MATHEMATICA, 63 (86), N° 1, 2021, pp. 111–118

ON STARLIKENESS OF RECIPROCAL ORDER

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Abstract. Sufficient conditions for functions defined in the unit disk to be starlike of reciprocal order are obtained. Also certain recent results are generalized.

MSC 2010. 30C45, 30C80.

Key words. Analytic functions, subordination, starlike function of reciprocal order.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of functions that are analytic in the open unit disk $\Delta = \{z : |z| < 1\}$ with the normalization $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ of univalent functions. If f and g are functions which are analytic in Δ, we say that f is subordinate to g, written $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in \Delta)$. In particular, if g is univalent in Δ, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$. Ma and Minda [6] introduced the following classes

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A} : \left(\frac{zf'(z)}{f(z)} \right) \prec \varphi(z) \right\},$$

$$\mathcal{C}^*(\varphi) = \left\{ f \in \mathcal{A} : \left(1 + \frac{zf''(z)}{f'(z)} \right) \prec \varphi(z) \right\},$$

where φ is univalent with positive real part which maps Δ onto a domain which is symmetric with respect to the real line and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. When $-1 \leq B < A \leq 1$, $\mathcal{S}^*[A, B] = \mathcal{S}^*[(1 + Az)/(1 + Bz)]$ denotes the class of Janowski [4] starlike functions and $\mathcal{C}[A, B]$ denotes the corresponding class of convex functions. The subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha), (0 \leq \alpha < 1)$ of $\mathcal{S}$, introduced by Robertson [13], consisting of starlike functions of order α and convex functions of order α respectively, are obtained for the case $A = 1 - 2\alpha$ and $B = -1$. The classes $\mathcal{S}^* = \mathcal{S}^*(0)$ and $\mathcal{C} = \mathcal{C}(0)$ denote the classes of starlike and convex functions respectively.

Several results on sufficient conditions for analytic functions in the open unit disk to be starlike of order α were obtained by various authors in [9, 10, 11] and

The authors are thankful to the referees for their insightful suggestions.

DOI: 10.24193/mathcluj.2021.1.10

the properties of various subclasses of starlike functions were investigated in [4, 5, 6, 14, 15]. A function $f \in \mathcal{S}^*$ is starlike of reciprocal order α $(0 \leq \alpha < 1)$ in Δ if

$$\operatorname{Re}\left\{\frac{f(z)}{zf'(z)}\right\} > \alpha \qquad (z \in \Delta).$$

A starlike function of reciprocal order 0 is same as a starlike functions, since $\operatorname{Re}\{p(z)\} > 0$ implies that

$$\operatorname{Re}\left\{\frac{1}{p(z)}\right\} = \operatorname{Re}\left\{\frac{p(z)}{|p(z)|^2}\right\} > 0.$$

Hence every starlike function of reciprocal order $\alpha \geq 0$ is starlike, and hence univalent (cf. [10, Example 1]). When $0 < \alpha < 1$, the function f is starlike of reciprocal order α if and only if $\left|\frac{zf'(z)}{f(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}$. Nunokawa et. al [10] presented examples of starlike functions of reciprocal order 0 and $1/(2-\alpha)$.

Coefficient bounds and Fekete - Szego inequality, for some families of starlike functions of reciprocal order, were obtained by M. Arif et.al [1]. Sufficient conditions for analytic functions to be starlike of reciprocal order α were derived by B.A. Frasin et. al. [3]. Ravichandran et.al [12] obtained the argument estimates for certain analytic functions, associated with starlike functions of reciprocal order. Recently, B.A. Frasin and M.Ab. Sabri [2] also derived some sufficient conditions involving starlikeness of reciprocal order. In this article we generalize certain results stated in [2] and establish various sufficient conditions for analytic functions to be starlike of reciprocal order. Also some properties of starlike functions of reciprocal order are obtained.

We require the following lemmas to prove our main results.

Lemma 1.1 ([7]). *Let $p(z)$ be analytic and non constant in Δ with $p(0) = 1$. If $0 < |z_0| < 1$ and $\operatorname{Re} p(z_0) = min[\operatorname{Re} p(z)]$, then $z_0 p'(z_0) \leq -\dfrac{|1-p(z_0)|^2}{2(1-\operatorname{Re} p(z_0))}$.*

Lemma 1.2 ([9]). *Let $p(z) = 1 + \sum_{n=1}^{\infty}(c_n z^n)$ be analytic in Δ and suppose that there exists a point $z_0 \in \Delta$ such that $\operatorname{Re} p(z) > 0$ for $|z| < |z_0|$ and $\operatorname{Re} p(z_0) = 0$. Then we have $z_0 p'(z_0) \leq -\frac{1}{2}(1 + |p(z_0)|^2)$, where $z_0 p'(z_0)$ is a negative real number.*

Lemma 1.3. [8] *Let q be univalent in Δ and let θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$, $w \in q(\Delta)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that*

(1) *Q is starlike in Δ.*
(2) $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0, \quad (z \in \Delta).$

If p is analytic in Δ and satisfies $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$, then $p(z) \prec q(z)$ and q is the best dominant.

2. MAIN RESULTS

THEOREM 2.1. *Let $f \in \mathcal{A}$ satisfy $f(z)f'(z) \neq 0$ in $0 < |z| < 1$ and let $\alpha \geq 0, \gamma \in \mathbb{C}$ with $\operatorname{Im}\gamma > \dfrac{1+\alpha}{2}$. If*

(1)
$$\operatorname{Im}\left\{\frac{f(z)}{zf'(z)}\left(1-\frac{\alpha zf''(z)}{f'(z)}+\frac{\gamma f(z)}{zf'(z)}\right)\right\} > (1+\alpha)\left|\frac{f(z)}{zf'(z)}\right| - (\operatorname{Im}\gamma)\left|\frac{f(z)}{zf'(z)}\right|^2,$$

then $f(z)$ is starlike of reciprocal order 0 in Δ, and thus $f(z)$ is starlike in Δ.

Proof. Let $p(z) = \dfrac{f(z)}{zf'(z)}$. Then $p(z)$ is analytic in Δ and $p(0) = 1$. Differentiating logarithmically we obtain

$$\frac{f(z)}{zf'(z)}\left(1-\frac{\alpha zf''(z)}{f'(z)}+\frac{\gamma f(z)}{zf'(z)}\right) = \alpha zp'(z) + (\alpha+1)p(z) - \alpha + \gamma p^2(z). \tag{2}$$

Suppose there exists a point $z_0 \in \Delta$ such that $\operatorname{Re}p(z) > 0$, for $|z| < |z_0|$. Then we have $p(z_0) = \mathrm{i}\beta$, where β is real and $\beta \neq 0$. Now applying Lemma 1.1, we get $z_0p'(z_0) \leq -(1+\beta^2)/2$. From the above inequality, we have

$$\begin{aligned}
&\operatorname{Im}\left\{\frac{f(z_0)}{z_0f'(z_0)}\left(1-\frac{\alpha z_0f''(z_0)}{f'(z_0)}+\frac{\gamma f(z_0)}{zf'(z_0)}\right)\right\}\\
&= \operatorname{Im}\left\{\alpha z_0p'(z_0) + (\alpha+1)p(z_0) - \alpha + \gamma p^2(z_0)\right\}\\
&\leq \operatorname{Im}\left\{-\alpha\left(\frac{1+\beta^2}{2}\right) + i(1+\alpha)\beta - \alpha - \gamma\beta^2\right\}\\
&= (1+\alpha)\beta - (\operatorname{Im}\gamma)\beta^2 = (1+\alpha)|p(z_0)| - (\operatorname{Im}\gamma)|p(z_0)|^2\\
&= (1+\alpha)\left|\frac{f(z_0)}{z_0f'(z_0)}\right| - (\operatorname{Im}\gamma)\left|\frac{f(z_0)}{z_0f'(z_0)}\right|^2,
\end{aligned}$$

which contradicts (1). Hence the result is proved. □

The proof of the following theorem being similar to that of Theorem 2.1, we omit the proof.

THEOREM 2.2. *Let $f \in \mathcal{A}$ satisfy $f(z)f'(z) \neq 0$ in $0 < |z| < 1$ and let $\alpha \geq 0, \gamma \in \mathbb{C}$. If*

$$\operatorname{Re}\left\{\frac{f(z)}{zf'(z)}\left(1-\frac{\alpha zf''(z)}{f'(z)}+\frac{\gamma f(z)}{zf'(z)}\right)\right\} > -\left|\frac{f(z}{zf'(z)}\right|^2\left(\operatorname{Re}\gamma+\frac{\alpha}{2}\right) - \frac{3\alpha}{2}, \tag{3}$$

then $f(z)$ is starlike of reciprocal order 0 in Δ and thus, $f(z)$ is starlike in Δ.

Taking $\gamma = 0$ in Theorem 2.2, we obtain the result of B.A. Frasin [2, Theorem 2, p. 872].

THEOREM 2.3. *Let $f(z) \in \mathcal{A}$ satisfy $f(z)f'(z) \neq 0$ in $0 < |z| < 1$. For $\alpha \neq 1, \alpha > 0$ and $z \in \Delta$, if*

(4)
$$\operatorname{Re}\left\{\frac{f(z)}{zf'(z)}\left(-1-\frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{1}{2}\left\{\frac{\alpha-1}{\alpha}+\frac{\alpha}{\alpha-1}\left|\frac{f(z)}{zf'(z)}-\frac{1}{\alpha}\right|^2+2\right\},$$

then $f(z)$ is starlike reciprocal of order $\dfrac{1}{\alpha}$.

Proof. Let
$$p(z) = \left(\frac{\alpha}{\alpha-1}\right)\left(\frac{f(z)}{zf'(z)}-\frac{1}{\alpha}\right).$$
Then $p(z)$ is analytic in Δ and $p(0) = 1$. After differentiation, we obtain
$$\frac{f(z)}{zf'(z)}\left[\frac{zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)}-1\right] = \left(\frac{\alpha-1}{\alpha}zp'(z)\right).$$
Suppose there exists a point $z_0 \in \Delta$ such that $\operatorname{Re}\{p(z)\} > 0$ for $|z| < |z_0|$ and $\operatorname{Re}\{p(z_0)\} = 0$. Then from Lemma 1.1, we have $z_0p'(z_0) \leq -\frac{1}{2}(1+|p(z_0)|^2)$. Now, we have
$$\begin{aligned}
&\operatorname{Re}\left\{\frac{f(z_0)}{z_0f'(z_0)}\left(-1-\frac{zf''(z_0)}{f'(z_0)}+\frac{zf''(z_0)}{f(z_0)}\right)\right\}\\
&= \left(\frac{\alpha-1}{\alpha}\right)\operatorname{Re}\{z_0p'(z_0)\}\\
&\leq \left(\frac{\alpha-1}{\alpha}\right)\left(-\frac{1}{2}[1+|p(z_0)|^2]\right)\\
&= -\frac{(\alpha-1)}{2\alpha}-\frac{(\alpha-1)}{2\alpha}\left\{\left(\frac{\alpha}{\alpha-1}\right)^2\left|\frac{f(z_0)}{zf'(z_0)}-\frac{1}{\alpha}\right|^2\right\}\\
&= -\frac{1}{2}\left(\frac{\alpha-1}{\alpha}\left|\frac{f(z_0)}{zf'(z_0)}-\frac{1}{\alpha}\right|^2\right).
\end{aligned}$$
Therefore,
$$\operatorname{Re}\left\{\frac{f(z_0)}{z_0f'(z_0)}\left(-1-\frac{zf''(z)}{f'(z)}\right)\right\} \leq -\frac{1}{2}\left\{\frac{\alpha-1}{\alpha}+\frac{\alpha}{\alpha-1}\left|\frac{f(z_0)}{zf'(z_0)}-\frac{1}{\alpha}\right|^2+2\right\},$$
which is a contradiction with (4). Hence $\operatorname{Re}[p(z)] > 0$, $(z \in \Delta)$. That is,

(5)
$$\operatorname{Re}\left\{\frac{f(z)}{zf'(z)}\right\} > \frac{1}{\alpha} \qquad (z \in \Delta),$$

which implies that $f(z)$ is starlike of reciprocal order $\dfrac{1}{\alpha}$. □

Taking $\alpha = 2$ in Theorem 2.3, we get the result by B.A. Frasin [2, Theorem 3, p. 873].

THEOREM 2.4. *If $f(z)\in\mathcal{A}$ satisfies $f(z)f'(z)\neq 0$ in $0<|z|<1$, and*

$$\begin{aligned}(6)\quad &\left|\mathrm{Im}\left\{\frac{zf''(z)}{f'(z)}\left(\frac{f(z)}{zf'(z)}+\gamma\right)+\frac{f(z)}{zf'(z)}\left(\frac{f(z)}{zf'(z)}-\gamma\left(\frac{zf'(z)}{f(z)}\right)^2-(\gamma+2)\right)\right.\right.\\ &\left.\left.+(1+2\gamma)\right\}\right|<\sqrt{\gamma(2+3\gamma)},\end{aligned}$$

for some real $\gamma>0$, then $f(z)$ is starlike of reciprocal order 0 in Δ, and thus $f(z)$ is starlike in Δ.

Proof. Let $p(z)=\dfrac{f(z)}{zf'(z)}$. Then $p(z)$ is analytic in Δ and $p(0)=1$. Differentiating logarithmically, we obtain

$$\begin{aligned}(p(z)-\gamma)\left(\frac{zp'(z)}{p(z)}+p(z)-1\right)&=\frac{zf''(z)}{f'(z)}\left(\frac{f(z)}{zf'(z)}+\gamma\right)\\ +\frac{f(z)}{zf'(z)}&\left(\frac{f(z)}{zf'(z)}-\gamma\left(\frac{zf'(z)}{f(z)}\right)^2-(\gamma+2)\right)+(1+2\gamma).\end{aligned}$$

Suppose there exists a point $z_0,(0<|z_0|<1)$ such that $\mathrm{Re}p(z)>0$ for $|z|<|z_0|$. Then $p(z_0)=i\beta$, where β is real and $\beta\neq 0$. Then by applying Lemma 1.1, we have $z_0p'(z_0)\leq\left(\frac{1+\beta^2}{2}\right)$. Thus,

$$\begin{aligned}I_0=\mathrm{Im}&\left\{\frac{zf''(z)}{f'(z)}\left(\frac{f(z)}{zf'(z)}+\gamma\right)+\frac{f(z)}{zf'(z)}\left(\frac{f(z)}{zf'(z)}-\gamma\left(\frac{zf'(z)}{f(z)}\right)^2-(\gamma+2)\right)\right.\\ &\left.+(1+2\gamma)\right\}=-\frac{\gamma+(2+3\gamma)\beta^2}{2\beta}.\end{aligned}$$

Therefore, in view of $\gamma>0$, we obtain

$$(7)\qquad I_0\geq-\frac{\gamma+(2+3\gamma)\beta^2}{2\beta}\geq\sqrt{\gamma(2+3\gamma)}\qquad(\beta<0),$$

$$(8)\qquad I_0\leq-\frac{\gamma+(2+3\gamma)\beta^2}{2\beta}\leq-\sqrt{\gamma(2+3\gamma)}\qquad(\beta>0).$$

Equations (7) and (8) are in contradiction with (6). Thus, $f(z)$ is starlike of reciprocal order 0, and hence starlike in Δ. □

THEOREM 2.5. *Let $f(z)\in\mathcal{A}$ satisfy $f(z)f'(z)\neq 0$ in $0<|z|<1$ and let $\alpha>0$. If*

$$(9)\qquad (1-\alpha)\frac{f(z)}{zf'(z)}+\alpha-\alpha\frac{f(z)f''(z)}{(f'(z))^2}\prec\alpha+(1-\alpha)\sqrt{1+z},$$

then $\frac{f(z)}{zf'(z)}\prec\alpha+(1-\alpha)\sqrt{1+z}$.

Proof. Let $p(z) = \dfrac{f(z)}{zf'(z)}$. Then $p(z)$ is analytic in Δ and $p(0) = 1$. A simple calculation yields

$$\frac{f(z)}{zf'(z)}\left(1 + \frac{\alpha z f'(z)}{f(z)} - \frac{\alpha z f''(z)}{f'(z)} - \alpha\right) = p(z) + \alpha z p'(z) \prec \alpha + (1-\alpha)\sqrt{1+z}.$$

Define the function $q : \Delta \to \mathbb{C}$ by $q(z) = \alpha + (1-\alpha)\sqrt{1+z}$ with $q(0) = 1$. Since $q(\Delta) = \left\{w : \left|\left(\dfrac{w(z)-\alpha}{1-\alpha}\right)^2 - 1\right| < 1\right\}$ is in the right half plane, $q(\Delta)$ is a convex set, and hence q is a convex function. This shows that $zq'(z)$ is starlike with respect to origin.

Now we claim that $p(z) + \alpha z p'(z) \prec q(z) + zq'(z)$. Let θ and ϕ be analytic functions in $\mathbb{C}$, defined by $\theta(w) = w$, such that $\phi(w) = \alpha$ and $\phi(w) \neq 0$. Let $Q(z) = zq'(z)\phi(q(z)) = \alpha z q'(z)$. Then

$$h(z) = \theta(q(z)) + Q(z) = q(z) + \alpha z q'(z) = \alpha + (1-\alpha)\sqrt{1+z} + \frac{\alpha(1-\alpha)z}{2\sqrt{1+z}}.$$

Now

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right) > 0.$$

Hence Q is a starlike univalent function in Δ, and for $\alpha > 0$,

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\frac{z(\theta(q(z))' + Q'(z)}{Q(z)}\right\} = \frac{1}{\alpha} + \operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > 0.$$

It follows from (9) that

$$p(z) + zp'(z) \prec q(z) = \alpha + (1-\alpha)\sqrt{1+z}. \tag{10}$$

Next we claim that $p(z) \prec q(z)$. Now,

$$q(z) = \alpha + (1-\alpha)\sqrt{1+z} \prec \alpha + (1-\alpha)\sqrt{1+z} + \frac{\alpha(1-\alpha)z}{2\sqrt{1+z}} = h(z).$$

Since $q^{-1}(w) = \left(\dfrac{w(z)-\alpha}{1-\alpha}\right)^2 - 1$. It follows that

$$q^{-1}(h(z)) = \left(\frac{h(z)-\alpha}{1-\alpha}\right)^2 - 1 = \left(\frac{\alpha + (1-\alpha)\sqrt{1+z} + \frac{\alpha(1-\alpha)}{2}\frac{z}{\sqrt{1+z}} - \alpha}{(1-\alpha)}\right)^2 - 1$$

$$= \left(\sqrt{1+z} + \frac{\alpha}{2}\frac{z}{\sqrt{1+z}}\right)^2 - 1 = z(1+\alpha) + \frac{\alpha^2 z^2}{4(1+z)}.$$

For $z = e^{i\psi}$, $(-\pi \le \psi \le \pi)$,

$$|q^{-1}(he^{i\psi})| = \left|e^{i\psi}(1+\alpha) + \frac{\alpha^2 e^{2i\psi}}{4(1+e^{i\psi})}\right|. \tag{11}$$

The minimum value of the right hand side of (11) is attained at $\psi = 0$. Hence $|q^{-1}(he^{i\psi})| \geq 1 + \alpha + \frac{\alpha^2}{8} > 1$. Thus, $q^{-1}(h(\Delta)) \supset \Delta$, that is, $h(\Delta) \supset q(\Delta)$. This shows that $q(z) \prec h(z)$. Now from equation (10), and by Lemma 1.3, we obtain $p(z) \prec q(z)$. That is, $\frac{f(z)}{zf'(z)} \prec \alpha + (1-\alpha)\sqrt{1+z}$. □

THEOREM 2.6. *Let $f \in \mathcal{A}$, let $g(z)$ be starlike of reciprocal order β, and let $\gamma \geq 0$, $0 \leq \delta < 1$. If*

(12)
$$\mathrm{Re}\left\{\frac{\gamma f'(z)}{g'(z)} + (1-\gamma)\frac{f(z)}{g(z)} - \delta\right\} > -\frac{\gamma\beta}{2}\left((1-\delta) + \frac{1}{(1-\delta)}\left|\frac{f(z)}{g(z)} - \delta\right|^2\right),$$

then $\mathrm{Re}\left(\frac{f(z)}{g(z)}\right) > \delta$.

Proof. Let $p(z) = \dfrac{1}{1-\delta}\left(\dfrac{f(z)}{g(z)} - \delta\right)$. Then $p(z)$ is analytic in Δ and $p(0) = 1$. Therefore
$$\frac{1}{1-\delta}\left(\frac{f'(z)}{g'(z)} - \delta\right) = p(z) + \frac{g(z)}{zg'(z)}zp'(z).$$
Hence we obtain
$$\frac{\gamma f'(z)}{g'(z)} + (1-\gamma)\frac{f(z)}{g(z)} - \delta = (1-\delta)p(z)\left(1 + \frac{\gamma z p'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right). \tag{13}$$
Suppose that there exists a point $z_0 \in \Delta$ such that $\mathrm{Re}\{p(z)\} > 0$, for $|z| < |z_0|$ and $\mathrm{Re}\{p(z_0)\} = 0$. Then using Lemma 1.2, we have $z_0 p'(z_0) \leq -\frac{1}{2}(1 + |p(z_0)|^2)$. From equation (13), we obtain
$$\begin{aligned}
&\mathrm{Re}\left\{\frac{\gamma f'(z_0)}{g'(z_0)} + (1-\gamma)\frac{f(z_0)}{g(z_0)} - \delta\right\}\\
&= \mathrm{Re}\left\{(1-\delta)p(z_0)\left(1 + \frac{\gamma z_0 p'(z_0)}{p(z_0)}\frac{g(z_0)}{z_0 g'(z_0)}\right)\right\}\\
&\leq \mathrm{Re}\left\{(1-\delta)\left(p(z_0) - \frac{\gamma}{2}(1 + |p(z_0)|^2)\frac{g(z_0)}{z_0 g'(z_0)}\right)\right\}\\
&\leq -\frac{\gamma\beta}{2}\left((1-\delta) + \frac{1}{(1-\delta)}\left|\frac{f(z)}{g(z)} - \delta\right|^2\right).
\end{aligned}$$
This is in contradiction with (12). Hence $\mathrm{Re}\left\{\dfrac{f(z)}{g(z)}\right\} > \delta$. □

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Received May 31, 2019
Accepted November 13, 2020

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