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GLOBAL EXISTENCE AND ENERGY DECAY OF SOLUTIONS FOR A WAVE EQUATION WITH A TIME-VARYING DELAY TERM

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Abstract. In this paper, we consider in a bounded domain the wave equation with a weak internal time-varying delay term:

$$u_{tt}(x,t) - \Delta_x u(x,t) + \mu_1(t)\, u_t(x,t) + \mu_2(t)\, u_t(x, t - \tau(t)) = 0.$$

Under appropriate conditions on the functions μ_1 and μ_2, we prove global existence of solutions by the Faedo-Galerkin method and establish a decay rate estimate for the energy using the multiplier method.

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Key words. Wave equation, delay term, decay rate, multiplier method.

1. INTRODUCTION

In this paper, we investigate the existence and decay properties of solutions to the following initial-boundary value problem for a linear wave equation of the from:

$$(P) \qquad \begin{cases} u_{tt}(x,t) - \Delta_x u(x,t) + \mu_1(t)\, u_t(x,t) & \\ +\mu_2(t)\, u_t(x,t-\tau(t)) = 0 & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{in } \Omega\times]0,\tau(0)[, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega = \Gamma$, $\tau(t) > 0$ is a time-varing delay term and the initial data (u_0, u_1, f_0) belong to a suitable function space.

In absence of delay ($\mu_2 = 0$), the energy of problem (P) is exponentially decaying to zero provided that μ_1 is constant, see, for instance, [5, 6, 9, 10, 14]. On the contrary, if $\mu_1 = 0$ and $\mu_2 > 0$ (a constant weight), that is, there exits only the internal delay, the system (P) becomes unstable (see, for instance [7]). In recent years, the PDEs with time delay effects have become an active area of research since they arise in many pratical problems (see, for example, [1, 21]). In [7], it has been shown that a small delay at the boundary can

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turn a well-behave hyperbolic system into a wild one and, therefore, delay becomes a source of instability. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see [15, 17, 22]). For instance, the authors of [15] studied the wave equation with a linear internal damping term with constant delay (τ=const. in the problem (P)) and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they also found a sequence of delays for which the corresponding solution of (P) will be instable if $\mu_2 \geq \mu_1$. The main approach used in [15] is an observability inequality obtained with a Carleman estimate. The same results were obtained if both the damping and the delay are acting on the boundary. We also recall the result by Xu, Yung and Li [22], where the authors proved a result similar to the one in [15] for the one-space dimension by adopting the spectral analysis approach.

In [19], Nicaise, Pignotti and Valein extended the above result to higher space dimensions and established an exponential decay.

Very recently, in [2], the energy of problem (P) is exponentially decaying under appropriate conditions on two functions μ_1 and μ_2 are time-varying.

The case of time-varying delay in the wave equation has been studied recently by Nicaise, Valein and Fridman [18] in one-space dimension. They proved an exponential stability result under the condition

$$\mu_2 < \sqrt{1-d}\mu_1,$$

where the fuction τ satisfies

$$\tau'(t) \leq d, \quad \forall t > 0,$$

for a constant $d < 1$.

In [19], Nicaise, Pignotti and Valein extended the above result to higher space dimensions and established an exponential decay.

Our purpose in this paper is to give an energy decay estimate of the solution of problem (P) in the presence of a time-varing delay term in the feedback. We use the Galerkin approximation scheme and the multiplier technique to prove our results.

2. PRELIMINARIES AND MAIN RESULTS

First assume the following hypotheses:

(H1) τ is a function such that

$$\tau \in W^{2,\infty}([0,T]), \quad \forall T > 0, \tag{1}$$

$$0 < \tau_0 \leq \tau(t) \leq \tau_1, \quad \forall t > 0, \tag{2}$$

$$\tau'(t) \leq d < 1, \quad \forall t > 0, \tag{3}$$

where τ_0 and τ_1 are two positive constants.

(H2) $\mu_1 : \mathbb{R}_+ \to]0, +\infty[$ is a non-increasing function of class $C^1(\mathbb{R}_+)$ satisfying

$$\left|\frac{\mu_1'(t)}{\mu_1(t)}\right| \le M, \tag{4}$$

such that $M > 0$.

(H3) $\mu_2 : \mathbb{R}_+ \to \mathbb{R}$ is a function of class $C^1(\mathbb{R}_+)$, which is not necessarily positive or monotone, such that

$$\left|\mu_2(t)\right| \le \beta\mu_1(t), \tag{5}$$

$$\left|\mu_2'(t)\right| \le \tilde{M}\mu_1(t), \tag{6}$$

for some $0 < \beta < \sqrt{1-d}$ and $\tilde{M} > 0$.

We now state a lemma needed later.

Lemma 2.1 ([12]). *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing C^1 function such that*

$$\phi(0) = 0 \quad \textit{and} \quad \phi(t) \to +\infty \quad \textit{as} \quad t \to +\infty.$$

Assume that there exist $\sigma > -1$ and $\omega > 0$ such that

$$\int_S^{+\infty} E^{1+\sigma}(t)\phi'(t)\,\mathrm{d}t \le \frac{1}{\omega}E^\sigma(0)E(S), \quad 0 \le S < +\infty. \tag{7}$$

Then

$$E(t) = 0 \quad \forall t \ge \frac{E(0)^\sigma}{\omega|\sigma|}, \quad \text{if} \quad -1 < \sigma < 0, \tag{8}$$

$$E(t) \le E(0)\left(\frac{1+\sigma}{1+\omega\sigma\phi(t)}\right)^{\frac{1}{\sigma}} \quad \forall t \ge 0, \quad \textit{if} \quad \sigma > 0, \tag{9}$$

$$E(t) \le E(0)\mathrm{e}^{1-\omega\phi(t)} \quad \forall t \ge 0, \quad \textit{if} \quad \sigma = 0. \tag{10}$$

We introduce, as in [15], the new variable

$$z(x,\rho,t) = u_t(x, t-\rho\,\tau(t)), \; x \in \Omega, \; \rho \in]0,1[, \quad t > 0. \tag{11}$$

Then, we have

$$\tau(t)z_t(x,\rho,t) + (1-\rho\,\tau'(t))z_\rho(x,\rho,t) = 0, \; \text{ in } \Omega\times]0,1[\times]0,+\infty[. \tag{12}$$

Therefore, problem (P) takes the form:

$$\begin{cases} u_{tt}(x,t) - \Delta_x u(x,t) + \mu_1(t)\,u_t(x,t) & \\ \qquad\qquad +\mu_2(t)\,z(x,1,t) = 0, & \text{in } \Omega\times]0,+\infty[, \\ \tau(t)z_t(x,\rho,t) + (1-\rho\tau'(t))z_\rho(x,\rho,t) = 0, & \text{in } \Omega\times]0,1[\times]0,+\infty[, \\ u(x,t) = 0, & \text{on } \Gamma\times]0,+\infty[, \\ z(x,0,t) = u_t(x,t), & \text{on } \Omega\times]0,+\infty[, \\ u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), & \text{in } \Omega, \\ z(x,\rho,0) = f_0(x,-\rho\,\tau(0)), & \text{in } \Omega\times]0,1[. \end{cases} \tag{13}$$

Let $\bar{\xi}$ be a positive constant such that

$$\frac{\beta}{\sqrt{1-d}} < \bar{\xi} < 2 - \frac{\beta}{\sqrt{1-d}}. \tag{14}$$

We define the energy associated to the solution of problem (13) by the following formula:

$$E(t) = \frac{1}{2}\|u_t(x,t)\|_2^2 + \frac{1}{2}\|\nabla_x u(x,t)\|_2^2 + \frac{\xi(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}x, \tag{15}$$

where $\xi(t) = \bar{\xi}\mu_1(t)$.

We have the following theorem.

Theorem 2.2. *Let* $(u_0, u_1, f_0) \in (H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)\times H_0^1(\Omega; H^1(0,1))$ *satisfy the compatibility condition*

$$f_0(\cdot, 0) = u_1.$$

Assume that the hypotheses $(H1)$-$(H3)$ *hold. Then problem* (P) *admits a unique global strong solution*

$$u \in L^\infty_{loc}((-\tau(0), +\infty); H^2(\Omega)\cap H_0^1(\Omega)),$$

$$u_t \in L^\infty_{loc}((-\tau(0), +\infty); H_0^1(\Omega)),$$

$$u_{tt} \in L^\infty_{loc}((-\tau(0), +\infty); L^2(\Omega)).$$

Moreover, for some positive constants c, ω, *we obtain the following decay property:*

$$E(t) \le cE(0)\mathrm{e}^{-\omega\int_0^t \mu_1(s)\,\mathrm{d}s}, \quad \forall\, t \ge 0. \tag{16}$$

We finish this section by giving an explicit upper bound for the derivative of the energy.

Lemma 2.3. *Let* (u,z) *be a solution to the problem* (13). *Then, the energy functional defined by* (15) *satisfies*

$$\begin{aligned} E'(t) \;\le\; & -\mu_1(t)\left(1 - \frac{\bar{\xi}}{2} - \frac{\beta}{2\sqrt{1-d}}\right)\|u_t\|_2^2 \\ & -\mu_1(t)\left(\frac{\bar{\xi}(1-\tau'(t))}{2} - \frac{\beta\sqrt{1-d}}{2}\right)\int_\Omega z^2(x,1,t)\,\mathrm{d}x. \end{aligned} \tag{17}$$

Proof. Multiplying the first equation in (13) by $u_t(x,t)$ and integrating the result over Ω, we obtain

$$\begin{aligned} &\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(\|u_t(x,t)\|_2^2 + \|\nabla u(x,t)\|_2^2) + \mu_1(t)\int_\Omega u_t^2(x,t)\,\mathrm{d}x \\ &\quad + \mu_2(t)\int_\Omega z(x,1,t)u_t(x,t)\,\mathrm{d}x = 0. \end{aligned} \tag{18}$$

We multiply the second equation in (13) by $\xi(t)z$ and integrate over $\Omega\times]0,1[$ to obtain:

(19)
$$\xi(t)\tau(t)\int_\Omega\int_0^1 z_t z(x,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}x = -\frac{\xi(t)}{2}\int_\Omega\int_0^1(1-\rho\tau'(t))\frac{\partial}{\partial\rho}(z(x,\rho,t))^2\,\mathrm{d}\rho\,\mathrm{d}x.$$

Consequently, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\xi(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}x\right)\\
&= -\frac{\xi(t)}{2}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}((1-\rho\tau'(t))z^2(x,\rho,t))\,\mathrm{d}\rho\,\mathrm{d}x\\
&+\frac{\xi'(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}x\\
&= \frac{\xi(t)}{2}\int_\Omega(z^2(x,0,t)-z^2(x,1,t))\,\mathrm{d}x+\frac{\xi(t)\tau'(t)}{2}\int_\Omega z^2(x,1,t)\,\mathrm{d}x\\
&+\frac{\xi'(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}x.
\end{aligned}\tag{20}$$

From (15), (18) and (20) we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\xi(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}x\right)\\
&= -\frac{\xi(t)}{2}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}((1-\rho\tau'(t))z^2(x,\rho,t))\,\mathrm{d}\rho\,\mathrm{d}x\\
&+\frac{\xi'(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}x\\
&= \frac{\xi(t)}{2}\int_\Omega(z^2(x,0,t)-z^2(x,1,t))\,\mathrm{d}x+\frac{\xi(t)\tau'(t)}{2}\int_\Omega z^2(x,1,t)\,\mathrm{d}x\\
&+\frac{\xi'(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}x.
\end{aligned}\tag{21}$$

Due to Young's inequality, we have

$$\begin{aligned}
\mu_2(t)\int_\Omega z(x,1,t)u_t(x,t)\,\mathrm{d}x \le{}& \frac{|\mu_2(t)|}{2\sqrt{1-d}}\|u_t\|_2^2\\
&+\frac{|\mu_2(t)|\sqrt{1-d}}{2}\int_\Omega z^2(x,1,t)\,\mathrm{d}x.
\end{aligned}\tag{22}$$

Inserting (22) into (21), we obtian

$$\begin{aligned}
E'(t) \le{}& -\left(\mu_1(t)-\frac{\xi(t)}{2}-\frac{|\mu_2(t)|}{2\sqrt{1-d}}\right)\|u_t\|_2^2\\
&-\left(\frac{\xi(t)}{2}-\frac{\xi(t)\tau'(t)}{2}-\frac{|\mu_2(t)|\sqrt{1-d}}{2}\right)\int_\Omega z^2(x,1,t)\,\mathrm{d}x
\end{aligned}$$

$$+ \frac{\xi'(t)\tau(t)}{2} \int_\Omega \int_0^1 z^2(x,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}x.$$

Then, we have

$$\begin{aligned} E'(t) &\le -\mu_1(t)\left(1 - \frac{\overline{\xi}}{2} - \frac{\beta}{2\sqrt{1-d}}\right)\|u_t\|_2^2 \\ &- \mu_1(t)\left(\frac{\overline{\xi}(1-\tau'(t))}{2} - \frac{\beta\sqrt{1-d}}{2}\right)\int_\Omega z^2(x,1,t)\,\mathrm{d}x \\ &\le 0. \end{aligned} \tag{23}$$

This completes the proof of the lemma. □

3. GLOBAL EXISTENCE

We are now ready to prove Theorem 2.2 in the next two sections.

Throughout this section we assume $u_0 \in H^2(\Omega) \cap H_0^1(\Omega), u_1 \in H_0^1(\Omega)$ and $f_0 \in L_0^1(\Omega; H^1(0,1))$.

We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \ldots, w_k\}$, where the set $\{w_k, k \in \mathbb{N}\}$ is a basis of $H^2(\Omega) \cap H_0^1(\Omega)$.

Now, we define, for $1 \le j \le k$, the sequence $\phi_j(x,\rho)$ as follows:

$$\phi_j(x,0) = w_j.$$

Then, we may extend $\phi_j(x,0)$ by $\phi_j(x,\rho)$ over $L^2(\Omega \times (0,1))$ such that $(\phi_j)_j$ form a basis of $L^2(\Omega; H^1(0,1))$ and denote by Z_k the space generated by $\{\phi_1, \phi_2, \ldots, \phi_k\}$.

We construct approximate solutions $(u_k, z_k), k = 1,2,3,\ldots$, in the form

$$u_k(t) = \sum_{j=1}^k g_{jk}(t) w_j, \qquad z_k(t) = \sum_{j=1}^k h_{jk}(t)\phi_j,$$

where g_{ik} and $h_{ik}, j = 1,2,\ldots,k$, are determined by the following ordinary differential equations:

$$\begin{cases} \big(u_k''(t), w_j) + (\nabla_x u_k(t), \nabla_x w_j) + \mu_1(t)(u_k'(t), w_j) + \mu_2(t)(z_k(\cdot,1), w_j\big) = 0, \\ 1 \le j \le k, \\ z_k(x,0,t) = u_k'(x,t), \end{cases} \tag{24}$$

$$u_k(0) = u_{0k} = \sum_{j=1}^k (u_0, w_j) w_j \to u_0 \;\text{ in }\; H^2(\Omega) \cap H_0^1(\Omega) \;\text{ as }\; k \to +\infty, \tag{25}$$

$$u_k'(0) = u_{1k} = \sum_{j=1}^k (u_1, w_j) w_j \to u_1 \;\text{ in }\; H_0^1(\Omega) \;\text{ as }\; k \to +\infty, \tag{26}$$

and

$$\left\{ \begin{array}{l} \left(\tau(t) z_{kt} + (1-\rho\tau'(t)) z_{k\rho}, \phi_j\right) = 0, \\ 1 \leq j \leq k, \end{array} \right. \tag{27}$$

$$z_k(\rho, 0) = z_{0k} = \sum_{j=1}^{k} (f_0, \phi_j)\phi_j \to f_0 \ \text{ in } \ H_0^1(\Omega; H^1(0,1)) \ \text{ as } \ k \to +\infty. \tag{28}$$

By virtue of the theory of ordinary differential equations the system (24)-(28) has a unique local solution which is extended to a maximal interval $[0, T_k[$ (with $0 < T_k \leq +\infty$) by Zorn lemma. Note that $u_k(t)$ is from the class C^2.

In the next step we obtain a priori estimates for the solution, such that it can be extended beyond $[0, T_k[$ to obtain a single solution defined for all $t > 0$.

In order to use a standard compactness argument for the limiting procedure, we will derive some a priori estimates for (u_k, z_k).

The first estimate. Since the sequences $(u_{0k}), (u_{1k})$ and (z_{0k}) converge, then standard calculations, using (24)-(28), similar to those used to derive (17), yield

$$E_k(t) + \int_0^t a_1(s)\|u_k'(x,s)\|_2^2 \mathrm{d}s + \int_0^t a_2(s)\|z_k(x,1,s)\|_2^2 \mathrm{d}s \leq E_k(0) \leq C, \tag{29}$$

where

$$E_k(t) = \frac{1}{2}\|u_k'(t)\|_2^2 + \frac{1}{2}\|\nabla_x u_k(t)\|_2^2 + \frac{\xi(t)\tau(t)}{2} \int_\Omega \int_0^1 z_k^2(x,\rho,t))\, \mathrm{d}\rho\, \mathrm{d}x,$$

$$a_1(t) = \mu_1(t)\left(1 - \frac{\bar{\xi}}{2} - \frac{\beta}{2\sqrt{1-d}}\right) \ \text{ and } \ a_2(t) = \mu_1(t)\left(\frac{\bar{\xi}(1-d)}{2} - \frac{\beta\sqrt{1-d}}{2}\right)$$

for some C independent of k.

Estimate (29) yields

$$(u_k) \text{ is bounded in } L^\infty_{loc}(0,\infty; H_0^1(\Omega)), \tag{30}$$

$$(u_k') \text{ is bounded in } L^\infty_{loc}(0,\infty; L^2(\Omega)), \tag{31}$$

$$(\mu_1(t) u_k'^2(t)) \text{ is bounded in } L^1(\Omega \times (0,T)), \tag{32}$$

$$(\mu_1(t) z_k^2(x,\rho,t)) \text{ is bounded in } L^\infty_{loc}(0,\infty; L^1(\Omega \times (0,1))), \tag{33}$$

$$(\mu_1(t) z_k^2(x,1,t)) \text{ is bounded in } L^1(\Omega \times (0,T)). \tag{34}$$

The second estimate. First, we estimate $u_k''(0)$. Testing (24) by $g_{jk}''(t)$ and choosing $t = 0$ we obtain

$$\|u_k''(0)\|_2 \leq \|\Delta_x u_{0k}\|_2 + \mu_1(0)\|u_{1k}\|_2 + |\mu_2(0)|\|z_{0k}\|_2.$$

Since (u_{0k}), (z_{1k}) are bounded in $L^2(\Omega)$, (25), (26) and (28) yield

$$\|u_k''(0)\|_2 \leq C.$$

Differentiating (24) with respect to t we get

$$\big(u_k'''(t)-\Delta_x u_k'(t)+\mu_1(t)u_k''(t)+\mu_1'(t)u_k'(t)+\mu_2(t)z_k'(1,t)+\mu_2'(t)z_k(1,t),w_j\big)=0.$$

Multiplying by $g_{jk}''(t)$ and summing over j from 1 to k implies

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u_k''(t)\|_2^2+\|\nabla_x u_k'(t)\|_2^2\big)+\mu_1(t)\int_\Omega u''^2_k(t)\,\mathrm{d}x\\
&+\mu_1'(t)\int_\Omega u''_k(t)u'_k(t)\,\mathrm{d}x+\mu_2(t)\int_\Omega u''_k(t)z_k'(x,1,t)\,\mathrm{d}x\\
&+\mu_2'(t)\int_\Omega u''_k(t)z_k(x,1,t)\,\mathrm{d}x=0.
\end{aligned}\tag{35}$$

Differentiating (27) with respect to t gives

$$\left(\left(\frac{\tau(t)}{1-\rho\tau'(t)}\right)'z_k'(t)+\frac{\tau(t)}{1-\rho\tau'(t)}z_k''(t)+\frac{\partial}{\partial\rho}z_k'(t),\phi_j\right)=0.$$

Multiplying by $h_{jk}'(t)$ and summing over j from 1 to k leads to

$$\left(\frac{\tau(t)}{1-\rho\tau'(t)}\right)'\|z_k'(t)\|_2^2+\frac{1}{2}\frac{\tau(t)}{1-\rho\tau'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\|z_k'(t)\|_2^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\rho}\|z_k'(t)\|_2^2=0.\tag{36}$$

Then, we have

$$\frac{1}{2}\left(\frac{\tau(t)}{1-\rho\tau'(t)}\right)'\|z_k'(t)\|_2^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\tau(t)}{1-\rho\tau'(t)}\|z_k'(t)\|_2^2\right)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\rho}\|z_k'(t)\|_2^2=0.\tag{37}$$

Consequently, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(\frac{\tau(t)}{1-\rho\tau'(t)}\|z_k'(t)\|_2^2\right)\mathrm{d}\rho+\frac{1}{2}\int_0^1\left(\frac{\tau(t)}{1-\rho\tau'(t)}\right)'\|z_k'(t)\|_2^2\,\mathrm{d}\rho\\
&+\frac{1}{2}\|z_k'(x,1,t)\|_2^2-\frac{1}{2}\|u_k''(t)\|_2^2\;=\;0.
\end{aligned}\tag{38}$$

Taking the sum of (35) and (38), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_k''(t)\|_2^2+\|\nabla_x u_k'(t)\|_2^2+\int_0^1\frac{\tau(t)}{1-\rho\tau'(t)}\|z_k'(x,\rho,t)\|_{L^2(\Omega)}^2\mathrm{d}\rho\right)\\
&+\mu_1(t)\int_\Omega u''^2_k(t)\,\mathrm{d}x+\frac{1}{2}\int_\Omega|z_k'(x,1,t)|^2\,\mathrm{d}x\\
&=-\frac{1}{2}\int_0^1\left(\frac{\tau(t)}{1-\rho\tau'(t)}\right)'\|z_k'(x,\rho,t)\|_2^2\mathrm{d}\rho-\mu_2(t)\int_\Omega u''_k(t)z_k'(x,1,t)\,\mathrm{d}x\\
&-\mu_1'(t)\int_\Omega u''_k(t)u_k'(t)\,\mathrm{d}x-\mu_2'(t)\int_\Omega u''_k(t)z_k(x,1,t)\,\mathrm{d}x+\frac{1}{2}\|u_k''(t)\|_2^2.
\end{aligned}$$

Using **(H1)-(H3)**, Cauchy-Schwarz and Young's inequalities, we conclude

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_k''(t)\|_2^2+\|\nabla_x u_k'(t)\|_2^2+\int_0^1\frac{\tau(t)}{1-\rho\tau'(t)}\|z_k'(x,\rho,t)\|_{L^2(\Omega)}^2\,\mathrm{d}\rho\right)$$

$$
\begin{aligned}
&+\mu_1(t)\int_\Omega {u''}_k^2(t)\,\mathrm{d}x+\frac{1}{2}\int_\Omega |z_k'(x,1,t)|^2\,\mathrm{d}x\\
&\le -\frac{1}{2}\int_0^1\left(\frac{\tau(t)}{1-\rho\tau'(t)}\right)'\|z_k'(x,\rho,t)\|_2^2\mathrm{d}\rho+|\mu_2(t)|\|{u''}_k(t)\|_2\|z_k'(x,1,t)\|_2\\
&+|\mu_1'(t)|\|{u''}_k(t)\|_2\|u_k'(t)\|_2+|\mu_2'(t)|\|{u''}_k(t)\|_2\|z_k(x,1,t)\|_2+\frac{1}{2}\|u_k''(t)\|_2^2\\
&\le c\int_0^1\frac{\tau(t)}{1-\rho\tau'(t)}\|z_k'(x,\rho,t)\|_{L^2(\Omega)}^2\,\mathrm{d}\rho+\frac{|\mu_2(t)|^2}{2}\|{u''}_k(t)\|_2^2+\frac{1}{2}\|z_k'(x,1,t)\|_2^2\\
&+\frac{|\mu_1'(t)|^2}{4}\|{u''}_k(t)\|_2^2+|\mu_1'(t)|\|u_k'(t)\|_2^2+\frac{|\mu_2'(t)|}{4}\|{u''}_k(t)\|_2^2\\
&+|\mu_2'(t)|\|z_k(x,1,t)\|_2^2+\frac{1}{2}\|u_k''(t)\|_2^2\\
&\le c'\|{u''}_k(t)\|_2^2+|\mu_1'(t)|\|u_k'(t)\|_2^2+c\int_0^1\frac{\tau(t)}{1-\rho\tau'(t)}\|z_k'(x,\rho,t)\|_{L^2(\Omega)}^2\,\mathrm{d}\rho\\
&+\frac{1}{2}\|z_k'(x,1,t)\|_2^2+|\mu_2'(t)|\|z_k(x,1,t)\|_2^2\\
&\le c'\|{u''}_k(t)\|_2^2+M\mu_1(t)\|u_k'(t)\|_2^2+c\int_0^1\frac{\tau(t)}{1-\rho\tau'(t)}\|z_k'(x,\rho,t)\|_{L^2(\Omega)}^2\,\mathrm{d}\rho\\
&+\frac{1}{2}\|z_k'(x,1,t)\|_2^2+\tilde{M}\mu_1(t)\|z_k(x,1,t)\|_2^2.
\end{aligned}
$$

Integrating the last inequality over $(0,t)$ and using (29), we get

$$
\begin{aligned}
&\left(\|u_k''(t)\|_2^2+\|\nabla_x u_k'(t)\|_2^2+\int_0^1\frac{\tau(t)}{1-\rho\tau'(t)}\|z_k'(x,\rho,t)\|_{L^2(\Omega)}^2\,\mathrm{d}\rho\right)\\
&\le\left(\|u_k''(0)\|_2^2+\|\nabla_x u_k'(0)\|_2^2+\int_0^1\frac{\tau(0)}{1-\rho\tau'(0)}\|z_k'(x,\rho,0)\|_{L^2(\Omega)}^2\,\mathrm{d}\rho\right)\\
&+2c'\int_0^t|{u''}_k(s)\|_2^2ds+2M\int_0^t\mu_1(s)\|u_k'(s)\|_2^2\mathrm{d}s+2\tilde{M}\int_0^t\mu_1(s)\|z_k(x,1,s)\|_2^2\mathrm{d}s\\
&2c\int_0^t\int_0^1\frac{\tau(s)}{1-\rho\tau'(s)}\|z_k'(x,\rho,s)\|_{L^2(\Omega)}^2\,\mathrm{d}\rho\,\mathrm{d}t+\int_0^t\|z_k'(x,1,s)\|_2^2\mathrm{d}s\\
&\le\left(\|u_k''(0)\|_2^2+\|\nabla_x u_k'(0)\|_2^2+\int_0^1\frac{\tau(0)}{1-\rho\tau'(0)}\|z_k'(x,\rho,0)\|_{L^2(\Omega)}^2\,\mathrm{d}\rho+C\right)\\
&+C'\int_0^t\left(\|u_k''(s)\|_2^2+\|\nabla u_k'(s)\|_2^2+\int_0^1\frac{\tau(s)}{1-\rho\tau'(s)}\|z_k'(x,\rho,s)\|_{L^2(\Omega)}^2\,\mathrm{d}\rho\right)\,\mathrm{d}s.
\end{aligned}
$$

Using Gronwall's lemma, we deduce that

$$
\begin{aligned}
&\|u_k''(t)\|_2^2+\|\nabla_x u_k'(t)\|_2^2+\int_0^1\frac{\tau(t)}{1-\rho\tau'(t)}\|z_k'(x,\rho,t)\|_{L^2(\Omega)}^2\,\mathrm{d}\rho\\
&\le \mathrm{e}^{C'T}\left(\|u_k''(0)\|_2^2+\|\nabla_x u_k'(0)\|_2^2+\int_0^1\frac{\tau(0)}{1-\rho\tau'(0)}\|z_k'(x,\rho,0)\|_{L^2(\Omega)}^2\,\mathrm{d}\rho+C\right)
\end{aligned}
$$

for all $t \in \mathbb{R}^+$, therefore, we conclude that

$$(u_k'') \text{ is bounded in } L^\infty_{loc}(0,+\infty;L^2(\Omega)), \tag{39}$$

$$(u_k') \text{ is bounded in } L^\infty_{loc}(0,+\infty;H_0^1(\Omega)), \tag{40}$$

$$(\tau(t)z_k') \text{ is bounded in } L^\infty_{loc}(0,+\infty;L^2(\Omega\times(0,1))). \tag{41}$$

Applying Dunford-Petti's theorem we conclude from (30), (31), (32), (33), (34),(39), (40) and (41), after replacing the sequences (u_k) and (z_k) by subsequence if necessary, that

$$u_k \to u \ \text{ weak-star in } \ L^\infty_{loc}(0,+\infty;H^2(\Omega)\cap H_0^1(\Omega)), \tag{42}$$

$$u_k' \to u' \ \text{ weak-star in } \ L^\infty_{loc}(0,+\infty;H_0^1(\Omega)),$$

$$u''{}_k \to u'' \ \text{ weak-star in } \ L^\infty_{loc}(0,+\infty;L^2(\Omega)), \tag{43}$$

$$u_k' \to \chi \ \text{ weak in } \ L^2(\Omega\times(0,T);\mu_1(t)),$$

$$z_k \to z \ \text{ weak-star in } \ L^\infty_{loc}(0,+\infty;H_0^1(\Omega;L^2(0,1)),$$

$$z_k' \to z' \ \text{ weak-star in } \ L^\infty_{loc}(0,+\infty;L^2(\Omega\times(0,1))), \tag{44}$$

$$z_k(x,1,t) \to \psi \ \text{ weak in } \ L^2(\Omega\times(0,T);\mu_1(t))$$

for suitable functions $u \in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)), z \in L^\infty(0,T;L^2(\Omega\times(0,1)))$, $\chi \in L^2(\Omega\times(0,T);\mu_1(t)), \psi \in L^2(\Omega\times(0,T);\mu_1(t))$ for all $T \geq 0$.

We have to show that (u,z) is a solution of (P).

From (30) and (31), we have that (u_k') is bounded in $L^\infty(0,T;H_0^1(\Omega))$. Then (u_k') is bounded in $L^2(0,T;H_0^1(\Omega))$. Since (u_k'') is bounded in $L^\infty(0,T;L^2(\Omega))$, then (u_k'') is bounded in $L^2(0,T;L^2(\Omega))$. Consequently, (u_k') is bounded in $H^1(Q)$.

Since the embedding $H^1(Q) \hookrightarrow L^2(Q)$ is compact, using Aubin-Lions theorem [11], we can extract a subsequence (u_ν) of (u_k) such that

$$u_\nu' \to u' \quad \text{strongly in } \ L^2(Q). \tag{45}$$

Therefore,

$$u_\nu' \to u' \ \text{ strongly and a.e in } Q. \tag{46}$$

Similarly we obtain

$$z_\nu \to z \quad \text{strongly in } \ L^2(\Omega\times(0,1)\times(0,T)) \tag{47}$$

and

$$z_\nu \to z \ \text{ strongly and a.e in } \Omega\times(0,1)\times(0,T). \tag{48}$$

It follows directly from (42), (43), (44), (45) and (47) that for each fixed $v \in L^2(0,T;L^2(\Omega))$ and $w \in L^2(0,T;L^2(\Omega\times(0,1)))$

$$\int_0^T \int_\Omega \Big(u_\nu'' - \Delta_x u_\nu + \mu_1(t)u_\nu' + \mu_2(t)z_\nu\Big) v \,\mathrm{d}x\,\mathrm{d}t$$

$$\to \int_0^T \int_\Omega \Big(u'' - \Delta_x u + \mu_1(t)u' + \mu_2(t)z\Big) v \,\mathrm{d}x\,\mathrm{d}t,$$

$$\int_0^T \int_0^1 \int_\Omega \Big(\tau(t)z_\nu' + (1-\rho\tau'(t))\frac{\partial}{\partial\rho}z_\nu\Big) w \,\mathrm{d}x\,\mathrm{d}\rho\,\mathrm{d}t$$

$$\to \int_0^T \int_0^1 \int_\Omega \Big(\tau(t)z' + (1-\rho\tau'(t))\frac{\partial}{\partial\rho}z\Big) w \,\mathrm{d}x\,\mathrm{d}\rho\,\mathrm{d}t$$

as $k \to +\infty$. Hence, for all $v \in L^2(0,T;L^2(\Omega))$,

$$\int_0^T \int_\Omega \Big(u'' + \Delta_x u + \mu_1(t)u' + \mu_2(t)z\Big) v \,\mathrm{d}x\,\mathrm{d}t = 0$$

and for all $w \in L^2(0,T;L^2(\Omega\times(0,1)))$,

$$\int_0^T \int_0^1 \int_\Omega \Big(\tau(t)z' + (1-\rho\tau'(t))\frac{\partial}{\partial\rho}z\Big) w \,\mathrm{d}x\,\mathrm{d}\rho\,\mathrm{d}t = 0.$$

Thus, the problem (P) admits a global weak solution u.

4. ASYMPTOTIC BEHAVIOR

From now on, we denote by c various positive constants which may be different at different occurrences. We multiply the first equation of (13) by $\phi' E^q u$, where ϕ is a bounded function satisfying all the hypotheses of Lemma 2.1. We obtain

$$\begin{aligned}
0 &= \int_S^T E^q\phi' \int_\Omega u\Big(u'' - \Delta u + \mu_1(t)u' + \mu_2(t)z(x,1,t)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \Big[E^q\phi' \int_\Omega uu'\,\mathrm{d}x\Big]_S^T - \int_S^T (qE'E^{q-1}\phi' + E^q\phi'')\int_\Omega uu'\,\mathrm{d}x\,\mathrm{d}t \\
&\quad -2\int_S^T E^q\phi' \int_\Omega u'^2\,\mathrm{d}x\,\mathrm{d}t + \int_S^T E^q\phi' \int_\Omega \left(u'^2 + |\nabla u|^2\right)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_S^T E^q\phi'\mu_1(t)\int_\Omega uu'\,\mathrm{d}x\,\mathrm{d}t + \int_S^T E^q\phi'\mu_2(t)\int_\Omega uz(x,1,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Similarly, we multiply the second equation of (13) by $E^q\phi'\xi(t)\mathrm{e}^{-2\rho\tau(t)}z(x,\rho,t)$ and get

$$0 = \int_S^T E^q\phi' \int_\Omega\int_0^1 \mathrm{e}^{-2\rho\tau(t)}\xi(t)z\Big(\tau(t)z_t + \big(1-\rho\tau'(t)\big)z_\rho\Big)\,\mathrm{d}\rho\,\mathrm{d}x\,\mathrm{d}t$$

$$
\begin{aligned}
&= \left[\frac{1}{2}E^q\phi'\xi(t)\tau(t)\int_\Omega\int_0^1 \tau(t)\mathrm{e}^{-2\rho\tau(t)}z^2\,\mathrm{d}x\,\mathrm{d}\rho\right]_S^T \\
&-\frac{1}{2}\int_S^T\int_\Omega\int_0^1 \left(E^q\phi'\xi(t)\tau(t)\mathrm{e}^{-2\rho\tau(t)}\right)' z^2\,\mathrm{d}\rho\,\mathrm{d}x\,\mathrm{d}t \\
&+\int_S^T E^q\phi'\int_\Omega\int_0^1 \xi(t)\bigg(\frac{1}{2}\frac{\partial}{\partial\rho}\left(\mathrm{e}^{-2\rho\tau(t)}\big(1-\rho\tau'(t)\big)z^2\right) \\
&+\tau(t)\big(1-\rho\tau'(t)\big)\mathrm{e}^{-2\rho\tau(t)}z^2+\frac{1}{2}\tau'(t)\mathrm{e}^{-2\rho\tau(t)}z^2\bigg)\,\mathrm{d}\rho\,\mathrm{d}x\,\mathrm{d}t \\
&= \left[\frac{1}{2}E^q\phi'\xi(t)\tau(t)\int_\Omega\int_0^1 \tau(t)\mathrm{e}^{-2\rho\tau(t)}z^2\,\mathrm{d}\rho\,\mathrm{d}x\right]_S^T \\
&-\frac{1}{2}\int_S^T \big(E^q\phi'\xi(t)\big)'\,\tau(t)\int_\Omega\int_0^1 \mathrm{e}^{-2\rho\tau(t)}z^2\,\mathrm{d}\rho\,\mathrm{d}x\,\mathrm{d}t \\
&+\frac{1}{2}\int_S^T E^q\phi'\xi(t)\int_\Omega\left(\mathrm{e}^{-2\tau(t)}(1-\tau'(t))z^2(x,1,t)-z^2(x,0,t)\right)\mathrm{d}x\,\mathrm{d}t \\
&+\int_S^T E^q\phi'\xi(t)\tau(t)\int_0^1\int_\Omega \mathrm{e}^{-2\rho\tau(t)}z^2\,\mathrm{d}x\,\mathrm{d}\rho\,\mathrm{d}t.
\end{aligned}
$$

Taking their sum, we obtain

$$
\begin{aligned}
&A\int_S^T E^{q+1}\phi'\,\mathrm{d}t \\
&\le -\left[E^q\phi'\int_\Omega uu'\,dx\right]_S^T+\int_S^T (qE'E^{q-1}\phi'+E^q\phi'')\int_\Omega uu'\,\mathrm{d}x\,\mathrm{d}t \\
&+2\int_S^T E^q\phi'\int_\Omega u'^2\,\mathrm{d}x\,\mathrm{d}t-\int_S^T \mu_1(t)E^q\phi'\int_\Omega uu'\,\mathrm{d}x\,\mathrm{d}t \\
&-\int_S^T \mu_2(t)E^q\phi'\int_\Omega uz(x,1,t)\,\mathrm{d}x\,\mathrm{d}t \\
&-\left[\frac{1}{2}E^q\phi'\xi(t)\tau(t)\int_\Omega\int_0^1 \mathrm{e}^{-2\rho\tau(t)}z^2\,\mathrm{d}\rho\,\mathrm{d}x\right]_S^T \\
&+\frac{1}{2}\int_S^T \big(E^q\phi'\xi(t)\big)'\,\tau(t)\int_\Omega\int_0^1 \mathrm{e}^{-2\rho\tau(t)}z^2\,\mathrm{d}\rho\,\mathrm{d}x\,\mathrm{d}t \\
&-\frac{1}{2}\int_S^T E^q\phi'\xi(t)\int_\Omega\left(\mathrm{e}^{-2\tau(t)}(1-\tau'(t))z^2(x,1,t)-z^2(x,0,t)\right)\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
\tag{49}
$$

where $A = 2\min\{1, \mathrm{e}^{-2\tau_1}\}$. Using the Cauchy-Schwarz and Poincaré's inequalities and the definition of E and assuming that ϕ' is a bounded non-negative function on $\mathbb{R}^+$, we get

$$
\left|E^q(t)\phi'\int_\Omega uu'\,\mathrm{d}x\right| \le cE(t)^{q+1}.
$$

By recalling (17), we have

$$\int_S^T \left| qE'E^{q-1}\phi' \int_\Omega uu' \, \mathrm{d}x \right| \mathrm{d}t \leq c \int_S^T E^q(t)|E'(t)| \, \mathrm{d}t$$
$$\leq c \int_S^T E^q(t)(-E'(t)) \, \mathrm{d}t \leq cE^{q+1}(S),$$

$$\int_S^T E^q \phi'' \int_\Omega uu' \, \mathrm{d}x \, \mathrm{d}t \leq c \int_S^T E^{q+1}(t)(-\phi'')\mathrm{d}t$$
$$\leq cE^{q+1}(S) \int_S^T (-\phi'') \, \mathrm{d}t \leq cE^{q+1}(S),$$

and

$$\begin{aligned} \int_S^T E^q \phi' \int_\Omega u'^2 \, \mathrm{d}x \, \mathrm{d}t &\leq c \int_S^T E^q \phi' \frac{1}{\mu_1(t)} \int_\Omega \mu_1(t) u'^2 \, \mathrm{d}x \, \mathrm{d}t \\ &\leq \int_S^T E^q \frac{\phi'}{\mu_1(t)} \left(-E'\right) \, \mathrm{d}t. \end{aligned} \tag{50}$$

Define

$$\phi(t) = \int_0^t \mu_1(s) \, \mathrm{d}s. \tag{51}$$

It is clear that ϕ is a non-decreasing function of class C^1 on $\mathbb{R}^+, \phi'$ is bounded and

$$\phi(t) \to +\infty \text{ as } t \to +\infty. \tag{52}$$

So, we deduce, from (50), that

$$\int_S^T E^q \phi' \int_\Omega u'^2 \, \mathrm{d}x \, \mathrm{d}t \leq c \int_S^T E^q(-E') \, \mathrm{d}t \leq cE^{q+1}(S). \tag{53}$$

By the hypothesis (**H1**), Young's and Poincaré's inequality and (17), we have

$$\begin{aligned} \left| \int_S^T E^q \phi' \int_\Omega uu' \, \mathrm{d}x \, \mathrm{d}t \right| &\leq c \int_S^T E^q \phi' \|u\|_2 \|u'\|_2 \, \mathrm{d}t \\ &\leq c\varepsilon' \int_S^T E^q \phi' \|u\|_2^2 \, \mathrm{d}t + c(\varepsilon') \int_S^T E^q \phi' \|u'\|_2^2 \, \mathrm{d}t \\ &\leq \varepsilon' c_* \int_S^T E^q \phi' \|\nabla_x u\|_2^2 \, \mathrm{d}t + c(\varepsilon') \int_S^T E^q \phi' \|u'\|_2^2 \, \mathrm{d}t \\ &\leq \varepsilon' c_* \int_S^T E^{q+1} \phi' \, \mathrm{d}t + cE^{q+1}(S). \end{aligned}$$

Recalling that $\xi' \leq 0$ and the definition of E, we have

$$\int_S^T \left(E^q \phi' \xi(t)\right)' \tau(t) \int_\Omega \int_0^1 \mathrm{e}^{-2\rho\tau(t)} z^2 \, \mathrm{d}\rho \, \mathrm{d}x \, \mathrm{d}t$$

$$
\begin{aligned}
&\leq \int_S^T \left(E^q\phi'\right)'\xi(t)\tau(t)\int_\Omega\int_0^1 \mathrm{e}^{-2\rho\tau(t)}z^2\,\mathrm{d}\rho\,\mathrm{d}x\,\mathrm{d}t\\
&\leq c\int_S^T E^q|E'|\phi'\,\mathrm{d}t\\
&\leq c\int_S^T E^q\phi'(-E'(t))\,\mathrm{d}t\\
&\leq cE^{q+1}(S),
\end{aligned}
$$

$$
\begin{aligned}
&\int_S^T E^q\phi'\xi(t)\int_\Omega \mathrm{e}^{-2\tau(t)}(1-\tau'(t))z^2(x,1,t)\,\mathrm{d}x\,\mathrm{d}t\\
\leq\ & c\int_S^T E^q\phi'\xi(t)\int_\Omega z^2(x,1,t)\,\mathrm{d}x\,\mathrm{d}t\\
\leq\ & c\int_S^T E^q\phi'\left(-E'\right)\,\mathrm{d}t\\
\leq\ & cE^{q+1}(S),
\end{aligned}
$$

$$
\begin{aligned}
\int_S^T E^q\phi'\xi(t)\int_\Omega z^2(x,0,t)\,\mathrm{d}x\,\mathrm{d}t &= \int_S^T E^q\phi'\xi(t)\int_\Omega u'^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\leq cE^{q+1}(S),
\end{aligned}
$$

where we also have used Cauchy-Schwarz inequality. Combining these estimates and choosing ε' sufficiently small, we conclude from (49) that

$$
\int_S^T E^{q+1}\phi'\,\mathrm{d}t \leq cE^{q+1}(S) \leq cE(S).
$$

Hence, we deduce from Lemma 2.1.

$$
E(t) \leq cE(0)\mathrm{e}^{-\omega\int_0^t \mu_1(s)\,\mathrm{d}s}, \qquad \forall\, t \geq 0.
$$

This ends the proof of Theorem 2.2.

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