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ON CONNECTED SPACES VIA m-STRUCTURES

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Abstract. In this paper, we introduce and investigate m-separated sets and m-connected sets in a topological space (X, τ) with a minimal structure m_X. As a special case, by setting $m_X = \tau^*$, we obtain properties of $*$-separated sets and $*_s$-connected sets.

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Key words. m-structure, m-separated, m-connected, ideal topological space, $*$-separated, $*_s$-connected.

1. INTRODUCTION

Ekici and Noiri [11] introduced and studied $*$-separated sets and $*_s$-connected sets in an ideal topological space $(X, \tau, \mathcal{I})$ [13]. Sathiyasundari and Renukadevi [19] obtained further properties of $*$-separated sets and $*_s$-connected sets. In this paper, we introduce and investigate the notions of m-separated sets and m-connected sets in a topological space (X, τ) with a minimal structure m_X. By setting $m_X = \tau^*$, we obtain properties of $*$- separated sets and $*_s$-connected sets established in [11] and [19] as a special case of results of this paper. Recently, papers [1–7] have introduced some new classes of sets via m-structures.

2. MINIMAL STRUCTURES

Definition 2.1. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily m_X of $\mathcal{P}(X)$ is called a minimal structure (briefly m-structure) on X [18] if $\emptyset \in m_X$ and $X \in m_X$. Each member of m_X is said to be m_X-open and the complement of an m_X-open set is said to be m_X-closed.

Definition 2.2. Let (X, τ) be a topological space. A subset A of X is said to be

(1) α-open [17] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(2) semi-open [14] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$,
(3) preopen [16] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$,
(4) b-open [10] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$,
(5) β-open [8] or semi-preopen [9] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$.

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The family of all α-open (resp. semi-open, preopen, b-open, semi-preopen) sets in (X,τ) is denoted by $\alpha(X)$ (resp. $SO(X)$, $PO(X)$, $BO(X)$, $SPO(X)$).

DEFINITION 2.3. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the m_X-closure of A and the m_X-interior of A are defined in [15] as follows:

(1) $m\mathrm{Cl}(A) = \cap\{F \subseteq X : A \subseteq F, X \setminus F \in m_X\}$,
(2) $m\mathrm{Int}(A) = \cup\{U \subseteq X : U \subseteq A, U \in m_X\}$.

REMARK 2.4. Let (X,τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. $SO(X), PO(X), BO(X), SPO(X)$), then we have

(1) $m\mathrm{Cl}(A) = \mathrm{Cl}(A)$ (resp. $s\mathrm{Cl}(A), p\mathrm{Cl}(A), b\mathrm{Cl}(A), sp\mathrm{Cl}(A)$),
(2) $m\mathrm{Int}(A) = \mathrm{Int}(A)$ (resp. $s\mathrm{Int}(A), p\mathrm{Int}(A), b\mathrm{Int}(A), sp\mathrm{Int}(A)$).

LEMMA 2.5 ([15]). *Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following properties hold:*

(1) $m\mathrm{Cl}(X \setminus A) = X \setminus m\mathrm{Int}(A)$ *and* $m\mathrm{Int}(X \setminus A) = X \setminus m\mathrm{Cl}(A)$,
(2) *If* $(X\setminus A) \in m_X$, *then* $m\mathrm{Cl}(A) = A$ *and if* $A \in m_X$, *then* $m\mathrm{Int}(A) = A$,
(3) $m\mathrm{Cl}(\emptyset) = \emptyset$, $m\mathrm{Cl}(X) = X$, $m\mathrm{Int}(\emptyset) = \emptyset$ *and* $m\mathrm{Int}(X) = X$,
(4) *If* $A \subseteq B$, *then* $m\mathrm{Cl}(A) \subseteq m\mathrm{Cl}(B)$ *and* $m\mathrm{Int}(A) \subseteq m\mathrm{Int}(B)$,
(5) $A \subseteq m\mathrm{Cl}(A)$ *and* $m\mathrm{Int}(A) \subseteq A$,
(6) $m\mathrm{Cl}(m\mathrm{Cl}(A)) = m\mathrm{Cl}(A)$ *and* $m\mathrm{Int}(m\mathrm{Int}(A)) = m\mathrm{Int}(A)$.

LEMMA 2.6 ([18]). *Let X be a nonempty set with an m-structure m_X and A a subset of X. Then $x \in m\mathrm{Cl}(A)$ if and only if $U \cap A \neq \emptyset$. for every $U \in m_X$ containing x.*

DEFINITION 2.7. An m-structure m_X on a nonempty set X is said to have property $\mathcal{B}$ [15] if the union of any family of subsets belong to m_X belongs to m_X.

REMARK 2.8. Let (X,τ) be a topological space. Then the families $\alpha(X)$, $SO(X), PO(X), BO(X)$ and $SPO(X)$ are m-structures on X with property $\mathcal{B}$.

LEMMA 2.9 ([18]). *Let X be a nonempty set and m_X an m-structure on X satisfying property $\mathcal{B}$. For a subset A of X, the following properties hold:*

(1) $A \in m_X$ *if and only if* $m\mathrm{Int}(A) = A$,
(2) A *is* m_X*-closed if and only if* $m\mathrm{Cl}(A) = A$,
(3) $m\mathrm{Int}(A) \in m_X$ *and* $m\mathrm{Cl}(A)$ *is* m_X*-closed.*

A subfamily $\mathcal{I}$ of the power set $\mathcal{P}(X)$ of a nonempty set X is called an ideal if the following properties are satisfied: (1) $A \in \mathcal{I}$ and $B \subseteq A$ imply $B \in \mathcal{I}$; (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$. A topological space (X,τ) with an ideal $\mathcal{I}$ on X is called an ideal topological space and is denoted by $(X,\tau,\mathcal{I})$. For an ideal topological space and a subset A of X, $A^*(\mathcal{I})$ is defined as follows: $A^*(\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I}$ for every open set U containing $x\}$. In [13], $A^*(\mathcal{I})$ (briefly A^*) is called the local function of A with respect to $\mathcal{I}$ and τ

and $Cl^*(A) = A^* \cup A$ defines a Kuratowski closure operator for a topology τ^* . which is finer than τ . A subset A is $*$-closed if and only if $A^* \subseteq A$. Naturally, the complement of a $*$-closed set is said to be $*$-open.

Definition 2.10 ([12]). An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $*$- connected if X cannot be written as the disjoint union of a nonempty open set and a nonempty $*$-open set.

Definition 2.11 ([11]). Nonempty subsets A, B of an ideal topological space $(X, \tau, \mathcal{I})$ are said to be $*$-separated if $Cl^*(A) \cap B = A \cap \mathrm{Cl}(B) = \emptyset$.

Definition 2.12 ([11]). A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $*_s$-connected if A is not the union of two $*$-separated sets in $(X, \tau, \mathcal{I})$.

3. m-SEPARATED SETS

A topological space (X, τ) with an m-structure m_X on X is called briefly a mixed space and is denoted by (X, τ, m_X).

Definition 3.1. Let (X, τ, m_X) be a mixed space. Nonempty subsets A, B of X are said to be m-separated if $\mathrm{Cl}(A) \cap B = \emptyset = A \cap m\mathrm{Cl}(B)$.

If $\tau \subseteq m_X$, then every separated sets are m-separated but the converse is not true as shown in the following example.

Example 3.2. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $m_X = \mathcal{P}(X)$. Let $A = \{a\}$ and $B = \{b\}$. Then A and B are m-separated sets but they are not separated.

Proposition 3.3. *Let (X, τ, m_X) be a mixed space. If A and B are nonempty disjoint subsets of X such that A is m_X-open and B is open, then A and B are m-separated.*

Proof. Since $A \cap B = \emptyset$, $A \subseteq X \setminus B$ and so $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(X \setminus B) = X \setminus B$. Then, $\mathrm{Cl}(A) \cap B = \emptyset$. Again $B \subseteq X \setminus A$ and so $m\mathrm{Cl}(B) \subseteq m\mathrm{Cl}(X \setminus A) = X \setminus A$. Thus, $m\mathrm{Cl}(B) \cap A = \emptyset$. Therefore, A and B are m-separated. □

Corollary 3.4. *Let (X, τ, m_X) be a mixed space and $\tau \subseteq m_X$. Then the disjoint nonempty open sets of X are m-separated.*

Proposition 3.5. *Let A and B be two m-separated sets in a mixed space (X, τ, m_X). If C and D are nonempty subsets such that $C \subseteq A$ and $D \subseteq B$, then C and D are also m-separated.*

Proof. Since A and B are m-separated, $\mathrm{Cl}(A) \cap B = \emptyset = A \cap m\mathrm{Cl}(B)$. Now, $C \cap m\mathrm{Cl}(D) \subseteq A \cap m\mathrm{Cl}(B) = \emptyset$ and so $C \cap m\mathrm{Cl}(D) = \emptyset$. Similarly, we can prove that $\mathrm{Cl}(C) \cap D = \emptyset$. Hence C and D are m-separated. □

Theorem 3.6. *Let (X, τ, m_X) be a mixed space, m_X have property $\mathcal{B}$ and $\tau \subseteq m_X$. If the union of two m-separated sets is a closed set, then one set is an m_X-closed and the other is closed.*

Proof. Let A and B be two m-separated sets such that $A \cup B$ is closed. Then $A \cap m\mathrm{Cl}(B) = \emptyset = \mathrm{Cl}(A) \cap B$. Since $A \cup B$ is closed, $A \cup B = \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$. Now, $\mathrm{Cl}(A) = \mathrm{Cl}(A) \cap [\mathrm{Cl}(A) \cup \mathrm{Cl}(B)] = \mathrm{Cl}(A) \cap [A \cup B] = [\mathrm{Cl}(A) \cap A] \cup [\mathrm{Cl}(A) \cap B] = A \cup \emptyset = A$ and so A is closed. Also, $B \subseteq A \cup B$ implies that $m\mathrm{Cl}(B) \subseteq mCl[A \cup B] \subseteq Cl[A \cup B] = A \cup B$ and so $m\mathrm{Cl}(B) = m\mathrm{Cl}(B) \cap [A \cup B] = [m\mathrm{Cl}(B) \cap A] \cup [\mathrm{Cl}(B) \cap B] = \emptyset \cup B = B$. Hence by Lemma 2.9, B is m_X-closed. □

THEOREM 3.7. *Let (X, τ, m_X) be a mixed space and τ and m_X satisfy the conditions:*

(1) *m_X has property $\mathcal{B}$.*
(2) *the intersection of an open set and an m_X-open set is m_X-open.*

If A and B are m-separated sets of X and $A \cup B \in \tau$, then A and B are m_X-open and open respectively.

Proof. Since A and B are m-separated in X, then $B = [A \cup B] \cap [X \setminus \mathrm{Cl}(A)]$. Since $A \cup B \in \tau$ and $\mathrm{Cl}(A)$ is closed in X, then B is open. Since A and B are m-separated in X, then $A = [A \cup B] \cap [X \setminus m\mathrm{Cl}(B)]$. Since $A \cup B \in \tau$ and $m\mathrm{Cl}(B)$ is m_X-closed in X, then A is m_X-open. □

LEMMA 3.8. *Let (X, τ) be topological space. $A \subseteq Y \subseteq X$ and $Y \in \tau$. Then the following are equivalent:*

(1) *A is open in Y;*
(2) *A is open in X.*

LEMMA 3.9. *Let (X, τ, m_X) be a mixed space and $B \subseteq Y \subseteq X$. Then $mCl^Y(B) = m\mathrm{Cl}(B) \cap Y$.*

LEMMA 3.10. *Let (X, τ) be topological space and $A, B \subseteq Y \subseteq X$. The following are equivalent:*

(1) *A, B are m-separated in Y;*
(2) *A, B are m-separated in X.*

Proof. It follows form Lemma 3.9 that $mCl^Y(A) \cap B = \emptyset = A \cap Cl^Y(B)$ if and only if $m\mathrm{Cl}(A) \cap B = \emptyset = A \cap \mathrm{Cl}(B)$. □

4. m-CONNECTED SPACES

In this section, we give the properties of m-separated sets and m-connected sets.

DEFINITION 4.1. A subset A of a mixed space (X, τ, m_X) is said to be m-connected if A is not the union of two m-separated sets in (X, τ, m_X).

If $\tau \subseteq m_X$, then every m-connected mixed space is connected but the converse is not true as shown in the following example.

EXAMPLE 4.2. Let $X = \mathbb{Q}$ be the set of all rational numbers with left ray topology τ_L and $m_X = \mathcal{P}(X)$. Then the mixed space (X, τ_L, m_X) is a connected space but it is not m-connected.

Definition 4.3. A mixed space (X, τ, m_X) is said to be m^*-connected if X cannot be written as the disjoint union of a nonempty m_X-open set and a nonempty open set.

Theorem 4.4. *Let Y be an open subset of a mixed space (X, τ, m_X). If m_X has property $\mathcal{B}$ and the intersection of an open set and an m_X-open set is m_X-open, then the following properties are equivalent:*

(1) *The subset Y is m-connected in X;*
(2) *The subspace $(Y, \tau_Y, (m_X)_Y)$ is m-connected in X.*

Proof. (1) $\Rightarrow$ (2): Suppose that Y is not m^*-connected. There exist nonempty disjoint an m_Y-open set A and an open set B in Y such that $Y = A \cup B$. Since Y is open in X, A is m_X-open and by Lemma 3.8, B open in X. Since A and B are disjoint, then $\mathrm{Cl}(A) \cap B = \emptyset = A \cap m\mathrm{Cl}(B)$. This implies that A, B are m-separated sets in X. Thus, Y is not m-connected in X. This is a contradiction.

(2) $\Rightarrow$ (1): Suppose that Y is not m-connected in X. There exist m-separated sets A, B such that $Y = A \cup B$. By Theorem 3.7, A and B are m_X-open and open in X, respectively. Since Y is open in X, A is m_Y -open and B is open in Y. Since A and B are m-separated in X, then A and B are nonempty disjoint. Thus, Y is not m^*-connected. This is a contradiction. □

Corollary 4.5 ([11]). *Let Y be an open subset of an ideal topological space $(X, \tau, \mathcal{I})$. The following are equivalent:*

(1) *Y is $*_s$-connected in $(X, \tau, \mathcal{I})$,*
(2) *Y is $*$-connected in $(X, \tau, \mathcal{I})$.*

Corollary 4.6. *Let (X, τ, m_X) be a mixed space such that m_X has property $\mathcal{B}$ and the intersection of an open set and an m_X-open set is m_X-open. Then X is m-connected if and only if X is m^*-connected.*

Proof. The proof follows from Theorem 4.4. □

Theorem 4.7. *Let (X, τ, m_X) be a mixed space, X not m-connected and A be a subset of X such that (i) $A \neq \emptyset$, X and (ii) A is open in X and m_X-closed in X. If Y is a nonempty m-connected subset of X, then either $Y \subseteq A$ or $Y \subseteq X \setminus A$.*

Proof. Let $X = A \cup B$, where $B = X \setminus A$. Then $Y = X \cap Y = [A \cup B] \cap Y = (A \cap Y) \cup (B \cap Y)$. Also $[A \cap Y] \cap Cl[B \cap Y] \subseteq A \cap \mathrm{Cl}(B) = \emptyset$, since $A \cap B = \emptyset$ and A is open. This implies that $[A \cap Y] \cap Cl[B \cap Y] = \emptyset$. Similarly, $mCl[A \cap Y] \cap [B \cap Y] \subseteq m\mathrm{Cl}(A) \cap B = A \cap B = \emptyset$ implies that $mCl[A \cap Y] \cap [B \cap Y] = \emptyset$. It is given that Y is m-connected subspace of X. Hence it cannot happen that $A \cap Y \neq \emptyset$ and $B \cap Y \neq \emptyset$, Since Y is an m-connected subspace of X, Y cannot admit any m-separation. Hence $A \cap Y = \emptyset$ or $B \cap Y = \phi$ implies that $Y \subseteq A$ or $Y \subseteq X - A$. □

Theorem 4.8. *Let (X, τ, m_X) be a mixed space. If A is an m-connected set of X and H, G are m-separated sets of X with $A \subseteq H \cup G$, then either $A \subseteq H$ or $A \subseteq G$.*

Proof. Let $A \subseteq H \cup G$. Since $A = [A \cap H] \cup [A \cap G]$, then $[A \cap H] \cap mCl[A \cap G] \subseteq H \cap m\mathrm{Cl}(G) = \emptyset$. By similar way, we have $[A \cap G] \cap Cl[A \cap H] \subseteq G \cap \mathrm{Cl}(H) = \emptyset$. Then $A \cap H$ and $A \cap G$ are m-separated sets. Suppose that $A \cap H$ and $A \cap G$ are nonempty. Then A is not m-connected. This is a contradiction. Thus, either $A \cap H = \emptyset$ or $A \cap G = \emptyset$. This implies that $A \subseteq H$ or $A \subseteq G$. □

Corollary 4.9 ([11]). *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If A is a $*_s$-connected set of X and H, G are $*$-separated sets of X with $A \subseteq H \cup G$, then either $A \subseteq H$ or $A \subseteq G$.*

Theorem 4.10. *Let A be an m-connected set of a a mixed space (X, τ, m_X). If $A \subseteq B \subseteq m\mathrm{Cl}(A)$, then B is m-connected.*

Proof. Suppose that B is not m-connected. There exist m-separated sets H and G such that $B = H \cup G$. This implies that H and G are nonempty and $G \cap \mathrm{Cl}(H) = \emptyset = H \cap m\mathrm{Cl}(G)$. By Theorem 4.8, we have either $A \subseteq H$ or $A \subseteq G$. Suppose that $A \subseteq G$. Then $m\mathrm{Cl}(A) \subseteq m\mathrm{Cl}(G)$ and $H \cap m\mathrm{Cl}(A) = \emptyset$. This implies that $H \subseteq B \subseteq m\mathrm{Cl}(A)$ and $H = m\mathrm{Cl}(A) \cap H = \emptyset$. Thus H is an empty set. Since H is nonempty, this is a contradiction. Suppose that $A \subseteq H$. By similar way, it follows that G is empty. This is a contradiction. Hence, B is m-connected. □

Corollary 4.11 ([11]). *If A is a $*_s$-connected set of an ideal topological space $(X, \tau, \mathcal{I})$ and $A \subseteq B \subseteq Cl^*(A)$, then B is $*_s$-connected.*

Corollary 4.12. *Let (X, τ, m_X) be a mixed space. If A is an m-connected set, then $m\mathrm{Cl}(A)$ is m-connected.*

Theorem 4.13. *Let $\{N_i : i \in I\}$ is a nonempty family of m-connected sets of a mixed space (X, τ, m_X). If $\cap_{i \in I} N_i \neq \emptyset$, then $\cup_{i \in I} N_i$ is m-connected.*

Proof. Suppose that $\cup_{i \in I} N_i$ is not m-connected. Then we have $\cup_{i \in I} N_i = H \cup G$, where H and G are m-separated sets in X. Since $\cap_{i \in I} N_i \neq \emptyset$, we have a point $x \in \cap_{i \in I} N_i$. Since $x \in \cup_{i \in I} N_i$, either $x \in H$ or $x \in G$. Suppose that $x \in H$. Since $x \in N_i$ for each $i \in I$, then N_i and H intersect for each $i \in I$. By Theorem 4.8 $N_i \subseteq H$ or $N_i \subseteq G$. Since H and G are disjoint, $N_i \subseteq H$ for all $i \in I$ and hence $\cup_{i \in I} N_i \subseteq H$. This implies that G is empty. This is a contradiction. Suppose $x \in G$. By similar way, we have that H is empty. This is a contradiction. Thus $\cup_{i \in I} N_i$ is m-connected. □

Corollary 4.14. *Let (X, τ, m_X) be a mixed space and $\{A_\alpha : \alpha \in \Delta\}$ be a family of m-connected subsets of X, and A be an m-connected subset of X. If $A \cap A_\alpha \neq \emptyset$ for every $\alpha \in \Delta$, then $A \cup (\cup A_\alpha)$ is m-connected.*

Proof. By Theorem 4.13, $A \cup A_\alpha$ is m-connected for each $\alpha \in \Delta$ and $\cap(A \cup A_\alpha) \supseteq A \neq \emptyset$. Therefore, by Theorem 4.13 $\cup(A \cup A_\alpha) = A \cup (\cup A_\alpha)$ is m-connected. □

COROLLARY 4.15 ([11]). *If $\{M_i : i \in I\}$ is a nonempty family of $*_s$-connected sets of an ideal topological space $(X, \tau, \mathcal{I})$ with $\cap_{i \in I} M_i \neq \emptyset$, then $\cup_{i \in I} M_i$ is $*_s$-connected.*

THEOREM 4.16. *Let (X, τ, m_X) be a mixed space and $\tau \subseteq m_X$. Every continuous image of an m-connected space is a connected space.*

Proof. Let $f : (X, \tau, m_X) \to (Y, \sigma)$ be a continuous function and X is m-connected space. If possible suppose that $f(X)$ is not a connected subset of Y. Then, there exists nonempty separated sets A and B such that $f(X) = A \cup B$. Since f is continuous and $A \cap \mathrm{Cl}(B) = \emptyset = \mathrm{Cl}(A) \cap B$, $\mathrm{Cl}(f^{-1}(A)) \cap f^{-1}(B) \subseteq f^{-1}(\mathrm{Cl}(A)) \cap f^{-1}(B) = f^{-1}[\mathrm{Cl}(A) \cap B] = \emptyset$, $f^{-1}(A) \cap m\mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}(A) \cap \mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}(A) \cap f^{-1}(\mathrm{Cl}(B)) = f^{-1}[A \cap \mathrm{Cl}(B)] = \emptyset$. Since A and B are nonempty, $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty. Therefore, $f^{-1}(A)$ and $f^{-1}(B)$ are m-separated and $X = f^{-1}(A) \cup f^{-1}(B)$. This is contrary to the assumption that X is m-connected. Therefore, $f(X)$ is connected. □

THEOREM 4.17. *Let (X, τ, m_X) be a mixed space and H a subset of X. If every pair of distinct points of H are elements of some m-connected subset of H, then H is an m-connected subset of X..*

Proof. Suppose H is not m-connected. Then there exist nonempty subsets A and B of X such that $\mathrm{Cl}(A) \cap B = \emptyset = A \cap m\mathrm{Cl}(B)$ and $H = A \cup B$. Since A and B are nonempty, there exists a point $a \in A$ and a point $b \in B$. By hypothesis, a and b must be elements of an m-connected subset C of H. Since $C \subseteq A \cup B$, by Theorem 4.8, either $C \subseteq A$ or $C \subseteq B$. Consequently, either a and b are both in A or both in B. Let $a, b \in A$. Then $A \cap B \neq \emptyset$. This is contrary to the fact that A and B are disjoint. Similarly, if we suppose that $a, b \in B$, then we have a contradiction. Therefore, H must be m-connected. □

THEOREM 4.18. *Let (X, τ, m_X) be a mixed space and X is m-connected. If A is an m-connected subset of X such that $X \setminus A$ is the union of two m-separated sets B and C, then $A \cup B$ and $A \cup C$ are m-connected.*

Proof. Suppose $A \cup B$ is not m-connected. Then there exist two nonempty m-separated sets G and H such that $A \cup B = G \cup H$. Since A is m-connected, $A \subseteq A \cup B = G \cup H$, by Theorem 4.8, either $A \subseteq G$ or $A \subseteq H$. Suppose $A \subseteq G$. Since $A \cup B = G \cup H$, $A \subseteq G$ implies that $A \cup B \subseteq G \cup B$ and so $G \cup H \subseteq G \cup B$. Hence $H \subseteq B$. Since B and C are m-separated, H and C are also m-separated. Thus, H is m-separated from G as well as C. Now, $m\mathrm{Cl}(H) \cap [G \cup C] = [m\mathrm{Cl}(H) \cap G] \cup [m\mathrm{Cl}(H) \cap C] = \emptyset$ and $H \cap Cl[G \cup C] = H \cap [\mathrm{Cl}(G) \cup \mathrm{Cl}(C)] = [H \cap \mathrm{Cl}(G)] \cup [H \cap \mathrm{Cl}(C)] = \emptyset$. Therefore, H is m-separated from $G \cup C$. Since $X \setminus A = B \cup C$, $X = A \cup [B \cup C] = [A \cup B] \cup C = [G \cup H] \cup C$, since $A \cup B = G \cup H$ and so $X = [G \cup C] \cup H$. Thus, X is union of two nonempty m-separated sets $G \cup C$ and H, which is a contradiction. Similar contradiction will arise if $A \subseteq H$. Hence, $A \cup B$ is m-connected. Similarly, we can prove that $A \cup C$ is m-connected. □

The following example shows that the union of two m-connected sets is not an m-connected set.

EXAMPLE 4.19. Consider the mixed space (X, τ, m_X) where $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{b\}, \{b, c\}, \{a, b, d\}, X\}$ and $m_X = \{\emptyset, \{b\}, \{b, c\}, \{c\}, \{a, d\}, \{a, c, d\}, X\}$. If $A = \{b\}$ and $B = \{a, d\}$, then A and B are m-connected. But $A \cup B = \{a, b, d\}$. Here $m\mathrm{Cl}(\{b\}) \cap \{a, d\} = \{b\} \cap \{a, d\} = \emptyset$ and $\{b\} \cap \mathrm{Cl}(\{a, d\}) = \{b\} \cap \{a, d\} = \emptyset$ and so $\{b\}$ and $\{a, d\}$ are m-separated sets. Hence, $A \cup B$ is not m-connected.

COROLLARY 4.20 ([19]). *If A is a $*_s$-connected subset of a $*_s$-connected ideal topological space $(X, \tau, \mathcal{I})$ such that $X \setminus A$ is the union of two $*$-separated sets B and C, then $A \cup B$ and $A \cup C$ are $*_s$-connected.*

THEOREM 4.21. *Let (X, τ, m_X) be a mixed space If A and B are m-connected sets of X such that none of them is m-separated, then $A \cup B$ is m-connected.*

Proof. Let A and B be m-connected in X. Suppose $A \cup B$ is not m-connected. Then, there exist two nonempty m-separated sets G and H such that $A \cup B = G \cup H$. Since A and B are m-connected, by Theorem 4.8, either $A \subseteq G$ and $B \subseteq H$ or $B \subseteq G$ and $A \subseteq H$. Let $A \subseteq G$ and $B \subseteq H$. Then, since G and H are m-separated, by Proposition 3.5 A and B are m-separated. This is a contradiction. Similarly, let $B \subseteq G$ and $A \subseteq H$. Then B and A are m-separated. This is a contradiction. Hence $A \cup B$ is m-connected. □

DEFINITION 4.22. Let (X, τ, m_X) be a mixed space and $x \in X$. The union of all m-connected subsets of X containing x is called the m-component of X containing x.

LEMMA 4.23. *The m-component of each point x of a mixed space (X, τ, m_X) is the maximal m-connected set of X that contains x.*

LEMMA 4.24. *The set of all distinct m-components of a mixed space (X, τ, m_X) forms a partition of X.*

Proof. Let A and B be two distinct m-components of X. Suppose A and B intersect. Then, by Theorem 4.13, $A \cup B$ is m-connected in X. Since $A \subseteq A \cup B$, then A is not maximal. Thus, A and B are disjoint. □

LEMMA 4.25. *Each m-component of a mixed space (X, τ, m_X), where m_X has property $\mathcal{B}$, is an m_X-closed in X.*

Proof. Let A be an m-component of X. By Corollary 4.12, $m\mathrm{Cl}(A)$ is m-connected and $A = m\mathrm{Cl}(A)$. Thus, by Lemma 2.9 A is m_X-closed in X. □

Theorem 4.26. *Let (X, τ, m_X) be a mixed space. Then each m-connected subset of X which both open and m_X-closed is m-component of X.*

Proof. Let A be an m-connected subset of X which both open and m_X-closed. Let $x \in A$. Since A is an m-connected subset of X containing x, if C is the m-component containing x, then $A \subseteq C$. Let A be a proper subset of C. Then C is nonempty and $C \cap (X \setminus A) \neq \emptyset$. Since A is open and m_X-closed, $X \setminus A$ is closed and m_X-open and $[A \cap C] \cap [(X \setminus A) \cap C] = \emptyset$. Also $[A \cap C] \cup [(X \setminus A) \cap C] = [A \cup (X \setminus A)] \cap C = C$. Again A and $X \setminus A$ are two nonempty disjoint open and m_X-open set respectively, such that $A \cap \mathrm{Cl}(X \setminus A) = \emptyset = m\mathrm{Cl}(A) \cap (X \setminus A)$. This implies $(A \cap C) \cap Cl[(X \setminus A) \cap C] = \emptyset = m\mathrm{Cl}(A \cap C) \cap [(X \setminus A) \cap C]$. This shows that A and $C \setminus A$ are m-separated sets. This is a contradiction. Hence, A is not a proper subset of C and $A = C$. Therefore, A is an m-component of X. □

Corollary 4.27 ([19]). *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then, each $*_s$-connected subset of X which is both open and $*$-closed is a $*$-component of X.*

Theorem 4.28. *Let (X, τ, m_X) be a mixed space such that $\tau \subseteq m_X$ and $A \subseteq X$. If C is an m-connected subset of X that intersects both A and $X \setminus A$, then C intersects $Bd(A)$, the boundary of A.*

Proof. Suppose $C \cap Bd(A) = \emptyset$. Then $C \cap \mathrm{Cl}(A) \cap \mathrm{Cl}(X \setminus A) = \emptyset$. Now, $C = C \cap X = C \cap (A \cup (X \setminus A)) = (C \cap A) \cup (C \cap (X \setminus A))$. Also, $m\mathrm{Cl}(C \cap A) \cap (C \cap (X \setminus A)) \subseteq m\mathrm{Cl}(C) \cap m\mathrm{Cl}(A) \cap C \cap (X \setminus A) = C \cap m\mathrm{Cl}(A) \cap (X \setminus A) = \emptyset$. and $(C \cap A) \cap \mathrm{Cl}(C \cap (X \setminus A)) \subseteq C \cap A \cap \mathrm{Cl}(C) \cap \mathrm{Cl}(X \setminus A) = C \cap \mathrm{Cl}(X \setminus A) \cap A = \emptyset$. Thus, $C \cap A$ and $C \cap (X \setminus A)$ form an m-separation for C, which is a contradiction. Hence, $C \cap Bd(A) \neq \emptyset$. □

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