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ON FI-RETRACTABLE MODULES

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Abstract. In this paper, we introduce the notion of FI-retractable modules which is a generalization of retractable modules. A module is called FI-retractable if for every nonzero fully invariant submodule N of M, $\mathrm{Hom}(M, N) \neq 0$. In this article, we continue the study of FI-retractable modules. Amongst other structural properties, we also deal direct sums and direct summands of FI-retractable modules. The last section of the paper is devoted to study of $\mathrm{End}(M)$, such that M is FI-retractable.

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1. INTRODUCTION

Throughout this paper R denotes an arbitrary associative ring with identity and all modules are unitary right R-module. For an R-module M, $S = \mathrm{End}(M)$ denotes the endomorphism ring of M. $E(M)$, $\mathrm{Soc}(M)$ and $\mathrm{Rad}(M)$ denote the injective hull, the socle and the Jacobson radical of M, respectively. Let M be a module and N be nonzero submodule of M. Then N is said to be an *essential* submodule of M denoted by $N \leq_e M$ if, $K \cap N \neq 0$ for every nonzero submodule K of M. A module M is called *uniform* if every submodule of M is essential in M. Recall that M is *singular (nonsingular)* provided that $Z(M) = M$ $(Z(M) = 0)$ where $Z(M) = \{x \in M; xI = 0$ for some essential ideal I of $R\}$. A submodule N of M is called *fully invariant*, if for every $f \in \mathrm{End}(M)$, $f(N) \subseteq N$. Clearly 0 and M are fully invariant submodules of M. There are some well-known fully invariant submodule of a module M such as $\mathrm{Rad}(M)$, $\mathrm{Soc}(M)$, $Z(M)$. A module M is called *Duo* if every submodule of M is fully invariant. It is clear that the sum and intersection of any collection of fully invariant submodules are also fully invariant. Thus the collection of fully invariant submodules of M is a sublattice of the complete modular lattice of all submodules of M. The concept of retractable modules introduced by Khuri [6] with the property that $\mathrm{Hom}(M, N) \neq 0$ for every nonzero submodule N of M. Often retractability condition combined with another conditions. For example, in the study of nonsingular modules

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satisfying CS condition, continuous, quasi-continuous and semi-projectivity. (see [5, 8, 9, 16]). In [13] as a generalization of retractable module, Vedadi studied essentially retractable modules with focus on essential submodules. In this work we present another generalization of retractable modules namely, FI-retractable module by focus just on nonzero fully invariant submodules. An R-module M is called *fully invariant (FI) retractable*, if for any nonzero fully invariant submodule N of M, $\mathrm{Hom}(M, N) \neq 0$. Clearly retractable modules are FI-retractable but the converse is not true in general. In section 2 we present some condition to prove that when two concepts of FI-retractable and retractable are equivalent. Also when the FI-retractability deduced essentially retractability and when essential retractable modules are FI-retractable. Also, we show that FI-retractability is preserved under direct sums and present some conditions to show that when the class of FI-retractable is closed under taking submodules and homomorphic image. Section 3 is devoted to the properties of FI-retractable modules and their endomorphism rings. We show that in some conditions the endomorphism ring of FI-retractable module is field and investigate FI-retractable modules which endomorphism ring are prime rings.

2. PRELIMINARY LEMMAS

We first recall the following elementary well known facts about fully invariant submodules.

PROPOSITION 2.1. *Let R be any ring and M be a nonzero R-module.*

(1) *Any sum or intersection of fully invariant submodules of M is again a fully invariant submodule.*
(2) *Let $K \leq N$ be submodules of M such that K is a fully invariant submodule of N and N is a fully invariant submodule of M. Then K is a fully invariant submodule of M.*
(3) *Let $M = \bigoplus_{i \in I} M_i$ and N be a fully invariant submodule of M. Then $N = \bigoplus_{i \in I}(N \cap M_i)$.*
(4) *Let $M = M_1 \oplus M_2$ be the direct sum of submodules M_1, M_2. Then M_1 is a fully invariant submodule of M if and only if $\mathrm{Hom}(M_1, M_2) = 0$.*

Proof. See [12, 2.1] and [12, 1.9]. □

DEFINITION 2.2. An R-module M is called *fully invariant retractable (FI-retractable)* provided for each nonzero fully invariant submodule N of M, $\mathrm{Hom}(M, N) \neq 0$.

EXAMPLE 2.3. Consider R-module M where $R = \begin{pmatrix} \mathbb{Z}_4 & 0 \\ \mathbb{Z}_4 & \mathbb{Z}_4 \end{pmatrix}$ and $M = \begin{pmatrix} 0 & 0 \\ \mathbb{Z}_4 & \mathbb{Z}_4 \end{pmatrix}$. Then $N = \begin{pmatrix} 0 & 0 \\ 2\mathbb{Z}_4 & 0 \end{pmatrix}$ is a fully invariant submodule of M such that $\mathrm{Hom}(M, N) = 0$.

It is clear that every retractable module is FI-retractable but the converse is not true in general, for example the $\mathbb{Z}$-module $\mathbb{Z}_4 \oplus \mathbb{Q}$ is FI-retractable but it is not retractable.

In the following result we present some conditions in which two concepts of retractable and FI-retractable are equivalent. Following [15] M has * condition if for any nonzero proper submodule K of M, there is an $r \in R \setminus \mathrm{ann}_R(M)$ with $Mr < K$.

Proposition 2.4. *Any FI-retractable module with * condition is retractable.*

Proof. Suppose that M is FI-retractable and N any nonzero submodule of M. By * condition there exists $r \in R \setminus \mathrm{ann}_R(M)$ such that $Mr < N$. Since Mr is fully invariant, $\mathrm{Hom}(M, Mr) \neq 0$ and so $\mathrm{Hom}(M, N) \neq 0$. □

A module M is called *cocyclic* provided it contains an essential simple submodule.

Proposition 2.5. *Let M be cocyclic FI-retractable module. Then M is retractable and* $\mathrm{Rad}(M) \neq M$.

Proof. Suppose that N is simple and essential submodule of M. We first show that N is a fully invariant submodule of M. Let $f \in \mathrm{End}(M)$. If $\mathrm{Ker}f = 0$, then $N \cong f(N)$. So $N = f(N)$. If $\mathrm{Ker}f \neq 0$, then $\mathrm{Ker}f \cap N \neq 0$. So $N \leq \mathrm{Ker}f$ and so $f(N) = 0$. Hence N is a nonzero fully invariant submodule of M, and by assumption $\mathrm{Hom}(M, N) \neq 0$. Hence M has a maximal submodule. Also since N is simple and essential submodule of M, it contained in any nonzero submodule of M. Therefore M is retractable. □

Vedadi in [13] studied essentially retractability for a module M by requiring that $\mathrm{Hom}(M, N) \neq 0$ for all $N \leq_e M$. The following results provide the condition that when FI-retractability deduce essentially retractability and vice versa.

Proposition 2.6. *Any FI-retractable module with nonzero socle is essentially retractable.*

Proof. Suppose that M is FI-retractable with $\mathrm{Soc}(M) \neq 0$ and N any nonzero essential submodule of M. Hence $\mathrm{Soc}(M) \leq N$. Since $\mathrm{Soc}(M)$ is nonzero fully invariant submodule of M, $\mathrm{Hom}(M, \mathrm{Soc}(M)) \neq 0$. Hence, $\mathrm{Hom}(M, N) \neq 0$. □

Following [4], a nonzero right R-module M is called *endoprime* if any nonzero fully invariant submodule of M is faithful as a left module over $\mathrm{End}(M)$.

Proposition 2.7. *Any essentially retractable endoprime module is FI-retractable.*

Proof. Suppose that M is essentially retractable endoprime and N any nonzero fully invariant submodule of M. Let $K \leq M$ such that $N \oplus K \leq_e M$. By assumption there exists nonzero $f \in \mathrm{Hom}(M, N \oplus K)$. So $\pi o f : M \to N$ is nonzero where $\pi : N \oplus K \to N$ is the canonical map . Because if $\pi o f = 0$ then $f(N) \leq N \cap K = 0$ which is contradiction with endoprimity of M. □

In general the class of FI-retractable modules is not closed under taking submodule and factor module. However, there are some special cases, as follows.

Proposition 2.8. *Let N be a fully invariant submodule of an FI-retractable module M such that $\mathrm{Hom}(\frac{M}{N}, N) = 0$. Then the module N is FI-retractable.*

Proof. Suppose that N is fully invariant submodule of M and K a nonzero fully invariant submodule of N. Then K is fully invariant submodule of M. So $\mathrm{Hom}(M, K) \neq 0$. Since $\mathrm{Hom}(\frac{M}{N}, N) = 0$, $foi \neq 0$ where i denotes the inclusion map of N to M. Because if $foi = 0$, then $N \leq \mathrm{Ker}f$. So $\mathrm{Hom}(\frac{M}{N}, \frac{M}{\mathrm{Ker}f}) \neq 0$ and so $\mathrm{Hom}(\frac{M}{N}, N) \neq 0$ which is in contradiction with our assumption. Hence $\mathrm{Hom}(N, K) \neq 0$. □

Corollary 2.9. *Let R be any ring and $M = M_1 \oplus M_2$ FI-retractable such that $\mathrm{Hom}(M_1, M_2) = 0$ or M_1 is fully invariant submodule in M. Then M_2 is a FI-retractable module.*

Proof. It follows that by Proposition 2.8 □

Proposition 2.10. *Let M be endoprime and FI-retractable module. Then any fully invariant submodule of M is FI-retractable.*

Proof. Suppose that N is a fully invariant submodule of M and K any nonzero fully invariant submodule of N. By FI-retractability of M there exists nonzero $f \in \mathrm{Hom}(M, K)$. Since M is endoprime, $f(N) \neq 0$. So $\mathrm{Hom}(N, K) \neq 0$. □

Proposition 2.11. *Let N be a fully invariant submodule of an FI-retractable module M. Then the module $\frac{M}{N}$ is FI-retractable.*

Proof. Let L be a submodule of M containing N such that $\frac{L}{N}$ is a fully invariant submodule of $\frac{M}{N}$. Let f be any endomorphism of M. Since $f(N) \leq N$, f induces an endomorphism $\bar{f} : \frac{M}{N} \to \frac{M}{N}$ defined by $\bar{f}(m+N) = f(m)+N$ for all $m \in M$. So $\bar{f}(\frac{L}{N}) \leq \frac{L}{N}$ and it follows that $f(L) \leq L$. Hence L is a fully invariant submodule of M. By hypothesis, $\mathrm{Hom}(M, L) \neq 0$. Since N is a fully invariant submodule of M, $\mathrm{Hom}(\frac{M}{N}, \frac{L}{N}) \neq 0$. It follows that $\frac{M}{N}$ is FI-retractable module. □

Proposition 2.12. *Let R be any ring and $M = \bigoplus_{i \in I} M_i$ be a direct sum of FI-retractable module M_i . Then M is FI-retractable.*

Proof. Let N be any fully invariant submodule of M. Then by Proposition 2.1, $N = \bigoplus_{i\in I}(N\cap M_i)$. Since $N\cap M_i$ is a fully invariant submodule of M_i, $\mathrm{Hom}(M_i, N\cap M_i)\neq 0$. Hence $\mathrm{Hom}(\bigoplus_{i\in I} M_i, \bigoplus_{i\in I}(N\cap M_i))\neq 0$ and so $\mathrm{Hom}(M,N)\neq 0$. □

PROPOSITION 2.13. *Let R be any ring and M_1, M_2 be R-modules such that $R = \mathrm{ann}_R(M_1) + \mathrm{ann}_R(M_2)$. Then the R-module $M = M_1\oplus M_2$ is FI-retractable if and only if M_1 and M_2 are FI-retractable modules.*

Proof. Suppose that $R = \mathrm{ann}_R(M_1) + \mathrm{ann}_R(M_2)$ and $f : M_1 \to M_2$ is any homomorphism. Then

$$\begin{aligned} f(M_1) &= f(M_1\mathrm{ann}(M_1)) + f(M_2\mathrm{ann}(M_2)) \\ &= f(0) + f(M_2)\mathrm{ann}(M_2) \\ &\leq M_2\mathrm{ann}(M_2) = 0. \end{aligned}$$

So $\mathrm{Hom}(M_1, M_2) = 0$. By Corollary 2.9, M_2 is FI-retractable module. Similarly, M_1 is FI-retractable. Conversely, by Proposition 2.12. □

COROLLARY 2.14. *Let R be any ring and n be a positive integer and $M_1,\ldots,$ M_n be R-modules such that $R = \mathrm{ann}_R(M_i) + \mathrm{ann}_R(M_j)$ for all $1\leq i\leq j\leq n$. Then the R-module $M = M_1\oplus ...\oplus M_n$ is FI-retractable if and only if M_i is FI-retractable for all $1\leq i\leq n$.*

3. MAIN RESULT

Recall that a submodule $U \lneq M$ is *rational* in M if for any $U\lneq V\lneq M$, $\mathrm{Hom}(\frac{V}{U}, M) = 0$. A module M is called *polyform* if every essential submodule of M is rational in M.

PROPOSITION 3.1. *Let M be projective FI-retractable module. Then M is nonsingular if and only if M is polyform.*

Proof. Suppose that M is nonsingular and U is an essential submodule of M. Let $U\lneq V\lneq M$ and $f\in\mathrm{Hom}(\frac{V}{U}, M)$. Since $\frac{V}{U}$ is singular, $f = 0$. Conversely, Suppose that M is polyform. If $Z(M)\neq 0$, then FI-retractability on M implies that $0\neq f\in\mathrm{Hom}(M, Z(M))$. Since $\mathrm{Im}f$ is singular, $\mathrm{Im}f\cong\frac{L}{K}$ for $K\leq_e L$. Now M is projective so, $f : M\to\frac{L}{K}$ can be extended by $g : M\to L$ such that $\pi o g = f$ where $\pi : L\to\frac{L}{K}$ is canonical map. Since $K\leq_e L$, $g^{-1}(K) = \mathrm{Ker}f$ is an essential submodule of M. By assumption M is polyform so $\mathrm{Hom}(\frac{M}{\mathrm{Ker}f}, M) = 0$. That is in contradiction with $\mathrm{Hom}(M, Z(M))\neq 0$. So $\mathrm{Hom}(M, Z(M)) = 0$ and $Z(M) = 0$. □

A ring R is called right *V-ring* if every simple right R-module is injective.

PROPOSITION 3.2. *Let R be V-ring. Then any cocyclic R-module is FI-retractable.*

Proof. Suppose that K is any nonzero fully invariant submodule of M and N an essential simple submodule of M. Since R is V-ring, N is injective and so is a direct summand of M. Also, $N \leq K$, because N is simple and essential submodule of M. Hence $\mathrm{Hom}(M, K) \neq 0$. □

Let M be an R-module and N submodule of M. We say that M is *N-FI-retractable* if for each nonzero fully invariant submodule K of N, $\mathrm{Hom}(M, K) \neq 0$.

LEMMA 3.3. *Let R be any ring and M be quasi-projective R-module. If $\frac{M}{N}$ is FI-retractable and M is N-FI-retractable, then M is FI-retractable.*

Proof. Let K be any nonzero fully invariant submodule of M. If $N \cap K \neq 0$, then $\mathrm{Hom}(M, N \cap K) \neq 0$ because M is N-FI-retractable. So $\mathrm{Hom}(M, K) \neq 0$. If $N \cap K = 0$, since M is quasi-projective $\frac{N+K}{N}$ is fully invariant submodule of $\frac{M}{N}$. So, $\mathrm{Hom}(\frac{M}{N}, \frac{N+K}{N}) \neq 0$. It follows that $\mathrm{Hom}(M, K) \neq 0$. □

PROPOSITION 3.4. *Let R be right V-ring and M be quasi-projective R-module. Then M is FI-retractable if and only if $\frac{M}{\mathrm{Soc}(M)}$ is FI-retractable.*

Proof. Suppose that $\frac{M}{\mathrm{Soc}(M)}$ is FI-retractable. By Lemma 3.3, it is enough to show that M is $\mathrm{Soc}(M)$-FI-retractable. Let N be any nonzero fully invariant submodule of $\mathrm{Soc}(M)$. So there exists a simple submodule K of M such that $K \leq N$. On the other hand since R is V-ring, K is a direct summand of M. Therefore $\mathrm{Hom}(M, N) \neq 0$. Conversely, by Proposition 2.11. □

LEMMA 3.5. *Let R be any ring and M be an R-module and M_1, M_2 submodules of M. If M is M_i-FI-retractable for $i = 1, 2$, then M is $M_1 \oplus M_2$-FI-retractable.*

Proof. Suppose that N is any nonzero fully invariant submodule of $M_1 \oplus M_2$. If $N \cap M_1 \neq 0$. Since $N \cap M_1$ is a fully invariant submodule of M_1, $\mathrm{Hom}(M, N \cap M_1) \neq 0$ and so $\mathrm{Hom}(M, N) \neq 0$. Similarly for $N \cap M_2 \neq 0$. □

LEMMA 3.6. *Let N be an essential submodule of M. If M is N-FI-retractable, then M is FI-retractable.*

Proof. Suppose that N is an essential submodule of M and M is N-FI-retractable. Let K be any nonzero fully invariant submodule of M. So $N \cap K$ is a nonzero fully invariant submodule of N and so $\mathrm{Hom}(M, N \cap K) \neq 0$. Hence $\mathrm{Hom}(M, K) \neq 0$. □

PROPOSITION 3.7. *Let R be right quasi-injective ring and M be R-module. M is FI-retractable if and only if M is $Z(M)$-FI-retractable.*

Proof. Suppose that M is $Z(M)$-FI-retractable. Let N be complemented of $Z(M)$. So, $Z(M) \oplus N \leq_e M$. By Lemma 3.6 it is enough to show that M is $(Z(M) \oplus N)$-FI-retractable. Suppose that K is any nonzero fully invariant submodule of N. Let $0 \neq x \in K$. Since R is right quasi-injective ring and xR

is nonsingular, xR is injective and hence is a direct summand. Consequently, $\mathrm{Hom}(M, xR) \neq 0$ and so $\mathrm{Hom}(M, K) \neq 0$. Therefore M is N-FI-retractable. Now, by Lemma 3.5, M is $(Z(M) \oplus N)$-FI-retractable. Conversely, suppose that M is FI-retractable. Since $Z(M)$ is fully invariant submodule of M, then M is $Z(M)$-FI-retractable. □

In the following M is a right R-module and $S = \mathrm{End}(M)$ is the ring of R-endomorphism.

Proposition 3.8. *Let M be finitely generated quasi-projective and FI-retractable module with $S = \mathrm{End}(M)$. If M is Noetherian (Artinian), then S is Noetherian (Artinian).*

Proof. Suppose that $I_1 \leq I_2 \leq \cdots$ is ascending chain of ideals in S. Therefore $I_1M \leq I_2M \leq \cdots$ is ascending chain of submodules in M. So $I_iM = I_{i+1}M = \cdots$ for some i. By FI-retractability on M and [14, 18.4] $0 \neq \mathrm{Hom}(M, I_jM) = I_j$ for any j. So $I_i = I_{i+1} = \cdots$ □

Proposition 3.9. *Let M be FI-retractable module. If $S = \mathrm{End}(M)$ is semisimple Artinian then any nonzero fully invariant submodule of M is a direct summand.*

Proof. We first prove that if I is minimal ideal of S, then IM has no non trivial fully invariant submodule. For it, let K be any nonzero fully invariant submodule of IM. So there exists a nonzero homomorphism $f : M \to K$ and so, $\mathrm{Hom}(M, \mathrm{Im}f) \leq \mathrm{Hom}(M, IM)$. On the other hand since I is a direct summand, $\mathrm{Hom}(M, IM) = I$. Therefore $\mathrm{Hom}(M, \mathrm{Im}f) = I$ and $\mathrm{Im}f = \mathrm{Hom}(M, \mathrm{Im}f)M = IM$. So $K = IM$. It follows that IM has no non trivial fully invariant submodule as desired. Now suppose that $S = I_1 \oplus \cdots \oplus I_n$ where each I_i $(1 \leq i \leq n)$ is minimal ideal of S. Then $M = SM = I_1M + \cdots + I_nM$ and each I_iM $(1 \leq i \leq n)$ has no non trivial fully invariant submodule. Also for each $i \neq j$ $(1 \leq i, j \leq n)$ if $I_iM \cap I_jM \neq 0$, then $I_iM = I_jM$. Consequently M is a finite direct sum of submodules of M where each of them has no non trivial fully invariant submodule. Now suppose that $M = M_1 \oplus \cdots \oplus M_n$ such that for each $1 \leq i \leq n$, M_i has no non trivial fully invariant submodule. Let K be any nonzero fully invariant submodule of M. Without loss of generality suppose that $K \cap M_1 \neq 0$. By assumption M_1 has no non trivial fully invariant submodule so $K \cap M_1 = M_1$. If for each $2 \leq i \leq n$, $K \cap M_i = 0$, then $K = M_1$. Suppose that $K \cap M_2 \neq 0$. So $K \cap M_2 = M_2$ and $K = M_1 \oplus M_2 \oplus (\bigoplus_{i=3}^{n} (K \cap M_i))$. Repeat this process for $(n-3)$-times we have $K = M_1 \oplus \cdots \oplus M_{n-1} \oplus (K \cap M_n)$. If $K \cap M_n = 0$, then $K = M_1 \oplus \cdots \oplus M_{n-1}$. If $K \cap M_n \neq 0$, then $K = M$. □

Corollary 3.10. *Let M be FI-retractable module. If $S = \mathrm{End}(M)$ is semisimple Artinian, then $M = Z(M) \oplus M'$ where M' is nonsingular FI-retractable.*

Proof. Suppose that M is FI-retractable and $S = \mathrm{End}(M)$ is semisimple Artinian. By Proposition 3.9, $Z(M)$ is a direct summand. So $M = Z(M) \oplus M'$. $M' \cong \frac{M}{Z(M)}$ is FI-retractable by Proposition 2.11. □

PROPOSITION 3.11. *Let M be an indecomposable quasi-injective module and $S = \mathrm{End}(M)$. In each of the following cases S is a field:*

(1) *M is FI-retractable and S is division ring.*
(2) *M is FI-retractable and nonsingular.*

Proof. (1) Let N be any nonzero fully invariant submodule of M. Then there exists a nonzero $f \in S$ such that $\mathrm{Im} f \le N$. Since S is division ring, there exists $g \in S$ such that $gf = 1$. So, $M = gf(M) \le g(N) \le N$. Therefore M has no non trivial fully invariant submodule. Hence by [3, exercise 29, page 183] S is a field.

(2) Let N be any nonzero fully invariant submodule of M. Then there exists a nonzero $f : M \to N$. Since M is nonsingular and quasi-injective, $\mathrm{Ker} f$ is a direct summand of M. So, $\mathrm{Im} f$ is isomorphic to a direct summand of M and so $\mathrm{Im} f$ is a direct summand of M because M is quasi-injective. Since M is indecomposable, $M = N$. It follows that M has no non trivial fully invariant submodule. So S is a field. □

Recall that a ring R is *prime* if for $a, b \in R$, $aRb = 0$ implies $a = 0$ or $b = 0$.

PROPOSITION 3.12. *Let M be a nonzero module with $S = \mathrm{End}(M)$.*

(1) *If M is FI-retractable and S is prime, then M is endoprime.*
(2) *If M is FI-retractable and endoprime, then $\mathrm{ann}_R(M)$ is prime.*

Proof. (1) Suppose that N is any fully invariant submodule of M such that $\mathrm{ann}_S(M) \neq 0$. Then there exists $f \in S$ such that $f(N) = 0$. Since M is FI-retractable, there exists nonzero $g \in S$ such that $\mathrm{Im} g \le N$. So, $fSg = 0$. Since S is prime, $f = 0$. It follows that M is endoprime.

(2) Suppose that M is FI-retractable and endoprime. Let $IJ \le \mathrm{ann}_R(M)$, $I \nleq \mathrm{ann}_R(M)$, $J \nleq \mathrm{ann}_R(M)$ for some right ideals I, J of R. Since M is FI-retractable, there exists nonzero $f \in \mathrm{Hom}(M, MI)$ such that $f(MJ) \le MIJ = 0$. Since M is endoprime, $f = 0$. That is a contradiction. □

LEMMA 3.13. *Let M be FI-retractable and N be any nonzero fully invariant submodule of M. If $\mathrm{End}(M)$ is prime ring, then the restriction map $\alpha : \mathrm{End}(M) \to \mathrm{End}(N)$ is injective homomorphism of rings.*

Proof. Suppose that $\alpha(f) = 0$ for some $f \in \mathrm{End}(M)$. So $N \le \mathrm{Ker} f$. By FI-retractability of M, there exists nonzero $g \in \mathrm{Hom}(M, N)$. Hence $fSg = 0$ and $f = 0$ because S is prime ring. □

REMARK 3.14. Let M is quasi-injective and N be any nonzero fully invariant submodule of M, it is easy to verify that the restriction map $\alpha : \mathrm{End}(M) \to \mathrm{End}(N)$ is surjective homomorphism of rings.

Corollary 3.15. *Let M be FI-retractable module and $S = \mathrm{End}(M)$ is prime ring. If M is quasi-injective, then the endomorphism ring of any nonzero fully invariant submodule of M is a prime ring.*

Proof. Suppose that M be FI-retractable and $\mathrm{End}(M)$ is prime ring. Let N be a nonzero fully invariant submodule of M. By Lemma 3.13 and Remark 3.14, $\mathrm{End}(M) \cong \mathrm{End}(N)$. So $\mathrm{End}(N)$ is prime ring. □

Recall that a ring R is *Dedekind-finite* if for any $x, y \in R$, $xy = 1$ implies that $yx = 1$. A module M is *Dedekind-finite* if $M \cong M \oplus N$ (for some R-modules N) implies that $N = 0$. Following [11, Exercise 1.8], an R-module M is Dedekind-finite if and only if the endomorphism ring of M is Dedekind-finite.

Proposition 3.16. *Let M be FI-retractable module and $S = \mathrm{End}(M)$ is prime ring. M is Dedekind-finite if and only if there exists a nonzero fully invariant submodule of M which is Dedekind-finite.*

Proof. Suppose that N is a fully invariant submodule of M and N Dedekind-finite. Since M is FI-retractable and $\mathrm{End}(M)$ is prime ring by Lemma 3.13, $\mathrm{End}(M)$ is isomorphic subring of $\mathrm{End}(N)$. On the other hand, since N is Dedekind-finite, $\mathrm{End}(N)$ is Dedekind-finite and so $\mathrm{End}(M)$ is Dedekind-finite. Consequently, M is Dedekind-finite. □

Recall that M is a homogeneous semisimple if all simple submodules are isomorphic.

Proposition 3.17. *Let M be FI-retractable module and $S = \mathrm{End}(M)$ is prime ring. If M is quasi-injective and M has a nonzero fully invariant submodule which is Dedekind-finite, then S is either simple Artinian or $\mathrm{Soc}(S) = 0$.*

Proof. By Proposition 3.16, M is Dedekind-finite. Suppose that $\mathrm{Soc}(S) \neq 0$. By [1, Exercise 11(1), page 164], $J(S) = 0$ and $\mathrm{Soc}(S)$ is homogeneous. On the other hand, since M is Dedekind-finite S is Dedekind-finite. Also, since M is quasi-injective and $J(S) = 0$, S is quasi-injective. Now, by [11, Exercise 31, page 244] for any nonzero right ideal I of S, $I \oplus I \oplus \cdots$ cannot be embedded in S. So $\mathrm{Soc}(S)$ is finitely generated. Hence, since S is prime ring and $\mathrm{Soc}(S)$ is finitely generated by [1, Exercise 11, p. 164], S is simple Artinian. □

Corollary 3.18. *Let M be FI-retractable module and $S = \mathrm{End}(M)$ is prime ring with $\mathrm{Soc}(S) \neq 0$. If M is quasi-injective and M has a nonzero fully invariant submodule which is Dedekind-finite, Then S is a division ring.*

Proof. By Proposition 3.17, $\mathrm{Soc}(S)$ is simple. Since S is prime, S is a division ring (see [1, Exercise 11, p. 164]). □

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