MATHEMATICA, 63 (86), N° 1, 2021, pp. 119–127

KERNEL STABLE AND UNIQUELY GENERATED MODULES

SERAP ŞAHINKAYA and TRUONG CONG QUYNH

Abstract. A module theoretic notion of annihilator-stable rings is defined and some characterizations of it are studied. A module M is called kernel-stable if every element $\alpha \in \text{End}(M)$ satisfies the following condition: if $\alpha(M) + Ker\beta = M$, $\beta \in \text{End}(M)$, then $(\alpha - \gamma)(m) \in Ker\beta$ for an automorphism γ of M and for all $m \in M$. For a pseudo-semi-projective module M, this notion is equivalent to that of uniquely generated module.

MSC 2010. 16U60, 16U99 16E50, 16L30, 19A13

Key words. Stable range, annihilator-stable rings, uniquely generated modules, von Neumann regular rings, unit-regular rings, matrix rings, pseudo-semi-projective module, kernel-stable modules.

1. INTRODUCTION

Following von Neumann, an element $x \in R$ is called a regular element if there exists $y \in R$ such that $x = xyx$ and R is called a regular ring if every element is regular [16]. The unit-regular rings were defined by Ehrlich [6] in 1968 as follows: if for every $x \in R$, there exists a unit $u \in R$ such that $x = xux$.

Let R be a ring. If, for $a, b \in R$, $a = bu$, where u is unit, then a and b are called right associated. Clearly, left (right) associated elements generate the same principal left (right) ideals Ra and Rb. But, the converse of this statement does not hold in general and this case was first considered by Kaplansky in 1949. An element $a \in R$ is called left (right) uniquely generated (UG), for $b \in R$, if $Ra = Rb$ ($aR = bR$) then a and b are left (right) associated. If every element of R has this condition then R is said to be a left (right) UG ring. The study of UG rings was started by Kaplansky in 1949 (see [7]). Kosan et al, in [9], defined the notion of uniquely generated modules as a module theoretic analogue of the notion of the uniquely generated ideal, as follows: a M-cyclic submodule N of M uniquely generated if for any two elements α and β of $\text{End}(M)$ such that $N = \alpha(M) = \beta(M)$, then α and β must be right associates in $\text{End}(M)$. In that work authors obtained a characterization of a unit-regular endomorphism ring of a module in terms of certain uniquely generated submodules of the module. They proved that $\text{End}(M)$ is unit-regular if and only if $\text{End}(M)$ is regular and all M-cyclic submodules of M are uniquely

The authors thank the referee for his helpful comments and suggestions.

DOI: 10.24193/mathcluj.2021.1.11

generated. As a result of this theorem, they obtained the main theorem of [10]: R is unit-regular if and only if R is regular and every principal right ideal is uniquely generated.

Rings of left stable ring range 1 were introduced in 1964 by Bass in his seminal work on K-theory in [2]. An element $a \in R$ has left stable range 1 (SR1) if $Ra + Rb = R$, $b \in R$, implies $a - u \in Rb$ for a unit u in R. Then Vaserstein showed that this notion is left-right symmetric. More precisely, he showed that a ring has left stable range 1 if and only if it has right stable range 1. But we do not know if this holds for elements.

Canfell [3, Corollary 4.4] obtained the following characterization of the left UG rings.

Canfell's Theorem: *The following are equivalent for a ring R:*

(1) *If $Ra + l(b) = R$, $a, b \in R$, then $a - u \in l(b)$ for some unit $u \in R$.*

(2) *R is left UG.*

(3) *If $Ra = Rb$, $a, b \in R$, then $a = vb$ for some left unit $v \in R$.*

Clearly, every SR1 ring is left (and right) UG (the converse fails as $\mathbb{Z}$ is UG). Nicholson considered the condition (1) of Canfell's theorem as a requirement on the element a. By analogy with the stable range 1 condition he called a ring with this property left annihilator-stable. More precisely, an element a in a ring R is called left annihilator-stable (left AS element) if the following condition holds: If $Ra + l(b) = R$, $b \in R$, then $a - u \in l(b)$ for some unit $u \in R$. The ring R is called a left annihilator-stable ring (a left AS ring) if every element of R is left AS. According to a Canfell's Theorem a ring R is left AS if and only if R is left UG.

In the present paper we defined the notion of kernel-stable modules as a module theoretic analogue of the notion of the annihilator-stable ring. We call an element $\alpha \in \mathrm{End}(M)$ is *kernel-stable* if the following condition holds: If $\alpha(M) + Ker\beta = M$, $\beta \in \mathrm{End}(M)$ then $(\alpha - \gamma)(m) \in Ker\beta$ for an automorphism γ of M and any $m \in M$. An R-module M is called a kernel-stable if every element of $\mathrm{End}(M)$ is kernel-stable. Clearly, R is annihilator-stable if and only if R_R is kernel-stable

In Section 2, we obtained the several characterization of kernel-stable module. We also give the relationship between the notions of kernel-stable module, uniquely generated module and stable range 1 property of $\mathrm{End}(M)$ (see Proposition 2.4 and Theorem 2.12).

In Section 3, we weaken the notion of kernel-stable element of $S = \mathrm{End}(M)$ and call it idempotent-kernel-stable. We obtained the module theoretic version of [8, Theorem 6.2] and [4, Lemma 1] in Theorem 3.5. More precisely, we proved that; for a pseudo-semi-projective module M over a ring R, M is idempotent kernel-stable if and only if $f(M)$ is an uniquely generated submodule of M for a regular element f of $\mathrm{End}(M)$ if and only if M is IC.

For the last section, we examine when the annihilator stable and kernel stable conditions pass to matrix rings.

Throughout this article, unless otherwise stated, all rings have unity and all modules are unital. For a subset X of a ring R, the left annihilator of X in R is $l(X) = \{r \in R : rx = 0 \text{ for all } x \in X\}$. For any $a \in R$, we write $l(a)$ for $l(\{a\})$. Right annihilators are defined similarly and indicated by r. We write $J(R)$ for the Jacobson radical of R and 1_M for an identity map from M to M. We also write $N \leq^e M$ and $N \leq^{\oplus} M$ to indicate that N is an essential submodule of M and a direct summand of M, respectively. General background material can be found in [1, 5, 11, 17].

2. RESULTS

The following lemma gives the characterization of the kernel-stable element of an endomorphism ring of a module M.

Lemma 2.1. *Let M be a module and $S = \text{End}(M)$. The following conditions are equivalent for an element $\alpha \in S$:*

(1) *α is kernel-stable.*
(2) *$\beta\alpha(M) = \beta(M)$ implies $\beta\alpha = \beta\gamma$ for an automorphism γ of M and $\gamma \in S$.*

Proof. (1) $\Rightarrow$ (2) : Let β be an endomorphism of M with $\beta\alpha(M) = \beta(M)$. It follows that $\alpha(M) + Ker\beta = M$. Then by hypothesis, $(\alpha - \gamma)(m) \in Ker\beta$ for every $m \in M$ and an automorphism γ of M. Hence $\beta(\alpha - \gamma)(m) = 0$, that is $\beta\alpha = \beta\gamma$.

(2) $\Rightarrow$ (1) : Let $\alpha(M) + Ker\beta = M$ for $\beta \in \text{End}(M)$. Then $\beta\alpha(M) = \beta(M)$. But then, (2) implies that for any $m \in M$, $\beta\alpha(m) = \beta\gamma(m)$, where γ is an automorphism of M. Then $\beta(\alpha - \gamma)(m) = 0$ and so $(\alpha - \gamma)(m) \in Ker\beta$. □

It is not surprising that there is a relation between the notion of kernel-stability and automorphism due to the following lemma.

Lemma 2.2. *If a kernel-stable element $\alpha \in \text{End}(M)$ is right invertible (that is a split epimorphism) or left invertible (that is a split monomorphism) then α is an automorphism.*

Proof. Let $\alpha \in \text{End}(M)$ be a right invertible kernel-stable element. Then, there exists $\beta \in \text{End}(M)$ such that $\alpha\beta = 1_M$ that gives $\alpha\beta(M) = 1_M(M)$. One gets $\alpha(M) + Ker1_M = M$ since $Ker1_M = 0$ and α is an epimorphism. It follows that, $(\alpha - \gamma)(m) \in Ker1_M = 0$ for an automorphism γ of M and for all $m \in M$ as α is kernel -stable. Then $\gamma = \alpha$ which completes the proof.

If $\alpha \in \text{End}(M)$ is a left invertible kernel-stable element. Then there exits β of $\text{End}(M)$ such that $\beta\alpha = 1_M$ and so $\beta\alpha(M) = M = \beta(M)$. By Lemma 2.1, $\beta\alpha = \beta\gamma$ for an automorphism γ of M. It follows that $\beta = \gamma^{-1}$ and so $\alpha = \gamma$ an automorphism γ of M. □

Recall that a module M is called a directly-finite module if $\text{End}(M)$ is directly-finite, that is, $gf = 1$ whenever $fg = 1$.

Corollary 2.3. *Every kernel-stable module is directly-finite.*

The converse of Corollary 2.3 is not true in general due to[12, Example 4].
A module M is pseudo-semi-projective if $\alpha S = \beta S$ for all $\alpha, \beta \in S = \mathrm{End}(M)$ with $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta)$ (see [14, Lemma 3.1]).

The next proposition contains several new characterization of the kernel-stable module.

Proposition 2.4. *The following conditions are equivalent for a module M:*

(1) *M is a kernel-stable module.*
(2) *If $\alpha(M) + Ker\beta = M$, $\alpha, \beta \in \mathrm{End}(M)$ then $\beta\alpha = \beta\gamma$ for an automorphism (split epimorphism) γ of M.*
(3) *If $\beta\alpha(M) = \beta(M)$ with $\alpha, \beta \in \mathrm{End}(M)$ then $\beta\alpha = \beta\gamma$ for an automorphism (split epimorphism) γ of M. Moreover, if M is a pseudo-semi-projective module, then the above conditions are equivalent to the following conditions:*
(4) *If $\beta(M) = \alpha(M)$ then $\beta = \alpha\gamma$ for an automorphism (split epimorphism) γ of M.*
(5) *M-cyclic submodules of M are uniquely generated.*

Proof. (1) $\Leftrightarrow$ (2) It is clear from the definition of kernel-stable module.

(2) $\Rightarrow$ (3) Let $\beta\alpha(M) = \beta(M)$ for $\alpha, \beta \in \mathrm{End}(M)$. It is easy to see that $M = Ker\beta + \alpha(M)$. Then by (2), $\beta\alpha = \beta\gamma$ for an automorphism (split epimorphism) γ of M.
(3) $\Rightarrow$ (1) Suppose that $\beta\alpha(M) = \alpha(M)$. Then there exists a split epimorphism γ of M such that $\beta\alpha = \beta\gamma$. Call γ' an endomorphism of M with $\gamma\gamma' = 1_M$. Then $\gamma(M) = \gamma\gamma'(M)$. By (3), there exists a split epimorphism γ" of M such that $\gamma\gamma$" $= \gamma\gamma' = 1$. Note that γ is a monomorphism. It shows that γ is an automorphism of M. We deduce that M is a kernel-stable module.

We now assume that M is a pseudo-semi-projective module.

(3) $\Rightarrow$ (4). Suppose that $\beta(M) = \alpha(M)$. Since M is a pseudo semi-projective module, $\beta S = \alpha S$ with $S = End(M)$. Then $\alpha = \beta t_1$ and $\beta = \alpha t_2$ for some $t_1, t_2 \in S$. This gives $\alpha t_2(M) = \alpha(M)$. By (3), $\beta = \alpha t_2 = \alpha\gamma$ for some automorphism (split epimorphism) γ of M.

(4) $\Rightarrow$ (5) Suppose that $\beta(M) = \alpha(M)$. By (4), there exists a split epimorphism γ of M such that $\alpha = \beta\gamma$. Take γ' as an endomorphism of M with $\gamma\gamma' = 1_M$. Then $\gamma(M) = \gamma\gamma'(M)$. By (4), there exists a split epimorphism γ'' of M such that $\gamma\gamma'' = \gamma\gamma' = 1_M$. Note that γ'' is a monomorphism. It shows that γ is an automorphism of M.

(5) $\Rightarrow$ (1) It is by Lemma 2.1. □

Example 2.5. (1) Every automorphism of M is kernel-stable.

(2) We consider the ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and $\alpha = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in R$. Then, α is kernel-stable.

One can use [12, Theorem 6] to show that there exists a module M such that M is uniquely generated but M is not a kernel-stable module.

Lemma 2.6. *If α is kernel-stable (or unit-regular) then $\alpha\gamma$ and $\gamma\alpha$ are kernel-stable (resp., unit-regular) for every automorphism γ of M.*

Proof. Let α be kernel-stable element and γ be an automorphism of M. Call β an endomorphism of M with $\alpha\gamma(M)+Ker(\beta) = M$. Then $\alpha(M)+Ker(\beta) = M$. By our assumption, $\beta\alpha = \beta\theta$ for some automorphism θ of M. It follows that $\beta(\alpha\gamma) = \beta(\theta\gamma)$ and $\theta\gamma$ is an automorphism of M. This shows that $\alpha\gamma$ is kernel-stable. We now assume that $\gamma\alpha(M) + Ker(\beta') = M$ with an endomorphism β' of M. It is easy to see that $\alpha(M) + Ker(\beta'\gamma) = M$. By Proposition 2.4 there exists an automorphism ψ of M with $(\beta'\gamma)\psi = (\beta'\gamma)\alpha$ or $\beta'(\gamma\psi) = \beta'(\gamma\alpha)$.

Unit-regularity case of α was proved similarly in [12, Lemma 12]. □

It is well-known that unit-regular elements are always regular. The following proposition provides us that the converse is true when it is also kernel-stable.

Proposition 2.7. *Let M be a module and α be an endomorphism of M. If α is regular and kernel-stable, then α is unit-regular.*

Proof. Assume that $\alpha = \alpha\beta\alpha$. Then $\alpha(M) = \alpha\beta(M)$ and $(\alpha\beta)^2 = \alpha\beta$. We have $M = \alpha(M) + Ker(\alpha\beta)$. As α is kernel-stable, $\alpha\beta\gamma = \alpha\beta\alpha = \alpha$ for some automorphism γ of M. It follows that $\alpha = \alpha\gamma^{-1}\alpha$. Thus α is unit-regular. □

The relationship between the notions of stable range 1 and kernel-stability is given by the following Lemma.

Lemma 2.8. *Let M be a pseudo-semi-projective module with $S = \mathrm{End}(M)$ and α be an endomorphism of M. If α has stable range 1 in S then α is kernel-stable.*

Proof. Let β be an endomorphism of M with $\beta\alpha(M) = \beta(M)$. Since M is a pseudo-semi-projective module, $\beta = \beta\alpha\theta$ for some $\theta \in S$. It means that $1_S = \alpha\theta + \psi$ for some $\psi \in S$ with $\beta\psi = 0$, and so $S = \alpha S + \psi S$. As α has right stable range 1 in S then there is a unit γ of S with $\alpha - \gamma \in \psi S$. Hence $\beta\alpha = \beta\gamma$ which completes the proof. □

Corollary 2.9. *Let M be a pseudo-semi-projective module such that $S = \mathrm{End}(M)$ and α is an endomorphism of M.*

(1) *α is unit-regular if and only if α is regular and kernel-stable.*
(2) *If $\alpha + J(S) \in S/J(S)$ is unit-regular then α is kernel-stable.*

Corollary 2.10. *Let M be a pseudo-semi-projective module with $S = \mathrm{End}(M)$. If $S/J(S)$ is unit-regular then M is kernel-stable.*

A element α of $\mathrm{End}(M)$ is called a quasi-morphic element if there exist $\beta, \gamma \in \mathrm{End}(M)$ such that $\alpha(M) = Ker\beta$ and $Ker\alpha = \gamma(M)$. A module M is called a quasi-morphic module if every element of End(M) is quasi-morphic (see [13]).

Lemma 2.11. *Let M be a right R-module with $S = End(M)$. Then the following are equivalent:*

(1) *S has stable range 1.*
(2) *Every left unit lifts modulo every left principal ideal of S.*
(3) *Every right unit lifts modulo every right principal ideal of S.*

Proof. By Theorem 3 in [15]. □

A right R-module M is semi-projective if and only if $fS = \mathrm{Hom}(M, fM)$ for every $f \in \mathrm{End}(M) = S$ ([17, page 260]).

It is well-known that every unit regular ring has stable range 1. Also a relationship between unit regularity of $S = \mathrm{End}(M)$ and uniquely generated submodules of M is given by [9, Theorem 2.8]. For a relation between stable range 1 property of End(M) and uniquely generated submodules of M, following theorem is obtained.

Theorem 2.12. *Let M be a semi-projective quasi-morphic module with $S =$ End(M). Then the following are equivalent:*

(1) *Every M-cyclic submodule of M is uniquely generated.*
(2) *S has stable range 1.*

Proof. (1) $\Rightarrow$ (2) By Lemma 2.11, we will show that every right unit lifts modulo every right principal ideal of S. Let α be a right unit lifts modulo to right principal ideal βS of S. There exists $\gamma \in S$ such that $\alpha\gamma - 1 \in \beta S$. As M is a quasi morphic module, there exist $f, g \in S$ such that $\beta(M) = Ker(f)$ and $f\alpha(M) = Ker(g)$. It follows that $\alpha(M) \leq Ker(gf)$ and $\beta(M) \leq Ker(gf)$. It means that $\alpha S \leq r_S(gf)$ and $\beta S \leq r_S(gf)$. On the other hand, we have $\alpha S + \beta S = S$. Thus $S = r_S(gf)$ and so $gf = 0$. This is proved that $f(M) = Ker(g) = f\alpha(M)$. Since every M-cyclic submodule of M is uniquely generated, there exists a unit θ of S such that $f\alpha = f\theta$. Thus, $(\alpha - \theta)(M) \leq \beta(M)$. Since M is a semi-projective module, $\alpha - \theta \in \beta S$.

(2) $\Rightarrow$ (1) By Lemma 2.8 and Lemma 2.2 (1) $\Rightarrow$ (5). □

3. IDEMPOTENT KERNEL STABLE

In this section, we weaken the notion of kernel-stable element of $S =$ End(M) and call it idempotent-kernel-stable.

Lemma 3.1. *Let M be a module with $S = \mathrm{End}(M)$ and α be an endomorphism of M. Then the following are equivalent:*

(1) *$\alpha(M) + Ker(e) = M, e^2 = e \in S$, implies $(\alpha - u)(M) \leq Ker(e)$ for some unit $u \in S$.*
(2) *$\alpha(M) + e(M) = M, e^2 = e \in S$, implies $(\alpha - u)(M) \leq e(M)$ for some unit $u \in S$.*

(3) *$\alpha(M)+\beta(M)=M$, $\beta \in S$ unit-regular, implies $(\alpha-u)(M) \leq \beta(M)$ for some unit $u \in S$.*
(4) *$\alpha(M)+Ker(\beta)=M$, $\beta \in S$ unit-regular, implies $(\alpha-u)(M) \leq Ker(\beta)$ for some unit $u \in S$.*

Proof. (1) $\Leftrightarrow$ (2) Let $\alpha(M)+e(M)=M, e^2=e \in S$. Since $e(M)=Ker(1-e)$ we can rewrite equation as following: $\alpha(M)+Ker(1-e)=M$. By (1), there exists unit $u \in S$ such that $(\alpha-u)(M) \leq Ker(1-e)=e(M)$. The converse can be proved in a similar way.

(2) $\Leftrightarrow$ (3) Let $\alpha(M)+\beta(M)=M$. Since β is unit-regular it can be written as $\beta=fv$, composition of a unit element $v \in S$ and an idempotent element $f \in S$. So $M=\alpha(M)+\beta(M)=\alpha(M)+f(M) \leq M$. By (2), $(\alpha-u)(M) \leq f(M)=\beta(M)$ for some unit $u \in S$. Converse is clear since every idempotent is unit-regular.

(1) $\Leftrightarrow$ (4) Let $\alpha(M)+Ker(\beta)=M$. Since β is unit-regular it can be written as $\beta=vf$ for some unit element $v \in S$ and an idempotent element $f \in S$. It is clear that $Ker(f)=Ker(\beta)$ and so $\alpha(M)+Ker(f)=M$. By (1), $(\alpha-u)(M) \leq Ker(f)=Ker(\beta)$ for some unit $u \in S$. □

We call an element $\alpha \in \mathrm{End}(M)$ *idempotent-kernel-stable* when the equivalent conditions of Lemma 3.1 hold for α and a module M is idempotent-kernel-stable if every element of $\mathrm{End}(M)$ is so.

Example 3.2. Let $D=\mathbb{Z}_5[x]$ and write $\bar{k}=k+5\mathbb{Z}$ for all $k \in \mathbb{Z}$ and $R=\{(z,\alpha): z \in \mathbb{Z}, \alpha \in D, \alpha(\bar{0})=\bar{z} \in \mathbb{Z}_5\}$. Then R_R is an idempotent-kernel-stable module but not kernel-stable.

Ring theoretic version of this Lemma was obtained by Nicholson in [12] as following:

Corollary 3.3 ([12, Lemma 23]). *The following are equivalent for an element $a \in R$:*

(1) *$Ra+l(e)=R$, $e^2=e$, implies $a-u \in l(e)$ for some unit $u \in R$.*
(2) *$Ra+Re=R$, $e^2=e$, implies $a-u \in Re$ for some unit $u \in R$.*
(3) *$Ra+Rb=R$, b unit-regular, implies $a-u \in Rb$ for some unit $u \in R$.*
(4) *$Ra+l(b)=R$, b unit-regular, implies $a-u \in l(b)$ for some unit $u \in R$.*

Nicholson called an element $a \in R$ left idempotent-annihilator-stable (left IAS) when these conditions hold for a, and a ring is a left IAS ring if every element is left IAS.

Lemma 3.4. *Let M be a pseudo-semi-projective module with $S=\mathrm{End}(M)$ and α be an endomorphism of M. The following are equivalent:*

(1) *α is regular and has stable range 1 in S.*
(2) *α is both regular and kernel-stable.*
(3) *α is both regular and idempotent kernel-stable.*
(4) *α is unit-regular.*

Proof. (1) $\Rightarrow$ (2) By Lemma 2.8.

(2) $\Rightarrow$ (3) It is a tautology.

(3) $\Rightarrow$ (4) By Corollary 2.9.

(4) $\Rightarrow$ (1) If α is unit-regular, we write $\alpha = ve$, $e^2 = e \in S$ and $v \in S$ is unit. By Lemma 2.6, it is enough to show that e is stable range 1. Let $S = eS + \beta S$ for $\beta \in S$. We show that there exists a unit $u \in S$ such that $e - u \in \beta S$. Take $s \in S$ with $1_S - es \in \beta S$. Call $u = 1_S - es(1-e)$. Then u is a unit with inverse $1_S + es(1-e)$. Then $e - u = (e - 1_S) + es(1_S - e) = (e - 1_S)(1_S - es) \in \beta S$. □

Recall that a right R-module M is said to have internal cancellation (IC) if, whenever $M = M_1 \oplus M_2 = M_1' \oplus M_2'$ with $M_1 \simeq M_1'$, then $M_2 \simeq M_2'$. It is well-known the Ehrlich's result ([6]) that a module M has IC if and only if every regular element in $\mathrm{End}(M)$ is unit-regular. Khurana and Lam called R an IC ring if R_R has IC equivalently R_R has IC ([8, page 5]). In the same paper, authors proved that that R is IC if and only if very regular element (unit-regular element, idempotent) in R has right UG ([8, Theorem 6.2]). Due to H. Chen [4, Lemma 1], R is left IAS (right IAS) if and only if R is IC. The next theorem is the module theoretic version of these results.

THEOREM 3.5. *The following conditions are equivalent for any pseudo-semi-projective module M over a ring R :*

(1) *M is idempotent kernel-stable.*
(2) *If f is a regular element of* $\mathrm{End}(M)$*, then $f(M)$ is an uniquely generated submodule of M.*
(3) *M is IC.*

Proof. (1) $\Rightarrow$ (2) Call $S = \mathrm{End}(M)$. Let f be a regular element of $\mathrm{End}(M)$. Suppose that $f(M) = g(M)$. There exists $e^2 = e \in \mathrm{End}(M)$ such that $f(M) = e(M)$. Since M is a pseudo-semi-projective module, $fS = eS$. Call $s_1, s_2 \in S$ with $f = es_1$ and $e = fs_2$. Then $es_1s_2 = fs_2 = e$ and so $M = s_1(M) + Ker(e)$. As M is idempotent kernel-stable, there is a unit u of S with $e(s_1 - u) = 0$. Thus $f = es_1 = eu$. Similarly, we have $g = ev$ for some unit v of S. It follows that $f = eu = g(v^{-1}u)$. Thus, (2) is required.

(2) $\Rightarrow$ (3) Call f is a regular element of $\mathrm{End}(M)$. There exists an idempotent e of $\mathrm{End}(M)$ such that $f(M) = e(M)$. But by (2), $f = eu$ for some unit u of $\mathrm{End}(M)$. It follows that f is unit-regular. By Ehrlich's result shown that M is IC.

(3) $\Rightarrow$ (1) Take $S = \mathrm{End}(M)$. Let α be an endomorphism of M and $\alpha(M) + e(M) = M, e^2 = e \in S$. Call $e_1 = 1 - e$ and $\alpha_1 = e_1\alpha$. We have $(1-e)\alpha(M) = (1-e)(M)$. Since M is a pseudo-semi-projective module, $(1-e)\alpha S = (1-e)S$. There exists $s \in S$ such that $e_1 = e_1\alpha s$ or $e_1 = \alpha_1 s$. But then $\alpha_1 s \alpha_1 = e_1\alpha_1 = \alpha_1$, so α_1 is regular. By (3), α_1 is unit-regular. It follows that α_1 is idempotent kernel-stable by Lemma 3.4. Moreover, since $\alpha_1 = e_1\alpha$, $\alpha_1(M) + e(M) = M$. Thus, $(\alpha_1 - u)(M) \leq e(M)$ for some unit u

of S. Finally, we have

$$(\alpha - u)(M) = [(\alpha_1 + e\alpha) - u](M) = [(\alpha_1 - u) + e\alpha](M) \leq e(M).$$

□

REFERENCES

[1] F.W. Anderson and K. R. Fuller, *Rings and categories of modules*, Springer-Verlag, New York, 1974.
[2] H. Bass, *K-Theory and stable algebra*, Publ. Math. Inst. Hautes Études Sci., **22** (1964), 5–70.
[3] M.J. Canfell, *Completion of diagrams by automorphisms and Bass first stable range condition*, J. Algebra, **176** (1995), 480–503.
[4] H. Chen, *On partially unit-regularity*, Kyungpook Math. J., **42** (2002), 13–19.
[5] N.V. Dung, D.V. Huynh, P.F. Smith and R. Wisbauer, *Extending modules*, Pitman, 1996.
[6] G. Ehrlich, *Unit-regular rings*, Port. Math., **27** (1968), 209–212.
[7] I. Kaplansky, *Elementary divisors and modules*, Trans. Amer. Math. Soc., **66** (1949), 464–491.
[8] D. Khurana and T.Y. Lam, *Rings with internal cancellation*, J. Algebra, **284** (2005), 203–235.
[9] M.T. Kosan, T.C. Quynh and S. Sahinkaya, *On rings with associated elements*, Comm. Algebra, **45** (2017), 2747–2756.
[10] G. Marks, *A criterion for unit-regularity*, Acta Math. Hungar., **111** (2006), 311–312.
[11] S.H. Mohammed and B.J. Müller, *Continous and discrete modules*, London Math. Soc., Vol. 147, Cambridge Univ. Press, 1990.
[12] W.K. Nicholson, *Annihilator-stability and unique generation*, J. Pure Appl. Algebra, **221** (2017), 2557–2572.
[13] W.K. Nicholson and E. Campos, *Morphic modules*, Comm. Algebra, **3** (2005), 2629–2647.
[14] T.C. Quynh, *On pseudo semi-projective modules*, Turkish J. Math., **37** (2013), 27–36.
[15] F. Siddique, *On two questions of Nicholson*, preprint.
[16] J. Von Neumann, *Regular ring*, Proc. Natl. Acad. Sci. USA, **22** (1936), 707-–713.
[17] R. Wisbauer, *Foundations of module and ring theory*, Gordon and Breach, Reading, 1991.

Received July 22, 2019
Accepted December 3, 2019

Tarsus University
Technology Faculty
Tarsus/Mersin Turkey
E-mail: serap@tarsus.edu.tr
https://orcid.org/0000-0002-2084-6260

Danang University
Department of Mathematics
459 Ton Duc Thang, DaNang city, Vietnam
E-mail: tcquynh@live.com
https://orcid.org/0000-0002-0845-0175