mathrm{B(S)}\|$*.*

Then the operator equation $x = \mathrm{A}x\mathrm{B}\omega + \mathrm{C}x$ *admits at least one solution in* S*.*

3. MAIN RESULTS

In this section, we prove an existence result for the boundary value problems for hybrid differential equations with fractional order on the closed bounded interval J.

LEMMA 3.1. *Let h be a continuous function on* J. *Then the solution of the boundary value problem*

$$\mathrm{D}^r_{1^+}\left[\frac{x(\tau)-\sum_{i=1}^m \mathrm{I}^{q_i}_{1^+} f_i(\tau,x(\tau))}{g(\tau,x(\tau))}\right] = h(\tau,x(\tau)), \quad \tau\in \mathrm{J}, \quad 1<r\le 2, \tag{15}$$

with boundary conditions

$$\begin{aligned}
&\alpha_1\left[\frac{x(\tau)-\sum_{i=1}^m \mathrm{I}^{q_i}_{1^+} f_i(\tau,x(\tau))}{g(\tau,x(\tau))}\right]_{\tau=1} \\
&+\beta_1\mathrm{D}^p_{1^+}\left[\frac{x(\tau)-\sum_{i=1}^m \mathrm{I}^{q_i}_{1^+} f_i(\tau,x(\tau))}{g(\tau,x(\tau))}\right]_{\tau=1} = \gamma_1, \\
&\alpha_2\left[\frac{x(\tau)-\sum_{i=1}^m \mathrm{I}^{q_i}_{1^+} f_i(\tau,x(\tau))}{g(\tau,x(\tau))}\right]_{\tau=T} \\
&+\beta_2\mathrm{D}^p_{1^+}\left[\frac{x(\tau)-\sum_{i=1}^m \mathrm{I}^{q_i}_{1^+} f_i(\tau,x(\tau))}{g(\tau,x(\tau))}\right]_{\tau=T} = \gamma_2,
\end{aligned} \tag{16}$$

satisfies the equation

$$x(\tau) = g(\tau,x(\tau))H(\tau) + \frac{\alpha_1 v_2(\log\tau)+\gamma_1 v_1}{\alpha_1 v_1}] + \sum_{i=1}^m \mathrm{I}^{q_i}_{1^+} f_i(\tau,x(\tau)), \tag{17}$$

where
$H(\tau) := \mathrm{I}^r_{1^+}h(\tau) - \frac{(\log\tau)}{v_1}\left\{\alpha_2\mathrm{I}^r_{1^+}h(T)+\beta_2\mathrm{I}^{r-p}_{1^+}h(T)\right\}$, *and*

$$v_1 = \left(\alpha_2(\log T)+\beta_2\frac{(\log T)^{1-p}}{\Gamma(2-p)}\right), \qquad v_2 = \frac{\gamma_2\alpha_1-\gamma_1\alpha_2}{\alpha_1}.$$

Proof. Applying the Hadamard fractional integral operator of order r to both sides of (15) and using Lemma 2.7, we get

$$\left[\frac{x(\tau)-\sum_{i=1}^m \mathrm{I}^{q_i}_{1^+} f_i(\tau,x(\tau))}{g(\tau,x(\tau))}\right] = \mathrm{I}^r_{1^+}h(\tau)+c_1+c_2(\log\tau), \quad c_1,c_2\in\mathbb{R}. \tag{18}$$

Consequently, the general solution of (15) is given by

$$\begin{aligned}
x(\tau) &= g(\tau,x(\tau))\,(\mathrm{I}^r_{1^+}h(\tau)+c_1 \\
&+c_2(\log\tau)) + \sum_{i=1}^m \mathrm{I}^{q_i}_{1^+} f_i(\tau,x(\tau)), \quad c_1,c_2\in\mathbb{R}.
\end{aligned} \tag{19}$$

Applying the boundary conditions (16) in (18), a simple calculation gives

$$\begin{aligned}
c_1 &= \frac{\gamma_1}{\alpha_1}, \\
c_2 &= \frac{1}{v_1}\left\{\gamma_2-\frac{\alpha_2\gamma_1}{\alpha_1}-\alpha_2\mathrm{I}^r_{1^+}h(T)-\beta_2\mathrm{I}^{r-p}_{1^+}h(T)\right\}.
\end{aligned}$$

Substituting the values of c_1, c_2 into (19), we get (17). □

Now we list the following hypotheses.

(H1) The functions $g : \mathrm{J} \times \mathbb{R} \to \mathbb{R}\ \{0\}$ and $h, f : \mathrm{J} \times \mathbb{R} \to \mathbb{R}$ are continuous.

(H2) There exist two positive functions ω_0, ϖ_1 with bounds $\|\omega_0\|$ and $\|\varpi_1\|$ respectively, such that

$$|g(\tau, x) - g(\tau, \omega)| \leq \omega_0(\tau)|x - \omega|, \tag{20}$$

and

$$|f_i(\tau, x) - f_i(\tau, \omega)| \leq \varpi_i(\tau)|x - \omega|, \tag{21}$$

for all $(\tau, x, \omega) \in \mathrm{J} \times \mathbb{R} \times \mathbb{R}$.

(H3) There exist a function $p \in L^\infty(\mathrm{J}, \mathbb{R}^+)$ and a continuous nondecreasing function $\varphi : [0, \infty) \to (0, \infty)$ such that

$$|h(\tau, x)| \leq p(\tau)\varphi\left(|x|\right), \tag{22}$$

for all $\tau \in \mathrm{J}$ and $x \in \mathbb{R}$.

(H4) There exists $R > 0$ such that

$$R \geq \frac{M g_0 + \sum_{i=1}^m \frac{f_i}{\Gamma(q_i+1)}}{1 - M\|\omega_0\| - \sum_{i=1}^m \frac{\|\varpi_i\|}{\Gamma(q_i+1)}}, \tag{23}$$

and

$$\|\omega_0\|M + \sum_{i=1}^m \frac{\|\varpi_i\|}{\Gamma(q_i + 1)} < 1, \tag{24}$$

where $g_0 = \sup_{\tau\in \mathrm{J}} |g(\tau, 0)|$, $f_i = \sup_{\tau\in \mathrm{J}} |f_i(\tau, 0)|$, $i = 1, ..., m$, and

$$M = \|p\|\varphi(R)K + \frac{|\alpha_1 v_2|(\log\tau) + |\gamma_1 v_1|}{|\alpha_1 v_1|}, \tag{25}$$

where

$$K = \left\{ \frac{(\log T)^r}{\Gamma(r + 1)} + \frac{|\alpha_2|}{|v_1|} \frac{(\log T)^{r+1}}{\Gamma(r + 1)} + \frac{|\beta_2|}{|v_1|} \frac{(\log T)^{r-p+1}}{\Gamma(r - p + 1)} \right\}.$$

THEOREM 3.2. *Assume that the assumptions* (H1)–(H4) *are fulfilled, then the problem* (1)–(2) *has at least one solution defined on* J.

Proof. First, we define the set

$$\mathrm{S} = \{x \in \mathrm{E} : \|x\|_{\mathrm{E}} \leq R\}.$$

Clearly, S is a closed convex bounded subset of the Banach space E. By Lemma 3.1 the boundary value problem (1)-(2) is equivalent to the equation

$$x(\tau) = \sum_{i=1}^m \mathrm{I}_{1^+}^{q_i} f_i(\tau, x(\tau)) + g(\tau, x(\tau))G(\tau) + \frac{\alpha_1 v_2(\log\tau) + \gamma_1 v_1}{\alpha_1 v_1}], \tau \in \mathrm{J}. \tag{26}$$

Where
$G(\tau) := [\mathrm{I}_{1^+}^r h(s, x(s))(\tau) - \frac{(\log\tau)}{v_1} \left\{ \alpha_2 \mathrm{I}_{1^+}^r h(s, x(s))(T) + \beta_2 \mathrm{I}_{1^+}^{r-p} h(s, x(s))(T) \right\}.$

Now, let us introduce the following three operators $\mathrm{A}, \mathrm{C} : \mathrm{E} \to \mathrm{E}$ and $\mathrm{B} : \mathrm{S} \to \mathrm{E}$ defined by

$$\mathrm{A}x(\tau) = g(\tau, x(\tau)), \tau \in \mathrm{J},$$

$$\mathrm{B}x(\tau) = G(\tau) + \frac{\alpha_1 v_2(\log \tau) + \gamma_1 v_1}{\alpha_1 v_1}, \tau \in \mathrm{J},$$

and

$$\mathrm{C}x(\tau) = \sum_{i=1}^{m} \mathrm{I}_{1+}^{q_i} f_i(\tau, x(\tau)), \tau \in \mathrm{J}.$$

Then, the integral equation (26) can be written in the operator form as

$$x(\tau) = \mathrm{A}x(\tau)\mathrm{B}x(\tau) + \mathrm{C}x(\tau), \tau \in \mathrm{J}.$$

We will show that the operators A, B and C satisfy all the conditions of Lemma 2.8. This will be achieved in the following steps.

Step1: First, we show that A and C are Lipschitzian on E. Let $x, \omega \in \mathrm{E}$. Then by(H2),for $\tau \in \mathrm{J}$, we have

$$|\mathrm{A}x(\tau) - Ay(\tau)| = |g(\tau, x(\tau)) - g(\tau, \omega(\tau))| \leq \omega_0(\tau)|x(\tau) - \omega(\tau)|,$$

for all $\tau \in \mathrm{J}$. Taking the supremum over τ, we obtain

$$\|\mathrm{A}x - \mathrm{A}\omega\| \leq \|\omega_0\| \|x - \omega\|,$$

for all $x, \omega \to \mathrm{E}$. Therefore A is Lipschitzian on E with Lipschitz constant $\|\omega_0\|$.

Analogously, for $\mathrm{C} : \mathrm{E} \to \mathrm{E}$, $x, \omega \in \mathrm{E}$, we have

$$\begin{aligned} |\mathrm{C}x(\tau) - \mathrm{C}\omega(\tau)| &= \left| \sum_{i=1}^{m} \mathrm{I}_{1+}^{q_i} f_i(\tau, x(\tau)) - \sum_{i=1}^{m} \mathrm{I}_{1+}^{q_i} f_i(\tau, \omega(\tau)) \right| \\ &\leq \sum_{i=1}^{m} \frac{1}{\Gamma(q_i)} \int_1^{\tau} \left(\log \frac{\tau}{s} \right) \varpi_i(s) |x(s) - \omega(s)| \frac{\mathrm{d}s}{s} \\ &\leq \|x(\tau) - \omega(\tau)\| \sum_{i=1}^{m} \frac{\|\varpi_i\|}{\Gamma(q_i + 1)}, \end{aligned}$$

which implies that

$$\|Cx - Cy\| \leq \sum_{i=1}^{m} \frac{\|\varpi_i\|}{\Gamma(q_i + 1)} \|x(\tau) - \omega(\tau)\|.$$

Hence $\mathrm{C} : \mathrm{E} \to \mathrm{E}$ is Lipschitzian on E with Lipschitz constant given by

$$\sum_{i=1}^{m} \frac{\|\varpi_i\|}{\Gamma(q_i + 1)}.$$

Step 2: The operator B is completely continuous on S. We first show that the operator B is continuous on E. Let x_n be a sequence in S converging to a point $x \in \mathrm{S}$. Then by Lebesgue dominated convergence theorem, for all $\tau \in \mathrm{J}$, we obtain

$$\begin{aligned}\lim_{n\to\infty} \mathrm{B}x_n(\tau) &= \frac{1}{\Gamma(r)} \lim_{n\to\infty} \int_1^\tau \left(\log\frac{\tau}{s}\right) h(s, x_n(s)) \frac{\mathrm{d}s}{s} \\ &- \frac{(\log\tau)}{v_1} \left\{ \frac{\alpha_2}{\Gamma(r)} \lim_{n\to\infty} \int_1^T \left(\log\frac{T}{s}\right) h(s, x_n(s)) \frac{\mathrm{d}s}{s} \right. \\ &\left. + \frac{\beta_2}{\Gamma(r-p)} \lim_{n\to\infty} \int_1^T \left(\log\frac{T}{s}\right) h(s, x_n(s)) \frac{\mathrm{d}s}{s} \right\} \\ &+ \frac{\alpha_1 v_2 (\log\tau) + \gamma_1 v_1}{\alpha_1 v_1} \\ &= \frac{1}{\Gamma(r)} \int_1^\tau \left(\log\frac{\tau}{s}\right) \lim_{n\to\infty} h(s, x_n(s)) \frac{\mathrm{d}s}{s} \\ &- \frac{(\log\tau)}{v_1} \left\{ \frac{\alpha_2}{\Gamma(r)} \int_1^T \left(\log\frac{T}{s}\right) \lim_{n\to\infty} h(s, x_n(s)) \frac{\mathrm{d}s}{s} \right. \\ &\left. + \frac{\beta_2}{\Gamma(r-p)} \int_1^T \left(\log\frac{T}{s}\right) \lim_{n\to\infty} h(s, x_n(s)) \frac{\mathrm{d}s}{s} \right\} \\ &+ \frac{\alpha_1 v_2 (\log\tau) + \gamma_1 v_1}{\alpha_1 v_1} \\ &= G(\tau) + \frac{\alpha_1 v_2 (\log\tau) + \gamma_1 v_1}{\alpha_1 v_1} \\ &= Bx(\tau),\end{aligned}$$

for all $\tau \in \mathrm{J}$. This shows that B is a continuous operator on S.

Next, we will prove that the set B(S) is a uniformly bounded in S. For any $x \in \mathrm{S}$, we have

$$\begin{aligned}|\mathrm{B}x(\tau)| &\le \frac{1}{\Gamma(r)} \int_1^\tau \left(\log\frac{\tau}{s}\right) |h(s, x(s))| \frac{\mathrm{d}s}{s} \\ &+ \frac{(\log\tau)}{|v_1|} \left\{ \frac{|\alpha_2|}{\Gamma(r)} \int_1^T \left(\log\frac{T}{s}\right) |h(s, x(s))| \frac{\mathrm{d}s}{s} \right. \\ &\left. + \frac{|\beta_2|}{\Gamma(r-p)} \int_1^T \left(\log\frac{T}{s}\right) |h(s, x(s))| \frac{\mathrm{d}s}{s} \right\} + \frac{|\alpha_1 v_2| (\log\tau) + |\gamma_1 v_1|}{|\alpha_1 v_1|}.\end{aligned}$$

Using (22), we can write

$$\begin{aligned}&\le \frac{1}{\Gamma(r)} \int_1^\tau \left(\log\frac{\tau}{s}\right) p(s)\varphi(|x|) \frac{\mathrm{d}s}{s} - \frac{(\log\tau)}{|v_1|} \left\{ \frac{|\alpha_2|}{\Gamma(r)} \int_1^T \left(\log\frac{T}{s}\right) p(s)\varphi(|x|) \frac{\mathrm{d}s}{s} \right. \\ &\left. + \frac{|\beta_2|}{\Gamma(r-p)} \int_1^T \left(\log\frac{T}{s}\right) p(s)\varphi(|x|) \frac{\mathrm{d}s}{s} \right\} + \frac{|\alpha_1 v_2| (\log\tau) + |\gamma_1 v_1|}{|\alpha_1 v_1|} \\ &\le \|p\|\varphi(R)K + \frac{|\alpha_1 v_2| (\log\tau) + |\gamma_1 v_1|}{|\alpha_1 v_1|}.\end{aligned}$$

Thus $\|\mathrm{B}x\| \leq M$ for all $x \in \mathrm{S}$ with M given in (25). This shows that B is uniformly bounded on S.

Let $\tau_1, \tau_2 \in \mathrm{J}$. Then for any $x \in \mathrm{S}$, by (22) we get

$$\begin{aligned}
&|\mathrm{B}x(\tau_2) - \mathrm{B}x(\tau_1)| \\
&\leq \frac{1}{\Gamma(r)} \left| \int_1^{\tau_2} \left(\log \frac{\tau_2}{s}\right)^{r-1} h(s) \frac{\mathrm{d}s}{s} - \int_1^{\tau_1} \left(\log \frac{\tau_1}{s}\right)^{r-1} h(s) \frac{\mathrm{d}s}{s} \right| \\
&+ \frac{|(\log \tau_2) - (\log \tau_1)|}{|v_1|} \left\{ |\alpha_2| \mathrm{I}_{1+}^{r} h(T) + |\beta_2| \mathrm{I}_{1+}^{r-p} h(T) \right\} \\
&+ \frac{|\alpha_1 v_2|}{|\alpha_1 v_1|} |(\log \tau_2) - (\log \tau_1)| \\
&\leq \frac{\varphi(R)\|p\|}{\Gamma(r)} \int_1^{\tau_1} \left[\left(\log \frac{\tau_2}{s}\right)^{r-1} - \left(\log \frac{\tau_1}{s}\right)^{r-1} \right] \frac{\mathrm{d}s}{s} \\
&+ \frac{\varphi(R)\|p\|}{\Gamma(r)} \int_{\tau_1}^{\tau_2} \left(\log \frac{\tau_2}{s}\right)^{r-1} \frac{\mathrm{d}s}{s} \\
&+ \frac{|(\log \tau_2) - (\log \tau_1)|}{|v_1|} \left\{ |\alpha_2| \mathrm{I}_{1+}^{r} h(T) + |\beta_2| \mathrm{I}_{1+}^{r-p} h(T) \right\} \\
&+ \frac{|\alpha_1 v_2|}{|\alpha_1 v_1|} |(\log \tau_2) - (\log \tau_1)|.
\end{aligned} \tag{27}$$

Obviously, the right-hand side of inequality (27) tends to zero independently of $x \in \mathrm{S}$ as $\tau_2 \to \tau_1$. As a consequence of the Ascoli-Arzela theorem, B is a completely continuous operator on S.

Step 3: Hypothesis (c) of Lemma 2.8 is satisfied.

Let $x \in \mathrm{E}$ and $\omega \in \mathrm{S}$ be arbitrary elements such that $x = \mathrm{A}x\mathrm{B}\omega + \mathrm{C}x$. Then we have

$$\begin{aligned}
&|x(\tau)| \leq |\mathrm{A}x(\tau)||\mathrm{B}\omega(\tau)| + |\mathrm{C}x(\tau)| \\
&\leq \sum_{i=1}^{m} \mathrm{I}_{1+}^{q_i} |f_i(\tau, x(\tau))| \\
&+ |g(\tau, x(\tau))| \left[\frac{1}{\Gamma(r)} \int_1^{\tau} \left(\log \frac{\tau}{s}\right) |h(s, x(s))| \frac{\mathrm{d}s}{s} \right. \\
&+ \frac{(\log \tau)}{|v_1|} \left\{ \frac{|\alpha_2|}{\Gamma(r)} \int_1^{T} \left(\log \frac{T}{s}\right) |h(s, x(s))| \frac{\mathrm{d}s}{s} \right. \\
&\left. \left. + \frac{|\beta_2|}{\Gamma(r-p)} \int_1^{T} \left(\log \frac{T}{s}\right) |h(s, x(s))| \frac{\mathrm{d}s}{s} \right\} + \frac{|\alpha_1 v_2|(\log \tau) + |\gamma_1 v_1|}{|\alpha_1 v_1|} \right] \\
&\leq \sum_{i=1}^{m} \frac{1}{\Gamma(q_i)} \int_1^{\tau} \left(\log \frac{\tau}{s}\right)^{q_i+1} (|f_i(s, x(s)) - f_i(s, 0)| + |f_i(s, 0)|) \frac{\mathrm{d}s}{s} \\
&+ (|g(s, x(s)) - g(s, 0)| + |g(s, 0)|) \left[\frac{1}{\Gamma(r)} \int_1^{\tau} \left(\log \frac{\tau}{s}\right) \varphi(R) p(s) \frac{\mathrm{d}s}{s} \right.
\end{aligned}$$

$$+\frac{(\log\tau)}{|v_1|}\left\{\frac{|\alpha_2|}{\Gamma(r)}\int_1^T\left(\log\frac{T}{s}\right)\varphi(R)p(s)\frac{\mathrm{d}s}{s}\right.$$
$$\left.+\frac{|\beta_2|}{\Gamma(r-p)}\int_1^T\left(\log\frac{T}{s}\right)\varphi(R)p(s)\frac{\mathrm{d}s}{s}\right\}+\frac{|\alpha_1 v_2|(\log\tau)+|\gamma_1 v_1|}{|\alpha_1 v_1|}\Bigg]$$
$$\leq (\|\omega_0\||x(\tau)|+g_0)M+\sum_{i=1}^{m}\frac{\|\varpi_i\|}{\Gamma(q_i+1)}|x(\tau)|+\sum_{i=1}^{m}\frac{f_i}{\Gamma(q_i+1)}.$$

Thus

$$|x(\tau)|\leq\frac{Mg_0+\sum_{i=1}^m\frac{f_i}{\Gamma(q_i+1)}}{1-M\|\omega_0\|-\sum_{i=1}^m\frac{\|\varpi_i\|}{\Gamma(q_i+1)}}.$$

Taking the supremum over τ, we get

$$\|x\|\leq\frac{Mg_0+\sum_{i=1}^m\frac{f_i}{\Gamma(q_i+1)}}{1-M\|\omega_0\|-\sum_{i=1}^m\frac{\|\varpi_i\|}{\Gamma(q_i+1)}}\leq R.$$

Step 4: Finally, we show that $\delta N+\rho<1$, that is, (d) of Lemma 2.8 holds. Since $N=\|\mathrm{B}(\mathrm{S})\|=\sup_{x\in\mathrm{S}}\{\sup_{\tau\in\mathrm{J}}|Bx(\tau)|\}\leq M$, we have

$$\|\omega_0\|N+\sum_{i=1}^m\frac{\|\varpi_i\|}{\Gamma(q_i+1)}\leq\|\omega_0\|M+\sum_{i=1}^m\frac{\|\varpi_i\|}{\Gamma(q_i+1)}<1,$$

with $\delta=\|\omega_0\|$ and $\rho=\sum_{i=1}^m\frac{\|\varpi_i\|}{\Gamma(q_i+1)}$. Thus, all the conditions of Lemma 2.8 are satisfied, and hence the operator equation $x=\mathrm{A}x\mathrm{B}\omega+\mathrm{C}x$ has a solution in S. As a result, problem (1)-(2) has a solution on J. □

4. EXAMPLE

Consider the following nonlocal hybrid boundary value problem:

$$(28)\quad\begin{cases}\mathrm{D}_{1^+}^{\frac{3}{2}}\left[\frac{x(\tau)-\mathrm{I}_{1^+}^{q_1}f_1(\tau,x(\tau))}{g(\tau,x(\tau))}\right]=\frac{\mathrm{e}^{-2(\log\tau)}}{\sqrt{9+\tau}}\sin x(\tau),\quad \tau\in\mathrm{J}:=[1,e],\\ 5\left[\frac{x(\tau)-\mathrm{I}_{1^+}^{\frac{1}{5}}f_1(\tau,x(\tau))}{g(\tau,x(\tau))}\right]_{\tau=1}+\frac{3}{8}\mathrm{D}_{1^+}^{\frac{1}{2}}\left[\frac{x(\tau)-\mathrm{I}_{1^+}^{q_1}f_1(\tau,x(\tau))}{g(\tau,x(\tau))}\right]_{\tau=1}=1,\\ \frac{2}{5}\left[\frac{x(\tau)-\mathrm{I}_{1^+}^{\frac{1}{5}}f_1(\tau,x(\tau))}{g(\tau,x(\tau))}\right]_{\tau=T}+\frac{2}{5}\mathrm{D}_{1^+}^{\frac{1}{2}}\left[\frac{x(\tau)-\mathrm{I}_{1^+}^{q_1}f_1(\tau,x(\tau))}{g(\tau,x(\tau))}\right]_{\tau=T}=1.\end{cases}$$

We take

$$f_1(\tau,x(\tau))=\frac{(\log\tau)^2}{100}\left(\frac{1}{2}\left(x(\tau)+\sqrt{x^2+1}\right)+\log\tau\right),$$
$$g(\tau,x(\tau))=\frac{\sqrt{\pi}(\log\tau)}{(7\pi+15(\log\tau)^2)^2}\frac{x(\tau)}{1+x(\tau)}+\frac{\log\tau}{10},$$

$$h(\tau, x(\tau)) = \frac{e^{-2(\log \tau)}}{\sqrt{9+\tau}} \sin x(\tau).$$

We show that

$$|f_1(\tau, x) - f_1(\tau, \omega)| \leq \frac{\tau^2}{100}|x-\omega|,$$
$$|g(\tau, x) - g(\tau, \omega)| \leq \frac{\sqrt{\pi}}{(7\pi + 15(\log \tau)^2)^2}|x-\omega|,$$
$$h(\tau, x(\tau)) \leq p(\tau)\varphi(|x|),$$

where $\varphi(|x|) = |x|$, $p(\tau) = e^{-2(\log \tau)}$. Hence, we have $\omega_0(\tau) = \frac{(\log \tau)^2}{100}$, $\varpi_1(\tau) = \frac{\sqrt{\pi}}{(7\pi+15(\log \tau)^2)^2}$. Then $\|\omega_0\| = \frac{1}{100}$, $\|\varpi_1\| = \frac{\sqrt{\pi}}{(7\pi+15)^2}$, $\|p\| = 0.1353$, and $g_0 = \sup_{\tau \in J} |g(\tau, 0)| = \frac{1}{10}$, $f_1 = \sup_{\tau \in J} |f_1(\tau, 0)| = \frac{1}{100}$. Using these values, it follows by (23) and (24) that the constant R satisfies the inequality $0.0035 < R < 3.2552$. As all the conditions of Theorem 3.2 are satisfied, problem (28) has at least one solution on J.

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