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WELL-POSEDNESS AND EXPONENTIAL DECAY FOR A LAMINATED BEAM IN THERMOELASTICITY OF TYPE III WITH DELAY TERM

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Abstract. In this article, we study the well-posedness and asymptotic behaviour of solutions to a laminated beam in thermoelasticity of type III with delay term in the first equation. We show that the system is well-posed by using Lumer-Philips theorem and prove that the system is exponentially stable if and only if the wave speeds are equal.

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Key words. Laminated beam, thermoelasticity of type III, delay term, exponential decay.

1. INTRODUCTION

In the present paper, we are concerned onedimensional laminated beam system in thermoelasticity of type III with delay term, which has the form

$$(1)\qquad \begin{cases} \rho_1\varphi_{tt} + G(\psi-\varphi_x)_x + \mu_1\varphi_t(x,t) + \mu_2\varphi_t(x,t-\tau) = 0, \\ \rho_2(3\omega-\psi)_{tt} - G(\psi-\varphi_x) - D(3\omega-\psi)_{xx} + \sigma\theta_{tx} = 0, \\ \rho_2\omega_{tt} + G(\psi-\varphi_x) + \frac{4}{3}\gamma\omega + \frac{4}{3}\beta\omega_t - D\omega_{xx} = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \sigma(3\omega-\psi)_{tx} - k\theta_{txx} = 0, \end{cases}$$

where $(x,t) \in (0,1)\times(0,+\infty)$, with the following initial and boundary conditions

$$(2)\qquad \begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), & x\in[0,1], \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), & x\in[0,1], \\ \omega(x,0) = \omega_0(x), \omega_t(x,0) = \omega_1(x), & x\in[0,1], \\ \theta(x,0) = \theta_0(x), \theta_t(x,0) = \theta_1(x), & x\in[0,1], \\ \varphi_t(x,t-\tau) = f_0(x,t-\tau), & x\in(0,1),\ t\in(0,\tau), \\ \varphi_x(0,t) = \psi(0,t) = \omega(0,t) = \theta(0,t) = 0, & \forall t\geq 0, \\ \varphi_x(1,t) = \psi(1,t) = \omega(1,t) = \theta_x(1,t) = 0, & \forall t\geq 0, \end{cases}$$

Here $\varphi = \varphi(x,t)$ denotes the transverse displacement of the beam which departs from its equilibrium position, $\psi = \psi(x,t)$ represents the rotation angle. $\omega = \omega(x,t)$ is proportional to the amount of slip along the interface at time t

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and longitudinal spatial variable x. $\theta=\theta(x,t)$ is the differential temperature, and ρ_1, ρ_2, ρ_3, G, D, σ, γ, β, δ, k, μ_1 are positive constants, μ_2 is a real number, and $\tau>0$ represents the time delay. With the increasing demand of advanced performance, the vibration suppression of the laminated beams has been one of the main research topics in smart materials and structures. Laminated beam describes that two identical homogeneous beams are allowed between the beams, which were placed on top of each and a slip at the interface. These composite laminates usually have superior structural properties such as adaptability. The design of their piezoelectric materials can be invoked as both actuators and sensors [30]. Hansen [9] proposed a model of laminated beam based on the Timoshenko system which is one of particular interest. Hansen and Spies [10] studied the boundary stabilization of laminated beams with structural damping, which is

$$\begin{cases} \rho_1\varphi_{tt}+G\left(\psi-\varphi_x\right)_x=0, \\ \rho_2\left(3\omega-\psi\right)_{tt}-D\left(3\omega-\psi\right)_{xx}-G\left(\psi-\varphi_x\right)=0, \\ 3\rho_2\omega_{tt}+3G\left(\psi-\varphi_x\right)+4\gamma\omega+4\beta\omega_t-3D\omega_{xx}=0, \end{cases}$$

where $(x,t)\in(0,1)\times(0,+\infty)$, and ρ_1, G, ρ_2, D, γ, β are positive constants coefficients. ρ_1 is the density of the beams, G is the shear stiffness, ρ_2 is the mass moment of inertia, D is the flexural rigidity, γ is the adhesive stiffness of the beams, and β is the adhesive damping parameter. For asymptotic behavior results to laminated beams, we refer the reader to [1, 15, 16, 17, 18, 29] and the references therein. For the Timoshenko system of thermo-viscoelasticity of type III, Messaoudi and Said-Houari [22] considered the following one-dimensional linear Timoshenko system of thermoelastic type:

$$\begin{cases} \rho_1\varphi_{tt}-K\left(\varphi_x+\psi\right)_x=0, \\ \rho_2\psi_{tt}-b\psi_{xx}+K\left(\varphi_x+\psi\right)+\beta\theta_x=0, \\ \rho_3\theta_{tt}-\delta\theta_{xx}+\gamma\psi_{ttx}-\kappa\theta_{txx}=0, \end{cases}$$

where $(x,t)\in(0,1)\times(0,+\infty)$, they used the energy method to prove an exponential decay under the condition $\frac{\rho_1}{K}=\frac{\rho_2}{b}$. A similar result was also obtained by Rivera and Racke [28]. Since this theory predicts an infinite speed of heat propagation, many theories have emerged, to overcome this physical paradox. A large number of interesting decay results depending on the stability number have been established, (see [7, 20, 21, 23] and references therein). In [19],Y. Luan, W. Liu and G. Li considered a coupled system of a laminated beam with thermoelasticity of type III, which has the form:

$$\begin{cases} \rho_1\varphi_{tt}+G\left(\psi-\varphi_x\right)_x=0, \\ I_{\rho_1}\left(3\omega-\psi\right)_{tt}-D\left(3\omega-\psi\right)_{xx}-G\left(\psi-\varphi_x\right)+\alpha\theta_x=0, \\ I_{\rho_1}\omega_{tt}-D\omega_{xx}+G\left(\psi-\varphi_x\right)+\frac{4}{3}\beta_1\omega+\frac{4}{3}\beta_2\omega_t=0, \\ \rho_2\theta_{tt}-\delta\theta_{xx}+\gamma\left(3\omega-\psi\right)_{ttx}-k\theta_{txx}=0, \end{cases}$$

where $(x,t)\in(0,1)\times(0,+\infty)$, they used the energy method to prove an exponential decay result for the case of equal wave speeds. Time delays arise in many applications because most phenomena naturally depend not only on the present state but also on some past occurrences. In recent years, the PDEs with time delay effects have become an active area of research. Many authors have focused on this problem (see [2, 4, 7, 24, 25, 26, 31, 32]). The presence of delay may lead to a source of instability. In [4] for example, R. Datko, J. Lagnese and M. P. Polis proved that a small delay may destabilize a system. Nicaise, Pignotti and Valein [26] replaced the constant delay term in the boundary condition of [24] by a time-varying delay term and obtained an exponential decay result under an appropriate assumption on the weights of the damping and delay. Moreover, Kafini et al. [13] examined a coupling Timoshenko-thermoelasticity of type III system with time delay and established exponential and polynomial stability results depending on the wave propagation speeds. For other related results, we refer the reader to [3, 5, 6, 11, 12, 14]. The purpose of this work is to study the well-posedness and asymptotic behaviour of solutions to the laminated beam (1)-(2) in thermoelasticity of type III with delay term appearing in the control term in the first equation. Introducing the delay term $\mu_2\varphi_t(x,t-\tau)$ makes the problem different from those considered in the literature. The plan of the paper is as follows. In Section 2, we introduce some preliminaries. In Section 3, we prove the well-posedness of the system. In Section 4, we prove that the system is exponentially stable in the case of equal wave speeds.

2. PRELIMINARIES

In order to prove the well-posedness result, we introduce as in [24] the new variable $z(x,\rho,t)=\varphi_t(x,t-\tau\rho)$, $(x,\rho,t)\in(0,1)\times(0,1)\times(0,\infty)$. Thus, we have $\tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0$, $(x,\rho,t)\in(0,1)\times(0,1)\times(0,\infty)$. Therefore, system (1) takes the form

$$
\begin{cases}
\rho_1\varphi_{tt}+G(\psi-\varphi_x)_x+\mu_1\varphi_t(x,t)+\mu_2 z(x,1,t)=0,\\
\rho_2(3\omega-\psi)_{tt}-G(\psi-\varphi_x)-D(3\omega-\psi)_{xx}+\sigma\theta_{tx}=0,\\
\rho_2\omega_{tt}+G(\psi-\varphi_x)+\frac{4}{3}\gamma\omega+\frac{4}{3}\beta\omega_t-D\omega_{xx}=0,\\
\rho_3\theta_{tt}-\delta\theta_{xx}+\sigma(3\omega-\psi)_{tx}-k\theta_{txx}=0,\\
\tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0,
\end{cases}
\tag{3}
$$

where $(x,\rho,t)\in(0,1)\times(0,1)\times(0,\infty)$, with the following initial and boundary conditions

$$
\begin{cases}
\varphi(x,0)=\varphi_0(x),\varphi_t(x,0)=\varphi_1(x), & x\in[0,1],\\
\psi(x,0)=\psi_0(x),\psi_t(x,0)=\psi_1(x), & x\in[0,1],\\
\omega(x,0)=\omega_0(x),\omega_t(x,0)=\omega_1(x), & x\in[0,1],\\
\theta(x,0)=\theta_0(x),\theta_t(x,0)=\theta_1(x), & x\in[0,1],\\
z(x,\rho,0)=f_0(x,-\tau\rho), & x\in(0,1),\ \rho\in(0,1),\\
z(x,0,t)=\varphi_t(x,t), & x\in(0,1),\ t\in(0,\infty),\\
\varphi_x(0,t)=\psi(0,t)=\omega(0,t)=\theta(0,t)=0, & \forall t\geq 0,\\
\varphi_x(1,t)=\psi(1,t)=\omega(1,t)=\theta_x(1,t)=0, & \forall t\geq 0.
\end{cases}
\tag{4}
$$

In order to be able to use Poincaré's inequality for θ, we introduce

$$\overline{\theta}(x,t) := \theta(x,t) - \int_0^1 \theta_0(x)\,\mathrm{d}x - t\int_0^1 \theta_1(x)\,\mathrm{d}x.$$

Then by $(3)_4$ we have

$$\int_0^1 \overline{\theta}(x,t)\,\mathrm{d}x = 0,\ \forall t > 0.$$

In this case, Poincaré's inequality is applicable for $\overline{\theta}$, furthermore, $(\varphi,\psi,\omega,\overline{\theta},z)$ satisfies the same equations and boundary conditions. In what follows, we will work with $\overline{\theta}$. For convenience, we write θ instead of $\overline{\theta}$.

We will assume that

$$\mu_1 > |\mu_2|\,. \tag{5}$$

and show the well-posedness of the problem and that this condition is sufficient to prove the uniform decay of the solution energy.

THEOREM 2.1 (Lumer-Philips). *Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be a linear operator with dense domain $D(\mathcal{A})$ in a Hilbert space $\mathcal{H}$. Then $\mathcal{A}$ is maximal monotone if and only if $-\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$* (see [8, 27]).

3. WELL-POSEDNESS OF THE PROBLEM

In this section, we prove the well-posedness of problem (3)–(4) by using the Lumer-Philips theorem.

From now on, we let

$$U = (\varphi,\varphi_t,3\omega-\psi,3\omega_t-\psi_t,\omega,\omega_t,\theta,\theta_t,z)^T\,,$$

then (3) and (4) can be written as an evolutionary equation

$$\begin{cases} \dfrac{\mathrm{dU}(t)}{\mathrm{d}t} = \mathcal{A}U(t)\,, \quad t>0,\\ U(0) = U_0 = (\varphi_0,\varphi_1,3\omega_0-\psi_0,3\omega_1-\psi_1,\omega_0,\omega_1,\theta_0,\theta_1,f_0)^T\,, \end{cases} \tag{6}$$

where $\mathcal{A}$ is a linear operator defined by

$$\mathcal{A}\begin{pmatrix} \varphi\\ \varphi_t\\ 3\omega-\psi\\ 3\omega_t-\psi_t\\ \omega\\ \omega_t\\ \theta\\ \theta_t\\ z \end{pmatrix} = \begin{pmatrix} \varphi_t\\ -\frac{G}{\rho_1}(\psi-\varphi_x)_x - \frac{\mu_1}{\rho_1}\varphi_t(x,t) - \frac{\mu_2}{\rho_1}z(x,1,t)\\ 3\omega_t-\psi_t\\ \frac{G}{\rho_2}(\psi-\varphi_x) + \frac{D}{\rho_2}(3\omega-\psi)_{xx} - \frac{\sigma}{\rho_2}\theta_{tx}\\ \omega_t\\ -\frac{G}{\rho_2}(\psi-\varphi_x) - \frac{4\gamma}{3\rho_2}\omega - \frac{4\beta}{3\rho_2}\omega_t + \frac{D}{\rho_2}\omega_{xx}\\ \theta_t\\ \frac{\delta}{\rho_3}\theta_{xx} - \frac{\sigma}{\rho_3}(3\omega-\psi)_{tx} + \frac{k}{\rho_3}\theta_{txx}\\ -\frac{1}{\tau}z_\rho \end{pmatrix}.$$

We consider the following spaces:

$$\begin{aligned} L_*^2(0,1) &= \left\{ w \in L^2(0,1) : \int_0^1 w(s)\,\mathrm{d}s = 0 \right\}, \\ H_*^1(0,1) &= H^1(0,1) \cap L_*^2(0,1), \\ H_*^2(0,1) &= \left\{ w \in H^2(0,1) : w_x(0) = w_x(1) = 0 \right\}, \end{aligned}$$

and the energy space:

$$\begin{aligned} \mathcal{H} &= H_*^1(0,1) \times L_*^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \\ &\quad \times L^2(0,1) \times H_*^1(0,1) \times L_*^2(0,1) \times L^2\left((0,1), L^2(0,1)\right). \end{aligned}$$

The inner product on Hilbert space $\mathcal{H}$ is defined by

$$\begin{aligned} \left\langle U, \widetilde{U} \right\rangle_{\mathcal{H}} &= \rho_1 \int_0^1 \varphi_t \tilde{\varphi}_t \mathrm{d}x + G \int_0^1 (\psi - \varphi_x)\left(\tilde{\psi} - \tilde{\varphi}_x\right) \mathrm{d}x + 4\gamma \int_0^1 \omega \tilde{\omega} \mathrm{d}x \\ &\quad + \rho_2 \int_0^1 (3\omega - \psi)_t \left(3\tilde{\omega} - \tilde{\psi}\right)_t \mathrm{d}x + 3\rho_2 \int_0^1 \omega_t \tilde{\omega}_t \mathrm{d}x \\ &\quad + D \int_0^1 (3\omega - \psi)_x \left(3\tilde{\omega} - \tilde{\psi}\right)_x \mathrm{d}x + 3D \int_0^1 \omega_x \tilde{\omega}_x \mathrm{d}x \\ &\quad + \rho_3 \int_0^1 \theta_t \tilde{\theta}_t \mathrm{d}x + \delta \int_0^1 \theta_x \tilde{\theta}_x \mathrm{d}x + \lambda \int_0^1 \int_0^1 z \widetilde{z} \mathrm{d}\rho \mathrm{d}x, \end{aligned}$$

where λ is the positive constant satisfying

$$\left\{ \begin{array}{lll} \tau|\mu_2| < \lambda < \tau(2\mu_1 - |\mu_2|), & \text{if} & |\mu_2| < \mu_1, \\ \lambda = \tau\mu_1, & \text{if} & |\mu_2| = \mu_1. \end{array} \right. \tag{7}$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \left\{ \begin{array}{c} U \in \mathcal{H} \mid \varphi, \theta \in H_*^2(0,1) \cap H_*^1(0,1), \\ \omega, \psi \in H^2(0,1) \cap H_0^1(0,1),\ \psi_t, \omega_t \in H_0^1(0,1), \\ \varphi_t, \theta_t \in H_*^1(0,1),\quad \delta\theta + k\theta_t \in H_*^2(0,1), \\ z, z_\rho \in L^2\left((0,1), L^2(0,1)\right),\quad z(x,0) = \varphi_t(x) \end{array} \right\}, \tag{8}$$

and it is dense in $\mathcal{H}$.

The well-posedness of problem (6) is ensured by

Theorem 3.1. *Assume that $U_0 \in \mathcal{H}$ and (5) holds. Then there exists a unique solution $U \in C(\mathbb{R}^+; \mathcal{H})$ of problem (6). Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C\left(\mathbb{R}^+; D(\mathcal{A}) \cap C^1\left(\mathbb{R}^+; \mathcal{H}\right)\right).$$

Proof. To obtain the above result, we need to prove that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is a maximal monotone operator. First, we prove that $\mathcal{A}$ is dissipative.

For any $U \in D(\mathcal{A})$, by using the inner product and integration by parts, we can imply that

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= -4\beta \int_0^1 \omega_t^2 \mathrm{d}x - k \int_0^1 \theta_{tx}^2 \mathrm{d}x - \mu_1 \int_0^1 \varphi_t^2 \mathrm{d}x \\ &- \mu_2 \int_0^1 \varphi_t z(x,1,t)\,\mathrm{d}x - \frac{\lambda}{\tau} \int_0^1 \int_0^1 z z_\rho (x,\rho,t)\,\mathrm{d}\rho \mathrm{d}x. \end{aligned} \tag{9}$$

By using Young's inequality, the fourth term on the right-hand side of Equation (9) gives

$$-\mu_2 \int_0^1 \varphi_t z(x,1,t)\,\mathrm{d}x \leq \frac{|\mu_2|}{2} \int_0^1 \varphi_t^2 \mathrm{d}x + \frac{|\mu_2|}{2} \int_0^1 z^2(x,1,t)\,\mathrm{d}x.$$

Also, using integration by parts and the fact that $z(x,0) = \varphi_t(x)$, the last term in the right-hand side of (9) gives

$$\begin{aligned} -\frac{\lambda}{\tau} \int_0^1 \int_0^1 z z_\rho (x,\rho,t)\,\mathrm{d}\rho \mathrm{d}x &= -\frac{\lambda}{2\tau} \int_0^1 \int_0^1 \frac{\partial}{\partial \rho} z^2(x,\rho,t)\,\mathrm{d}\rho \mathrm{d}x \\ &= \frac{\lambda}{2\tau} \int_0^1 \left(\varphi_t^2 - z^2(x,1,t)\right) \mathrm{d}x. \end{aligned}$$

Consequently, (9) yields

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathcal{H}} &\leq -4\beta \int_0^1 \omega_t^2 \mathrm{d}x - k \int_0^1 \theta_{tx}^2 \mathrm{d}x - \left(\mu_1 - \frac{\lambda}{2\tau} - \frac{|\mu_2|}{2}\right) \int_0^1 \varphi_t^2 \mathrm{d}x \\ &- \left(\frac{\lambda}{2\tau} - \frac{|\mu_2|}{2}\right) \int_0^1 z^2(x,1,t)\,\mathrm{d}x. \end{aligned}$$

Keeping in mind condition (7), we observe that $\mu_1 - \frac{\lambda}{2\tau} - \frac{|\mu_2|}{2} \geq 0$, $\frac{\lambda}{2\tau} - \frac{|\mu_2|}{2} \geq 0$. Consequently, $\mathcal{A}$ is a dissipative operator. Next, we prove that the operator $Id - \mathcal{A}$ is surjective. Given $F = (f_1, ..., f_9)^T \in \mathcal{H}$, we prove that there exists a unique $U = (\varphi, \varphi_t, 3\omega - \psi, (3\omega - \psi)_t, \omega, \omega_t, \theta, \theta_t, z) \in D(\mathcal{A})$ such that

$$(Id - \mathcal{A})\,U = F \tag{10}$$

which is equivalent to

$$\begin{cases} \varphi - \varphi_t = f_1, \\ \rho_1 \varphi_t + G(\psi - \varphi_x)_x + \mu_1 \varphi_t + \mu_2 z(x,1,t) = \rho_1 f_2, \\ (3\omega - \psi) - (3\omega - \psi)_t = f_3, \\ \rho_2 (3\omega - \psi)_t - G(\psi - \varphi_x) - D(3\omega - \psi)_{xx} + \sigma \theta_{tx} = \rho_2 f_4, \\ \omega - \omega_t = f_5, \\ 3\rho_2 \omega_t + 3G(\psi - \varphi_x) + 4\gamma\omega + 4\beta\omega_t - 3D\omega_{xx} = 3\rho_2 f_6, \\ \theta - \theta_t = f_7, \\ \rho_3 \theta_t - \delta\theta_{xx} + \sigma(3\omega - \psi)_{tx} - k\theta_{txx} = \rho_3 f_8, \\ \tau z(x,\rho,t) + z_\rho(x,\rho,t) = \tau f_9. \end{cases} \tag{11}$$

The last equation in (11) and the fact that $z(x,0)=\varphi_t(x,t)$, we get

$$z(x,\rho)=\varphi(x)e^{-\tau\rho}-e^{-\tau\rho}f_1+\tau e^{-\tau\rho}\int_0^\rho e^{\tau s}f_9(x,s)\,\mathrm{d}s, \tag{12}$$

$(11)_1$, $(11)_3$, $(11)_5$ and $(11)_7$ give

$$\begin{cases}\varphi_t=\varphi-f_1,\\ (3\omega-\psi)_t=(3\omega-\psi)-f_3,\\ \omega_t=\omega-f_5,\\ \theta_t=\theta-f_7.\end{cases} \tag{13}$$

Inserting (13) into $(11)_2$, $(11)_4$, $(11)_6$ and $(11)_8$, we get

$$\begin{cases}\left(\rho_1+\mu_1+\mu_2e^{-\tau}\right)\varphi-G\varphi_{xx}-G(3\omega-\psi)_x+3G\omega_x\\ =\left(\rho_1+\mu_1+\mu_2e^{-\tau}\right)f_1+\rho_1f_2-\mu_2\tau e^{-\tau}\int_0^1 e^{\tau s}f_9\mathrm{d}s,\\ (\rho_2+G)(3\omega-\psi)+G\varphi_x-3G\omega-D(3\omega-\psi)_{xx}+\sigma\theta_x\\ =\rho_2(f_3+f_4)+\sigma\partial_xf_7,\\ (3\rho_2+4\beta+4\gamma+9G)\omega-3G(3\omega-\psi)-3G\varphi_x-3D\omega_{xx}\\ =(3\rho_2+4\beta)f_5+3\rho_2f_6,\\ \rho_3\theta-(\delta+k)\theta_{xx}+\sigma(3\omega-\psi)_x=\rho_3(f_7+f_8)+\sigma\partial_xf_3-k\partial_{xx}f_7,\end{cases} \tag{14}$$

Multiplying the forth equation of system (14) by $\tilde{\varphi}$, $\left(3\tilde{\omega}-\tilde{\psi}\right)$, $\tilde{\omega}$ and $\tilde{\theta}$ respectively, and integrating over $(0,1)$, we arrive

$$\begin{cases}\left(\rho_1+\mu_1+\mu_2e^{-\tau}\right)\int_0^1\varphi\tilde{\varphi}\mathrm{d}x+G\int_0^1\varphi_x\tilde{\varphi}_x\mathrm{d}x-G\int_0^1(3\omega-\psi)_x\tilde{\varphi}\mathrm{d}x\\ +3G\int_0^1\omega_x\tilde{\varphi}\mathrm{d}x=\left(\rho_1+\mu_1+\mu_2e^{-\tau}\right)\int_0^1f_1\tilde{\varphi}\mathrm{d}x+\rho_1\int_0^1f_2\tilde{\varphi}\mathrm{d}x\\ \qquad -\mu_2\tau e^{-\tau}\int_0^1\int_0^1e^{\tau s}f_9\tilde{\varphi}\mathrm{d}s\mathrm{d}x,\\ \\ (\rho_2+G)\int_0^1(3\omega-\psi)\left(3\tilde{\omega}-\tilde{\psi}\right)\mathrm{d}x+D\int_0^1(3\omega-\psi)_x\left(3\tilde{\omega}-\tilde{\psi}\right)_x\mathrm{d}x\\ +G\int_0^1\varphi_x\left(3\tilde{\omega}-\tilde{\psi}\right)\mathrm{d}x-3G\int_0^1\omega\left(3\tilde{\omega}-\tilde{\psi}\right)\mathrm{d}x+\sigma\int_0^1\theta_x\left(3\tilde{\omega}-\tilde{\psi}\right)\mathrm{d}x\\ =\rho_2\int_0^1(f_3+f_4)\left(3\tilde{\omega}-\tilde{\psi}\right)\mathrm{d}x+\sigma\int_0^1\partial_xf_7\left(3\tilde{\omega}-\tilde{\psi}\right)\mathrm{d}x,\\ \\ (3\rho_2+4\beta+4\gamma+9G)\int_0^1\omega\tilde{\omega}\mathrm{d}x-3G\int_0^1(3\omega-\psi)\tilde{\omega}\mathrm{d}x-3G\int_0^1\varphi_x\tilde{\omega}\mathrm{d}x\\ +3D\int_0^1\omega_x\tilde{\omega}_x\mathrm{d}x=(3\rho_2+4\beta)\int_0^1f_5\tilde{\omega}\mathrm{d}x+3\rho_2\int_0^1f_6\tilde{\omega}\mathrm{d}x,\\ \\ \rho_3\int_0^1\theta\tilde{\theta}\mathrm{d}x+(\delta+k)\int_0^1\theta_x\tilde{\theta}_x\mathrm{d}x+\sigma\int_0^1(3\omega-\psi)_x\tilde{\theta}\mathrm{d}x\\ =\rho_3\int_0^1(f_7+f_8)\tilde{\theta}\mathrm{d}x+\sigma\int_0^1\partial_xf_3\tilde{\theta}\mathrm{d}x-k\int_0^1\partial_{xx}f_7\tilde{\theta}\mathrm{d}x,\end{cases} \tag{15}$$

The sum of the equations in (15) gives the following variational formulation:

$$B\left((\varphi,3\omega-\psi,\omega,\theta)^T,\left(\tilde{\varphi},3\tilde{\omega}-\tilde{\psi},\tilde{\omega},\tilde{\theta}\right)^T\right)=L\left(\left(\tilde{\varphi},3\tilde{\omega}-\tilde{\psi},\tilde{\omega},\tilde{\theta}\right)^T\right),$$
(16)
$$\forall\left(\tilde{\varphi},3\tilde{\omega}-\tilde{\psi},\tilde{\omega},\tilde{\theta}\right)^T\in H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_*^1(0,1),$$

where $B:\left(H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_*^1(0,1)\right)^2\longrightarrow\mathbb{R}$ is the bilinear form defined by

$$\begin{aligned}
&B\left((\varphi,3\omega-\psi,\omega,\theta)^T,\left(\tilde{\varphi},3\tilde{\omega}-\tilde{\psi},\tilde{\omega},\tilde{\theta}\right)^T\right)\\
&=\int_0^1\Big[G(\psi-\varphi_x)\left(\tilde{\psi}-\tilde{\varphi}_x\right)+\left(\rho_1+\mu_1+\mu_2e^{-\tau}\right)\varphi\tilde{\varphi}+3D\omega_x\tilde{\omega}_x\\
&\quad+\rho_2(3\omega-\psi)\left(3\tilde{\omega}-\tilde{\psi}\right)+D(3\omega-\psi)_x\left(3\tilde{\omega}-\tilde{\psi}\right)_x+\rho_3\theta\tilde{\theta}\\
&\quad+(3\rho_2+4\beta+4\gamma)\,\omega\tilde{\omega}+\sigma(3\omega-\psi)_x\tilde{\theta}+(\delta+k)\,\theta_x\tilde{\theta}_x\\
&\quad+\sigma\theta_x\left(3\tilde{\omega}-\tilde{\psi}\right)\Big]\,\mathrm{d}x,
\end{aligned}$$

and $L:\left(H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_*^1(0,1)\right)\longrightarrow\mathbb{R}$ is the linear functional given by

$$\begin{aligned}
&L\left(\left(\tilde{\varphi},3\tilde{\omega}-\tilde{\psi},\tilde{\omega},\tilde{\theta}\right)^T\right)\\
&=\int_0^1\Big[\left(\rho_1+\mu_1+\mu_2e^{-\tau}\right)f_1\tilde{\varphi}+\rho_1f_2\tilde{\varphi}-\mu_2\tau e^{-\tau}\int_0^1e^{\tau s}f_9\tilde{\varphi}\,\mathrm{d}s\\
&\quad+\rho_2(f_3+f_4)\left(3\tilde{\omega}-\tilde{\psi}\right)+\sigma(\partial_xf_7)\left(3\tilde{\omega}-\tilde{\psi}\right)+(3\rho_2+4\beta)\,f_5\tilde{\omega}\\
&\quad+3\rho_2f_6\tilde{\omega}+\rho_3(f_7+f_8)\,\tilde{\theta}+\sigma(\partial_xf_3)\,\tilde{\theta}+k(\partial_xf_7)\,\tilde{\theta}_x\Big]\,\mathrm{d}x.
\end{aligned}$$

Now, for

$$V=H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_*^1(0,1),$$

equipped with the norm

$$\begin{aligned}
\|\varphi,3\omega-\psi,\omega,\theta\|_V^2 &= \|-\varphi_x-(3\omega-\psi)+3\omega\|_2^2+\|\varphi\|_2^2+\|(3\omega-\psi)_x\|_2^2\\
&\quad+\|\omega_x\|_2^2+\|\theta\|_2^2+\|\theta_x\|_2^2,
\end{aligned}$$

It is clear that B and L are bounded. Furthermore, using integration by parts, we have

$$\begin{aligned}
&B\left((\varphi,3\omega-\psi,\omega,\theta)^T,(\varphi,3\omega-\psi,\omega,\theta)^T\right)\\
=&\int_0^1\Big[G(\psi-\varphi_x)^2+\left(\rho_1+\mu_1+\mu_2e^{-\tau}\right)\varphi^2+\rho_2(3\omega-\psi)^2+D(3\omega-\psi)_x^2\\
&+(3\rho_2+4\beta+4\gamma)\,\omega^2+3D\omega_x^2+\rho_3\theta^2+(\delta+k)\,\theta_x^2\Big]\,\mathrm{d}x
\end{aligned}$$

$$\geq \quad m\left\|\varphi, 3\omega-\psi, \omega, \theta\right\|_V^2,$$

for some m Thus, B is coercive.

Hence, we assert that $B(\cdot,\cdot)$ is a bilinear continuous coercive form on $V\times V$, and $L(\cdot)$ is a linear continuous form on V. Applying the Lax-Milgram theorem [27], we obtain that (16) has a unique solution

$$(\varphi, 3\omega-\psi, \omega, \theta) \in H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_*^1(0,1),$$

The substitution of $\varphi, 3\omega-\psi, \omega$ and θ into (13) yields

$$(\varphi_t, 3\omega_t-\psi_t, \omega_t, \theta_t) \in H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_*^1(0,1),$$

Next, it remains to show that

$$\begin{aligned} \varphi &\in \left(H_*^2(0,1)\cap H_*^1(0,1)\right), \ (3\omega-\psi), \omega \in \left(H^2(0,1)\cap H_0^1(0,1)\right), \\ \theta &\in \left(H_*^2(0,1)\cap H_*^1(0,1)\right). \end{aligned}$$

Furthermore, if $\left(3\tilde{\omega}-\tilde{\psi}, \tilde{\omega}, \tilde{\theta}\right) = (0,0,0) \in H_0^1(0,1)\times H_0^1(0,1)\times H_*^1(0,1)$, then (16) reduces to

$$\begin{aligned} &B\left((\varphi, 3\omega-\psi, \omega, \theta)^T, (\tilde{\varphi}, 0,0,0)^T\right) \\ &\qquad = \int_0^1 \Big[-G(3\omega-\psi)_x\tilde{\varphi} - G\varphi_{xx}\tilde{\varphi} + 3G\omega_x\tilde{\varphi} \\ &\qquad\qquad + \left(\rho_1+\mu_1+\mu_2 e^{-\tau}\right)\varphi\tilde{\varphi}\Big]\mathrm{d}x \qquad (17) \\ &= \int_0^1 \left[\left(\rho_1+\mu_1+\mu_2 e^{-\tau}\right) f_1\tilde{\varphi} + \rho_1 f_2\tilde{\varphi} - \mu_2\tau e^{-\tau}\int_0^1 e^{\tau s} f_9\tilde{\varphi}\mathrm{d}s\right]\mathrm{d}x, \end{aligned}$$

for all $\forall\tilde{\varphi}\in H_*^1(0,1)$,which implies

$$\begin{aligned} G\varphi_{xx} &= \left(\rho_1+\mu_1+\mu_2 e^{-\tau}\right)\varphi + 3G\omega_x - G(3\omega-\psi)_x \\ &\quad - \left(\rho_1+\mu_1+\mu_2 e^{-\tau}\right) f_1 - \rho_1 f_2 + \mu_2\tau e^{-\tau}\int_0^1 e^{\tau s} f_9\mathrm{d}s. \end{aligned} \qquad (18)$$

Consequently, by the regularity theory for the linear elliptic equations, it follows that

$$\varphi \in H^2(0,1)\cap H_*^1(0,1).$$

Moreover, (17) is also true for any $\phi\in C^1[0,1]\subset H_*^1(0,1)$. Hence, we have

$$\begin{aligned} &\int_0^1 G\varphi_x\phi_x\mathrm{d}x + \int_0^1 \left[\left(\rho_1+\mu_1+\mu_2 e^{-\tau}\right)\varphi - G(3\omega-\psi)_x + 3G\omega_x \right. \\ &\left. - \left(\rho_1+\mu_1+\mu_2 e^{-\tau}\right) f_1 - \rho_1 f_2 + \mu_2\tau e^{-\tau}\int_0^1 e^{\tau s} f_9\mathrm{d}s\right]\phi\mathrm{d}x = 0, \end{aligned}$$

for all $\phi\in C^1[0,1]$. Thus, using integration by parts and bearing in mind (18), we obtain

$$\varphi_x(1)\phi(1) - \varphi_x(0)\phi(0) = 0, \forall\phi\in C^1[0,1].$$

Therefore, $\varphi_x(0) = \varphi_x(1) = 0$.Consequently, we obtain

$$\varphi \in H^2_*(0,1) \cap H^1_*(0,1).$$

Similarly, we obtain $(3\omega - \psi)$, $\omega \in H^2(0,1) \cap H^1_0(0,1)$. Also, if we take $\left(\tilde{\varphi}, 3\tilde{\omega} - \tilde{\psi}, \tilde{\omega}\right) = (0,0,0) \in H^1_*(0,1) \times H^1_0(0,1) \times H^1_0(0,1)$ in (16), then using $(11)_3$ and $(11)_7$, we get$\delta\theta_{xx} + k\theta_{txx} = \rho_3\theta_t - \rho_3 f_8 + \sigma(3\omega - \psi)_{tx}$, and we conclude that$\delta\theta + k\theta_t \in H^2(0,1)$. Furthermore, it is obvious from

$$\delta\theta_x + k\theta_{tx} = \rho_3 \int_0^x \theta_t \mathrm{d}x - \rho_3 \int_0^x f_8 \mathrm{d}x + \sigma(3\omega - \psi)_t$$

that $(\delta\theta_x + k\theta_{tx})(0) = (\delta\theta_x + k\theta_{tx})(1) = 0$, then, we get $\delta\theta + k\theta_t \in H^2_*(0,1)$. Finally, it follows, from (12), that

$$z(x,0) = \varphi_t(x) \quad \text{and} \quad z, z_\rho \in L^2\left((0,1), L^2(0,1)\right).$$

Hence, there exists a unique $U \in D(\mathcal{A})$ such that (16) is satisfied, the operator $Id - \mathcal{A}$ is surjective. Moreover, it is easy to see that $D(\mathcal{A})$ is dense in $\mathcal{H}$. Consequently, the result of Theorem 3.1 follows from Lumer-Philips theorem. □

4. EXPONENTIAL STABILITY

In this section, we show that, under the assumption $|\mu_2| \leq \mu_1$ and $\dfrac{\rho_1}{G} = \dfrac{\rho_2}{D}$ for the solution of problem (3)-(4) decays exponentially to the study state. To achieve our goal we use the perturbed energy method to produce a suitable Lyapunov functional. We define the energy functional $E(t)$ as

$$\begin{aligned} E(t) := \frac{1}{2}\int_0^1 \Big[&\rho_1\varphi_t^2 + \rho_2(3\omega - \psi)_t^2 + 3\rho_2\omega_t^2 + \rho_3\theta_t^2 + G(\psi - \varphi_x)^2 \\ &+ D(3\omega - \psi)_x^2 + 4\gamma\omega^2 + 3D\omega_x^2 + \delta\theta_x^2 + \frac{\lambda\tau}{2}\int_0^1 z^2(x,\rho,t)\,\mathrm{d}s\Big]\,\mathrm{d}x. \end{aligned} \tag{19}$$

If the wave speeds are equal, we have the following exponentially stable result.

THEOREM 4.1. *Assume that $\frac{\rho_1}{G} = \frac{\rho_2}{D}$ and (5) holds. Let $U^0 \in \mathcal{H}$, then there exists positive constants c_0, c_1 such that the energy $E(t)$ associated with problem (3)-(4) satisfies $E(t) \leq c_0 e^{-c_1 t}$, $t \geq 0$.*

To prove our this result, we will state and prove some useful lemmas in advance.

LEMMA 4.2. *Let $(\varphi, \psi, \omega, \theta, z)$ be the solution of (3)–(4) with (7). Then the energy functional satisfies*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq& -4\beta\int_0^1 \omega_t^2 \mathrm{d}x - k\int_0^1 \theta_{tx}^2 \mathrm{d}x \\ &- C_1\int_0^1 \varphi_t^2 \mathrm{d}x - C_2\int_0^1 z^2(x,1,t)\,\mathrm{d}x \leq 0, \end{aligned} \tag{20}$$

where $C_1 = \mu_1 - \dfrac{\lambda}{2\tau} - \frac{|\mu_2|}{2} \geq 0,\ C_2 = \dfrac{\lambda}{2\tau} - \frac{|\mu_2|}{2} \geq 0.$

Proof. First, multiplying $(3)_1$ by φ_t, integrating over $(0,1)$, using integration by parts and the boundary conditions in (4), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\rho_1 \int_0^1 \varphi_t^2 \mathrm{d}x\right) - G\int_0^1 (\psi - \varphi_x)\,\varphi_{tx}\mathrm{d}x \\ &= -\mu_1 \int_0^1 \varphi_t^2 \mathrm{d}x - \mu_2 \int_0^1 \varphi_t z\,(x,1,t)\,\mathrm{d}x, \end{aligned} \tag{21}$$

note that

$$\begin{aligned} G\int_0^1 (\psi - \varphi_x)\,\varphi_{tx}\mathrm{d}x &= -G\int_0^1 (\psi - \varphi_x)\,(\psi - \varphi_x - \psi)_t\,\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{1}{2}G\int_0^1 (\psi - \varphi_x)^2\,\mathrm{d}x\right) + G\int_0^1 (\psi - \varphi_x)\,\psi_t \mathrm{d}x. \end{aligned}$$

Hence, equation (21) becomes

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_1 \int_0^1 \varphi_t^2 \mathrm{d}x + G\int_0^1 (\psi - \varphi_x)^2\,\mathrm{d}x\right) \\ &= G\int_0^1 (\psi - \varphi_x)\,\psi_t \mathrm{d}x - \mu_1 \int_0^1 \varphi_t^2 \mathrm{d}x - \mu_2 \int_0^1 \varphi_t z\,(x,1,t)\,\mathrm{d}x, \end{aligned} \tag{22}$$

Similarly, multiplying $(3)_2$, $(3)_3$, $(3)_4$ by $(3\omega - \psi)_t$, $3\omega_t$, θ_t and integrating over $(0,1)$, using integration by parts and the boundary conditions in (4), we can get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_2 \int_0^1 (3\omega - \psi)_t^2\,\mathrm{d}x + D\int_0^1 (3\omega - \psi)_x^2\,\mathrm{d}x\right) \\ &= G\int_0^1 (\psi - \varphi_x)\,(3\omega - \psi)_t\,dx - \sigma\int_0^1 \theta_{tx}\,(3\omega - \psi)_t\,\mathrm{d}x, \end{aligned} \tag{23}$$

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(3\rho_2 \int_0^1 \omega_t^2 \mathrm{d}x + 4\gamma\int_0^1 \omega^2 \mathrm{d}x + 3D\int_0^1 \omega_x^2 \mathrm{d}x\right) \\ &= -3G\int_0^1 (\psi - \varphi_x)\,\omega_t \mathrm{d}x - 4\beta\int_0^1 \omega_t^2 \mathrm{d}x, \end{aligned} \tag{24}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_3 \int_0^1 \theta_t^2 \mathrm{d}x + \delta\int_0^1 \theta_x^2 \mathrm{d}x\right) = \sigma\int_0^1 (3\omega - \psi)_t\,\theta_{tx}\mathrm{d}x - k\int_0^1 \theta_{tx}^2 \mathrm{d}x. \tag{25}$$

Now, multiplying $(3)_5$, by $\dfrac{\lambda}{\tau}z$ and integrating over $(0,1)\times(0,1)$, using integration by parts and the boundary conditions in (4), we can get

$$\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^1 z^2\,(x,\rho,t)\,\mathrm{d}\rho\mathrm{d}x = -\frac{\lambda}{2\tau}\int_0^1 \left(z^2\,(x,1,t) - \varphi_t^2\right)\mathrm{d}x. \tag{26}$$

Finally, adding (22), (23), (24), (25) and (26), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho_1\int_0^1 \varphi_t^2\mathrm{d}x + G\int_0^1 (\psi-\varphi_x)^2\,\mathrm{d}x + D\int_0^1 (3\omega-\psi)_x^2\,\mathrm{d}x\right. \\ &+\rho_2\int_0^1 (3\omega-\psi)_t^2\,\mathrm{d}x + \rho_3\int_0^1 \theta_t^2\mathrm{d}x + \delta\int_0^1 \theta_x^2\mathrm{d}x + 4\gamma\int_0^1 \omega^2\mathrm{d}x \\ &\left.+3\rho_2\int_0^1 \omega_t^2\mathrm{d}x + 3D\int_0^1 \omega_x^2\mathrm{d}x\right] + \frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^1 z^2(x,\rho,t)\,\mathrm{d}\rho\mathrm{d}x \\ = \;& -4\beta\int_0^1 \omega_t^2\mathrm{d}x - k\int_0^1 \theta_{tx}^2\mathrm{d}x - \mu_1\int_0^1 \varphi_t^2\mathrm{d}x - \mu_2\int_0^1 \varphi_t z(x,1,t)\,\mathrm{d}x \\ &-\frac{\lambda}{2\tau}\int_0^1 z^2(x,1,t)\,\mathrm{d}x + \frac{\lambda}{2\tau}\int_0^1 \varphi_t^2\mathrm{d}x. \end{aligned}$$

Meanwhile, using Young's inequality, we have

$$-\mu_2\int_0^1 \varphi_t z(x,1,t)\,\mathrm{d}x \le \frac{|\mu_2|}{2}\int_0^1 \varphi_t^2\mathrm{d}x + \frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)\,\mathrm{d}x.$$

Hence,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E(t) \;\le\; & -4\beta\int_0^1 \omega_t^2\mathrm{d}x - k\int_0^1 \theta_{tx}^2\mathrm{d}x - \left(\mu_1 - \frac{\lambda}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^1 \varphi_t^2\mathrm{d}x \\ & - \left(\frac{\lambda}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^1 z^2(x,1,t)\,\mathrm{d}x, \end{aligned}$$

using (7), we obtain the result. □

Next, in order to construct a Lyapunov functional equivalent to the energy, we will prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy.

LEMMA 4.3. *Let $(\varphi,\psi,\omega,\theta,z)$ be the solution of (3)-(4). Then the functional*

$$I_1(t) := \rho_2\int_0^1 (3\omega-\psi)(3\omega-\psi)_t\,\mathrm{d}x \tag{27}$$

satisfies the estimate

$$\begin{aligned} I_1'(t) \le & -\frac{D}{2}\int_0^1 (3\omega-\psi)_x^2\,\mathrm{d}x + \rho_2\int_0^1 (3\omega-\psi)_t^2\,\mathrm{d}x \\ & + \frac{\sigma^2}{D}\int_0^1 (\psi-\varphi_x)^2\,\mathrm{d}x + \frac{G^2}{D}\int_0^1 \theta_t^2\mathrm{d}x, \end{aligned} \tag{28}$$

Proof. Taking the derivative of $I_1(t)$ with respect to t, using $(3)_2$ and integrating by parts, we get

$$I_1'(t) \;=\; \rho_2\int_0^1 (3\omega-\psi)_t^2\,\mathrm{d}x + G\int_0^1 (\psi-\varphi_x)(3\omega-\psi)\,\mathrm{d}x$$

$$-D\int_0^1 (3\omega-\psi)_x^2\,\mathrm{d}x+\sigma\int_0^1 \theta_t\,(3\omega-\psi)_x\,\mathrm{d}x.$$

Using Young's and Poincaré inequalities, we arrive at (28). □

LEMMA 4.4. *Let* $(\varphi,\psi,\omega,\theta,z)$ *be the solution of* (3)–(4). *Then the functional*

$$I_2(t):=\rho_2\int_0^1 \omega\omega_t\mathrm{d}x \tag{29}$$

satisfies the estimate

$$\begin{aligned} I_2'(t)\le -\frac{2}{3}\gamma\int_0^1\omega^2\mathrm{d}x-D\int_0^1\omega_x^2\mathrm{d}x+C_3\int_0^1\omega_t^2\mathrm{d}x+&\\ \frac{3G^2}{4\gamma}\int_0^1(\psi-\varphi_x)^2\,\mathrm{d}x,&\end{aligned}\tag{30}$$

where $C_3=\rho_2+\dfrac{4\beta^2}{3\gamma}$.

Proof. By differentiating (29) with respect to t, using $(3)_3$ and integrating by parts, we obtain

$$\begin{aligned} I_2'(t) &= \rho_2\int_0^1\omega_t^2\mathrm{d}x-G\int_0^1(\psi-\varphi_x)\,\omega\mathrm{d}x-\frac{4}{3}\gamma\int_0^1\omega^2\mathrm{d}x-\frac{4}{3}\beta\int_0^1\omega_t\omega\mathrm{d}x\\ &\quad -D\int_0^1\omega_x^2\mathrm{d}x.\end{aligned}$$

Using Young's inequality, we establish (30). □

LEMMA 4.5. *Let* $(\varphi,\psi,\omega,\theta,z)$ *be the solution of* (3)-(4). *Then the functional*

$$I_3(t):=\rho_2\rho_3\int_0^1(3\omega-\psi)_t\int_0^x\theta_t(y,t)\,\mathrm{d}y\mathrm{d}x-\rho_2\delta\int_0^1\theta_x\,(3\omega-\psi)\,\mathrm{d}x \tag{31}$$

satisfies the estimate

$$\begin{aligned} I_3'(t)\le -\frac{\rho_2\sigma}{2}\int_0^1(3\omega-\psi)_t^2\,\mathrm{d}x+\varepsilon_1\int_0^1(\psi-\varphi_x)^2\,\mathrm{d}x&\\ +C_4(\varepsilon_1)\int_0^1\theta_{tx}^2\mathrm{d}x+\varepsilon_1\int_0^1(3\omega-\psi)_x^2\,\mathrm{d}x,&\end{aligned}\tag{32}$$

for any $\varepsilon_1>0$, *where* $C_4(\varepsilon_1)=\sigma\rho_3+\dfrac{\rho_3^2G^2}{4\varepsilon_1}+\dfrac{\delta^2\rho_2^2}{2\varepsilon_1}+\dfrac{\rho_2k^2}{2\sigma}+\dfrac{D^2\rho_3^2}{2\varepsilon_1}$.

Proof. Taking the derivative of $I_3(t)$ with respect to t, using $(3)_2$, $(3)_4$ and integrating by parts, we get

$$\begin{aligned} I_3'(t)=\rho_3\int_0^1\left[G(\psi-\varphi_x)+D(3\omega-\psi)_{xx}-\sigma\theta_{tx}\right]\int_0^x\theta_t(y,t)\,\mathrm{d}y\mathrm{d}x&\\ +\rho_2\int_0^1(3\omega-\psi)_t\int_0^x\left[\delta\theta_{xx}+k\theta_{txx}-\sigma(3\omega-\psi)_{tx}\right]\mathrm{d}y\mathrm{d}x&\end{aligned}$$

$$-\delta\rho_2\int_0^1\theta_{xt}\left(3\omega-\psi\right)\mathrm{d}x-\delta\rho_2\int_0^1\theta_x\left(3\omega-\psi\right)_t\mathrm{d}x=\rho_3\int_0^1 G\psi\int_0^x\theta_t\left(y,t\right)\mathrm{d}y\mathrm{d}x$$
$$+\rho_2\int_0^1\left(3\omega-\psi\right)_t\left[\delta\theta_x+k\theta_{tx}-\sigma\left(3\omega-\psi\right)_t\right]\mathrm{d}x+\rho_3\int_0^1\Big[-G\varphi_x+D\left(3\omega-\psi\right)_{xx}$$
$$-\sigma\theta_{tx}\Big]\int_0^x\theta_t\left(y,t\right)\mathrm{d}y\mathrm{d}x-\delta\rho_2\int_0^1\theta_{xt}\left(3\omega-\psi\right)\mathrm{d}x-\delta\rho_2\int_0^1\theta_x\left(3\omega-\psi\right)_t\mathrm{d}x$$
$$=\rho_3\int_0^1 G\left(\psi-\varphi_x\right)\int_0^x\theta_t\left(y,t\right)\mathrm{d}y\mathrm{d}x-\delta\rho_2\int_0^1\theta_{xt}\left(3\omega-\psi\right)\mathrm{d}x$$
$$+\left[\rho_3\left(-G\varphi+D\left(3\omega-\psi\right)_x-\sigma\theta_t\right)\int_0^x\theta_t\left(y,t\right)\mathrm{d}y\right]_{x=0}^{x=1}+\sigma\rho_3\int_0^1\theta_t^2\mathrm{d}x$$
$$-\rho_2\sigma\int_0^1\left(3\omega-\psi\right)_t^2\mathrm{d}x+\rho_2k\int_0^1\left(3\omega-\psi\right)_t\theta_{tx}\mathrm{d}x-D\rho_3\int_0^1\theta_t\left(3\omega-\psi\right)_x\mathrm{d}x.$$

Note that $\int_0^1\theta_t\left(y,t\right)dy=\frac{d}{dt}\int_0^1\theta\left(y,t\right)dy=0$, then, by Young's and Poincaré inequalities with $\varepsilon_1>0$ to obtain (32). □

LEMMA 4.6. *Let $(\varphi,\psi,\omega,\theta,z)$ be the solution of problem (3)-(4). the functional*

$$I_4\left(t\right):=\int_0^1\left[\rho_3\theta_t\theta+\frac{k}{2}\theta_x^2+\sigma\left(3\omega-\psi\right)_x\theta\right]\mathrm{d}x \tag{33}$$

satisfies the estimate

$$I_4'\left(t\right)\le-\delta\int_0^1\theta_x^2\mathrm{d}x+C_5\left(\varepsilon_2\right)\int_0^1\theta_t^2\mathrm{d}x+\varepsilon_2\int_0^1\left(3\omega-\psi\right)_x^2\mathrm{d}x, \tag{34}$$

for any $\varepsilon_2>0$, where $C_5\left(\varepsilon_2\right)=\rho_3+\dfrac{\sigma^2}{4\varepsilon_2}$.

Proof. By differentiating I_4 with respect to t, using $(3)_4$ and integrating by parts, we obtain

$$\begin{aligned}I_4'\left(t\right)&=\int_0^1\rho_3\theta_{tt}\theta\mathrm{d}x+\int_0^1\rho_3\theta_t^2\mathrm{d}x+\int_0^1\frac{k}{2}\left(\theta_{xt}\theta_x+\theta_x\theta_{xt}\right)\mathrm{d}x\\&\quad+\int_0^1\sigma\left(3\omega-\psi\right)_{xt}\theta\mathrm{d}x+\int_0^1\sigma\left(3\omega-\psi\right)_x\theta_t\mathrm{d}x\\&=\int_0^1\left[\delta\theta_{xx}+k\theta_{txx}-\sigma\left(3\omega-\psi\right)_{tx}\right]\theta\mathrm{d}x+\int_0^1\rho_3\theta_t^2\mathrm{d}x\\&\quad-\int_0^1 k\theta_{xxt}\theta\mathrm{d}x+\int_0^1\sigma\left(3\omega-\psi\right)_{xt}\theta\mathrm{d}x+\int_0^1\sigma\left(3\omega-\psi\right)_x\theta_t\mathrm{d}x\\&=\int_0^1\delta\theta_{xx}\theta\mathrm{d}x+\int_0^1\rho_3\theta_t^2\mathrm{d}x+\int_0^1\sigma\left(3\omega-\psi\right)_x\theta_t\mathrm{d}x.\end{aligned}$$

Using Young's inequality with $\varepsilon_2>0$, we establish (34). □

Lemma 4.7. *Let $(\varphi,\psi,\omega,\theta,z)$ be the solution of* (3)–(4)*. Then the functional*

$$I_5(t) := \rho_2\int_0^1 (3\omega-\psi)_t(\varphi_x-\psi)\,\mathrm{d}x + \frac{D\rho_1}{G}\int_0^1 (3\omega-\psi)_x\varphi_t\mathrm{d}x \tag{35}$$

satisfies the estimate

$$\begin{aligned} I_5'(t) \le & -\frac{G}{2}\int_0^1 (\varphi_x-\psi)^2\,\mathrm{d}x + \frac{\sigma^2}{2G}\int_0^1 \theta_{tx}^2\mathrm{d}x \\ & + \frac{9\rho_2^2}{4\varepsilon_3}\int_0^1 \omega_t^2\mathrm{d}x + (\rho_2+\varepsilon_3)\int_0^1 (3\omega-\psi)_t^2\,\mathrm{d}x \\ & + \left(\frac{D\rho_1}{G}-\rho_2\right)\int_0^1 (3\omega-\psi)_{xt}\varphi_t\mathrm{d}x + \varepsilon_4\int_0^1 (3\omega-\psi)_x^2\,\mathrm{d}x \\ & + \frac{D^2\mu_1^2}{2G^2\varepsilon_4}\int_0^1 \varphi_t^2\mathrm{d}x + \frac{D^2\mu_2^2}{2G^2\varepsilon_4}\int_0^1 z^2(x,1,t)\,\mathrm{d}x, \end{aligned} \tag{36}$$

for any $\varepsilon_3,\varepsilon_4>0$.

Proof. By differentiating I_5 with respect to t, using $(3)_1$, $(3)_2$ and integrating by parts, we obtain

$$\begin{aligned} I_5'(t) = {} & \rho_2\int_0^1 (3\omega-\psi)_{tt}(\varphi_x-\psi)\,\mathrm{d}x + \rho_2\int_0^1 (3\omega-\psi)_t(\varphi_x-\psi)_t\,\mathrm{d}x \\ & + \frac{D\rho_1}{G}\int_0^1 (3\omega-\psi)_{xt}\varphi_t\mathrm{d}x + \frac{D\rho_1}{G}\int_0^1 (3\omega-\psi)_x\varphi_{tt}\mathrm{d}x \\ = {} & -\int_0^1 G(\varphi_x-\psi)^2\,\mathrm{d}x + \int_0^1 D(3\omega-\psi)_{xx}(\varphi_x-\psi)\,\mathrm{d}x \\ & - \int_0^1 \sigma\theta_{tx}(\varphi_x-\psi)\,\mathrm{d}x + \rho_2\int_0^1 (3\omega-\psi)_t(\varphi_x-\psi)_t\,\mathrm{d}x \\ & + \frac{D\rho_1}{G}\int_0^1 (3\omega-\psi)_{xt}\varphi_t\mathrm{d}x - D\int_0^1 (3\omega-\psi)_x(\psi-\varphi_x)_x\,\mathrm{d}x \\ & - \frac{D\mu_1}{G}\int_0^1 (3\omega-\psi)_x\varphi_t\mathrm{d}x - \frac{D\mu_2}{G}\int_0^1 (3\omega-\psi)_x z(x,1,t)\,\mathrm{d}x \\ = {} & -G\int_0^1 (\varphi_x-\psi)^2\,\mathrm{d}x - \sigma\int_0^1 \theta_{tx}(\varphi_x-\psi)\,\mathrm{d}x \\ & - \rho_2\int_0^1 (3\omega-\psi)_t\psi_t\mathrm{d}x + \left(\frac{D\rho_1}{G}-\rho_2\right)\int_0^1 (3\omega-\psi)_{xt}\varphi_t\mathrm{d}x \\ & - \frac{D\mu_1}{G}\int_0^1 (3\omega-\psi)_x\varphi_t\mathrm{d}x - \frac{D\mu_2}{G}\int_0^1 (3\omega-\psi)_x z(x,1,t)\,\mathrm{d}x. \end{aligned}$$

Using Young's inequality with $\varepsilon_3,\varepsilon_4>0$, we establish (36). □

LEMMA 4.8. *Let* $(\varphi, \psi, \omega, \theta, z)$ *be the solution of* (3)–(4). *Then the functional*

$$I_6(t) := \int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\,\mathrm{d}\rho\mathrm{d}x \tag{37}$$

satisfies the estimate

$$I_6'(t) \leq -m \int_0^1 \int_0^1 z^2(x,\rho,t)\,\mathrm{d}\rho\mathrm{d}x - \frac{c}{\tau}\int_0^1 z^2(x,1,t)\,\mathrm{d}x + \frac{1}{\tau}\int_0^1 \varphi_t^2\mathrm{d}x, \tag{38}$$

for any $m, c > 0$.

Proof. By differentiating I_6 with respect to t, using $(3)_5$ and integrating by parts, we obtain

$$\begin{aligned} I_6'(t) &= -\frac{2}{\tau}\int_0^1 \int_0^1 e^{-2\tau\rho} z(x,\rho,t)\, z_\rho(x,\rho,t)\,\mathrm{d}\rho\mathrm{d}x \\ &= -2\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,\mathrm{d}\rho\mathrm{d}x - \frac{1}{\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau\rho}z^2(x,\rho,t)\right)\mathrm{d}\rho\mathrm{d}x \\ &\leq -m\int_0^1\int_0^1 z^2(x,\rho,t)\,\mathrm{d}\rho\mathrm{d}x - \frac{c}{\tau}\int_0^1 z^2(x,1,t)\,\mathrm{d}x + \frac{1}{\tau}\int_0^1 \varphi_t^2\mathrm{d}x. \end{aligned}$$

This gives (38). □

Proof of Theorem 4.1. To finalize the proof, we assume $\frac{G}{\rho_1} = \frac{D}{\rho_2}$ and define a Lyapunov functional $\mathcal{L}$ as follows

$$\mathcal{L}(t) := NE(t) + N_1F_1(t) + F_2(t) + N_3F_3(t) + F_4(t) + N_5F_5(t) + F_6(t),$$

where N, N_1, N_3, N_5 are positive constants to be chosen properly later. Using Cauchy-Schwarz inequality and the Poincare inequality, one can easily see that all $F_i(t)$, $i = 1,2,3,4,5,6$ are bounded by an expression with the existing terms in the energy $E(t)$. This leads to the equivalence of $\mathcal{L}(t)$ and $E(t)$. Gathering the estimates in the previous lemmas and using $\int_0^1 \theta_t^2\mathrm{d}x \leq \int_0^1 \theta_{tx}^2\mathrm{d}x$, we arrive at

$$\begin{aligned} \mathcal{L}'(t) \leq &-\left[C_1N - \frac{D^2\mu_1^2}{2G^2\varepsilon_4}N_5 - \frac{1}{\tau}\right]\int_0^1 \varphi_t^2\mathrm{d}x - D\int_0^1 \omega_x^2\mathrm{d}x \\ &-\left[4\beta N - C_3 - \frac{9\rho_2^2}{4\varepsilon_3}N_5\right]\int_0^1 \omega_t^2\mathrm{d}x - \delta\int_0^1 \theta_x^2\mathrm{d}x \\ &-\left[kN - \frac{G^2}{D}N_1 - C_4(\varepsilon_1)N_3 - C_5(\varepsilon_2) - \frac{\sigma^2}{2G}N_5\right]\int_0^1 \theta_{tx}^2\mathrm{d}x \\ &-\left[\frac{G}{2}N_5 - \frac{\sigma^2}{D}N_1 - \frac{3G^2}{4\gamma} - \varepsilon_1N_3\right]\int_0^1 (\varphi_x - \psi)^2\,\mathrm{d}x \end{aligned} \tag{39}$$

$$- \left[\frac{D}{2}N_1 - \varepsilon_1 N_3 - \varepsilon_2 - \varepsilon_4 N_5\right] \int_0^1 (3\omega - \psi)_x^2 \, \mathrm{d}x - \frac{2}{3}\gamma \int_0^1 \omega^2 \mathrm{d}x$$
$$- \left[\frac{\rho_2 \sigma}{2}N_3 - \rho_2 N_1 - (\rho_2 + \varepsilon_3) N_5\right] \int_0^1 (3\omega - \psi)_t^2 \, \mathrm{d}x$$
$$- \left[C_2 N + \frac{c}{\tau} - \frac{D^2 \mu_2^2}{2G^2 \varepsilon_4} N_5\right] \int_0^1 z^2 (x,1,t) \, \mathrm{d}x - m \int_0^1 \int_0^1 z^2 (x, \rho, t) \, \mathrm{d}\rho \mathrm{d}x.$$

At this point, we choose our constants carefully. First, we take N_1 large enough and ε_2 small, such that $\frac{D}{2}N_1 - \varepsilon_2 > 0$. Then, we choose N_5 large enough, so that $\frac{G}{2}N_5 - \frac{\sigma^2}{D}N_1 - \frac{3G^2}{4\gamma} > 0$. Next, we pick ε_3 small and choose N_3 large enough such that $\frac{\rho_2\sigma}{2}N_3 - \rho_2 N_1 - (\rho_2 + \varepsilon_3) N_5 > 0$. Furthermore, we select ε_1 and ε_4 so small that

$$\frac{G}{2}N_5 - \frac{\sigma^2}{D}N_1 - \frac{3G^2}{4\gamma} - \varepsilon_1 N_3 > 0 \ , \quad \frac{D}{2}N_1 - \varepsilon_1 N_3 - \varepsilon_2 - \varepsilon_4 N_5 > 0.$$

Finally, we choose N so large such that

$$C_1 N - \frac{D^2 \mu_1^2}{2G^2 \varepsilon_4} N_5 - \frac{1}{\tau} > 0 \ , \quad 4\beta N - C_3 - \frac{9\rho_2^2}{4\varepsilon_3} N_5 > 0,$$
$$kN - \frac{G^2}{D}N_1 - C_4(\varepsilon_1) N_3 - C_5(\varepsilon_2) - \frac{\sigma^2}{2G}N_5 > 0.$$

From the above, we deduce that for some positive constants α_1, α_2 one has

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t).$$

Therefore, (39) becomes $\mathcal{L}'(t) \leq -cE(t)$. For $c_1 = \dfrac{c}{\alpha_2}$, we get

$$\mathcal{L}'(t) \leq -c_1 \mathcal{L}(t), \forall t \geq 0. \tag{40}$$

A simple integration of (40) over $(0,t)$ leads to $\mathcal{L}(t) \leq \mathcal{L}(0) e^{-c_1 t}, \forall t \geq 0$. It gives the desired result Theorem 4.1 when combined with the equivalence of $\mathcal{L}(t)$ and $E(t)$. □

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