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GENERALIZATIONS OF REGULAR AND NORMAL SPACES II

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Abstract. A family m_X of subsets of a nonempty set X is called an m-structure. A set X with a topology τ and m-structure m_X is called a mixed-space and is denoted by (X, τ, m_X). As a generalization of g-closed sets due to Levine, we introduce the notion of m_g-closed sets in (X, τ, m_X). By using m_g-open sets, we define and investigate mixed-regularity and mixed-normality in (X, τ, m_X). As special cases, we obtain $\mathcal{I}_g$-regular spaces and s-normal spaces.

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1. INTRODUCTION

In 1970, Levine [10] introduced the notion of generalized closed (briefly g-closed) sets in a topological space. Since then, many modifications of g-closed sets have been defined and investigated in topological spaces and ideal topological spaces (see Definitions 2.1 and 2.2 of [18]). Popa and Noiri [17] introduced the notion of a minimal structure (briefly m-structure) m_X on a nonempty set X. A subfamily m_X of the power set of a nonempty set X is called an m-structure if $\emptyset, X \in m_X$.

In this paper, a nonempty set X equipped with a topology τ and an m-structure m_X is called a mixed-space and is denoted by (X, τ, m_X). We define m_g-closed sets and m_g-open sets in (X, τ, m_X) and, by using m_g-open sets, we define mixed-regularity and mixed-normality on (X, τ, m_X). In Section 3, we obtain a sufficient condition for a mixed-regular space to be regular. As a special case of mixed-regular spaces, we obtain a characterization of $\mathcal{I}_g$-regular spaces [13]. In Section 4, we obtain several characterizations of mixed-normal spaces. We show that let m_X have property $\mathcal{B}$ and $\tau \subseteq m_X \subseteq \tau^\alpha$, then mixed-normal and normal are equivalent. By setting $m_X = SO(X, \tau)$, we obtain characterizations of s-normal spaces [8]. Moreover, we obtain some preservation theorems of mixed-normal spaces. Recently, papers [1–7] have introduced some new classes of sets via m-structures.

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2. MINIMAL STRUCTURES

Definition 2.1. A subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [17] on X if $\emptyset \in m$ and $X \in m$.

By (X, m) we denote a nonempty set X with a minimal structure m on X and call it an *m-space*. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*. For a point $x \in X$, the family $\{U : x \in U \text{ and } U \in m\}$ is denoted by $m(x)$.

Definition 2.2. Let (X, m) be an m-space and A a subset of X. The *m-closure* $mCl(A)$ and the *m-interior* $mInt(A)$ of A [12] are defined as follows:
(1) $mCl(A) = \cap\{F \subseteq X : A \subseteq F, X \setminus F \in m\}$,
(2) $mInt(A) = \cup\{U \subseteq X : U \subseteq A, U \in m\}$.

Lemma 2.3 ([12]). *Let X be a nonempty set and m a minimal structure on X. For subsets A and B of X, the following properties hold:*
(1) *$A \subset mCl(A)$ and $mCl(A) = A$ if A is m-closed,*
(2) *$mCl(\emptyset) = \emptyset$, $mCl(X) = X$,*
(3) *If $A \subseteq B$, then $mCl(A) \subseteq mCl(B)$,*
(4) *$mCl(A) \cup mCl(B) \subseteq mCl(A \cup B)$,*
(5) *$mCl(mCl(A)) = mCl(A)$,*
(6) *$mCl(X - A) = X - mInt(A)$.*

Definition 2.4. A minimal structure m of a set X is said to have *property $\mathcal{B}$* [12] if the union of any collection of elements of m is an element of m.

Lemma 2.5 ([17]). *Let (X, m) be an m-space and A a subset of X.*
(1) *$x \in mCl(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m(x)$.*
(2) *Let m have property $\mathcal{B}$. Then the following properties hold:*
  (i) *A is m-closed (resp. m-open) if and only if $mCl(A) = A$ (resp. $mInt(A) = A$)*
  (ii) *$mCl(A)$ is m-closed and $mInt(A)$ is m-open.*

3. GENERALIZATIONS OF REGULAR SPACES

By (X, τ, m_X) we denote a nonempty set X with a topology τ and an m-structure m_X on X and call it a mixed-space. In this section, we introduce and characterize the notion of mixed-regularity in a mixed space (X, τ, m_X).

Definition 3.1. Let (X, τ, m_X) be a mixed space. A subset A of X is said to be *m_g-closed* if $mCl(A) \subset U$ whenever $A \subset U$ and $U \in \tau$. A subset A of X is said to be *m_g-open* if the complement of A is *m_g-closed*.

In [16], m_g-closed (resp. m_g-open) sets are said to be *gm*-closed (resp. *gm*-open).

Proposition 3.2 ([16]). *A subset A of a a mixed space (X, τ, m_X) is m_g-open if and only if $F \subset mInt(A)$ whenever $F \subset A$ and F is closed.*

Definition 3.3. A mixed space (X, τ, m_X) is said to be mixed-regular if for each closed set A and each point $x \notin A$, there exist disjoint m_g-open sets U, V such that $x \in U$ and $A \subseteq V$.

Theorem 3.4. *For a mixed space (X, τ, m_X) such that m_X has property $\mathcal{B}$, the following properties are equivalent:*

(1) *X is mixed-regular.*
(2) *For any open set U containing $x \in X$, there exists an m_g-open set V such that $x \in V \subseteq mCl(V) \subseteq U$.*
(3) *For any closed set A, the intersection of all m_g-closed neighborhoods of A is A.*
(4) *For any set A and any open set B such that $A \cap B \neq \emptyset$, there exists an m_g-open set U such that $A \cap U \neq \emptyset$ and $mCl(U) \subseteq B$.*
(5) *For any nonempty set A and any closed set B such that $A \cap B = \emptyset$, there exist disjoint m_g-open sets U, V such that $A \cap U \neq \emptyset$ and $B \subseteq V$.*

Proof. (1) $\Rightarrow$ (2): Let U be an open set such that $x \in U$. Then $X - U$ is a closed set not containing x. By hypothesis, there exist disjoint m_g-open sets V, W such that $x \in V$ and $X - U \subseteq W$. By Proposition 3.2, $X - U \subseteq mInt(W)$ and so $X - mInt(W) \subseteq U$. Now $V \cap W = \emptyset$ implies that $V \cap mInt(W) = \emptyset$ and hence $mCl(V) \subseteq X - mInt(W)$. Thus, $x \in V \subseteq mCl(V) \subseteq U$.

(2) $\Rightarrow$ (3): Let A be any closed set and $x \notin A$. Then $X - A$ is an open set containing x. By hypothesis, there exists an m_g-open set V such that $x \in V \subseteq mCl(V) \subseteq X - A$. Thus, $A \subseteq X - mCl(V) \subseteq X - V$. Since $X - mCl(V)$ is m-open, then $X - V$ is an m_g-closed neighborhood of A and $x \notin X - V$. This shows that A is the intersection of all the m_g-closed neighborhoods of A.

(3) $\Rightarrow$ (4): Let A be any set and B be any open set such that $A \cap B \neq \emptyset$. Let $x \in A \cap B$. Then, $X - B$ is closed and $x \notin X - B$. By hypothesis, there exists an m_g-closed neighborhood V of $X - B$ such that $x \notin V$. Let $X - B \subseteq G \subseteq V$, where G is m-open. Then $U = X - V$ is an m_g-open set such that $x \in U$ and $A \cap U \neq \emptyset$. Furthermore $X - G$ is m-closed and $mCl(U) = mCl(X - V) \subseteq mCl(X - G) \subseteq B$.

(4) $\Rightarrow$ (5): Let A be any nonempty set and B any closed set such that $A \cap B = \emptyset$. Then $X - B$ is an open set and $A \cap (X - B) \neq \emptyset$. By hypothesis, there exists an m_g-open set U such that $A \cap U \neq \emptyset$ and $U \subseteq mCl(U) \subseteq X - B$. Let $V = X - mCl(U)$. Then U and V are disjoint m_g-open sets such that $B \subseteq X - mCl(U) = V$.

(5) $\Rightarrow$ (1): Let B be a closed set and $x \notin B$. Put $A = \{x\}$. Then, there exist disjoint m_g-open sets U, V such that $A \cap U \neq \emptyset$ and $B \subseteq V$, hence $x \in U$. Thus X is mixed-regular. □

An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}_g$-*regular* [13] if for each closed set A and each point $x \notin A$, there exist disjoint $\mathcal{I}_g$-open sets U, V such that $x \in U$ and $A \subseteq V$. By Theorem 3.4, if $m_X = \tau^*$ we obtain the following corollary.

Corollary 3.5 ([13]). *An ideal topological space $(X, \tau, \mathcal{I})$ is $\mathcal{I}_g$-regular if and only if for any open set V containing any $x \in X$, there exists an $\mathcal{I}_g$-open set U such that $x \in U \subseteq \mathrm{Cl}^\star(U) \subseteq V$.*

Definition 3.6. Let (X, m_X) be an m-space and A a subset of X.

(1) A is said to be *mg-closed* [15] if $\mathrm{mCl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in m$. The complement of an mg-closed set is said to be mg-open.

(2) X is *mg-regular* if for each m-closed set A and each point $x \notin A$, there exist disjoint mg-open sets U, V such that $x \in U$ and $A \subseteq V$.

Similarly with Theorem 3.4 we obtain the following corollary.

Corollary 3.7. *For an m-space (X, m), the following properties are equivalent:*

(1) *X is mg-regular.*
(2) *For any m-open set U containing $x \in X$, there exists an mg-open set V such that $x \in V \subseteq mCl(V) \subseteq U$.*
(3) *For any m-closed set A, the intersection of all mg-closed neighborhoods of A is A.*
(4) *For any set A and any m-open set B such that $A \cap B \neq \emptyset$, there exists an mg-open set U such that $A \cap U \neq \emptyset$ and $mCl(U) \subseteq B$.*
(5) *For any nonempty set A and any m-closed set B such that $A \cap B = \emptyset$, there exist disjoint mg-open sets U, V such that $A \cap U \neq \emptyset$ and $B \subseteq V$.*

A topological space (X, τ) is said to be *g-regular* if for each closed set A and each point $x \notin A$, there exist disjoint g-open sets U, V such that $x \in U$ and $A \subseteq V$. In Corollary 3.7, put $m_X = \tau$ (topology), then we obtain the following corollary.

Corollary 3.8. *For a topological space (X, τ), the following properties are equivalent:*

(1) *X is g-regular.*
(2) *For any open set U containing $x \in X$, there exists a g-open set V such that $x \in V \subseteq Cl(V) \subseteq U$.*
(3) *For any closed set A, the intersection of all g-closed neighborhoods of A is A.*
(4) *For any set A and any open set B such that $A \cap B \neq \emptyset$, there exists a g-open set U such that $A \cap U \neq \emptyset$ and $Cl(U) \subseteq B$.*
(5) *For any nonempty set A and any closed set B such that $A \cap B = \emptyset$, there exist disjoint g-open sets U, V such that $A \cap U \neq \emptyset$ and $B \subseteq V$.*

Theorem 3.9. *If a a mixed space (X, τ, m_X) is a mixed-regular, T_1-space such that $m_X \subseteq \tau^\alpha$ and m_X has property $\mathcal{B}$. Then X is regular space.*

Proof. Let B be a closed set not containing $x \in X$. By Theorem 3.4, there exists an m_g-open set U of X such that $x \in U \subseteq mCl(U) \subseteq X - B$. Since X is a T_1-space, $\{x\}$ is closed and so $\{x\} \subseteq mInt(U)$, by Proposition 3.2. Since $m_X \subseteq \tau^\alpha$ and m_X has property $\mathcal{B}$ and so $mInt(U)$ and $X - mCl(U)$ are τ^α-open sets. Now $mInt(U) \subseteq Int(Cl(Int(mInt(U)))) = G$ and $B \subseteq X - mCl(U) \subseteq Int(Cl(Int(X - mCl(U)))) = H$. Then G and H are disjoint open sets containing x and B, respectively. Therefore, X is regular. □

THEOREM 3.10. *If every open subset of a mixed space (X, τ, m_X) is m-closed, then (X, τ, m_X) is mixed-regular.*

Proof. Suppose every open subset of X is m-closed. Let $A \subseteq X$ and U be any open set containing A. Then U is m-closed and $mCl(A) \subseteq mCl(U) = U$. Hence every subset of X is m_g-closed and every subset of X is m_g-open. If B is a closed set not containing x, then $\{x\}$ and B are the required disjoint m_g-open sets containing x and B, respectively. Therefore, (X, τ, m_X) is mixed-regular. □

THEOREM 3.11. *Let (X, τ, m_X) be a mixed space such that m_X has property $\mathcal{B}$ and $\tau \subseteq m_X \subseteq \tau^\alpha$. Then the following properties are equivalent:*

(1) *X is regular.*
(2) *For every closed set A and each $x \notin A$, there exist disjoint m-open sets U and V such that $x \in U$ and $A \subseteq V$.*
(3) *For every open set V of X and $x \in V$, there exists an m-open set U such that $x \in U \subseteq mCl(U) \subseteq V$.*

Proof. (1) ⇒ (2): Let A be a closed subset of X and let $x \in X - A$. Then there exist disjoint open sets U and V such that $x \in U$ and $A \subseteq V$. But every open set is m-open. Then there exist disjoint m-open sets U and V such that $x \in U$ and $A \subseteq V$.

(2) ⇒ (3): Let V be an open set containing $x \in X$. Then $X - V$ is closed and $x \in V$. By hypothesis, there exist disjoint m-open sets U and W such that $x \in U$ and $X - V \subseteq W$. Since $U \cap W = \emptyset$, we have $U \subseteq X - W$ and $X - W$ is m-closed. So $mCl(U) \subseteq X - W \subseteq V$. Therefore, U is the required m-open set such that $x \in U \subseteq mCl(U) \subseteq V$.

(3) ⇒ (1): Let A be a closed set and $x \notin A$. By (3), there exists an m-open set U such that $x \in U \subseteq mCl(U) \subseteq X - A$. Let $V = X - mCl(U)$. Then $A \subseteq V$, and U and V are disjoint m-open sets. Since $m_X \subseteq \tau^\alpha$ and so U and V are τ^α-open sets. Therefore, $x \in U \subseteq Int(Cl(Int(U))) = G$ and $A \subseteq V \subseteq Int(Cl(Int(V))) = H$. Then G and H are disjoint open sets such that $x \in G$ and $A \subseteq H$. Hence X is regular. □

COROLLARY 3.12 ([14]). *For a topological space (X, τ), the following properties are equivalent:*

(1) *X is regular.*

(2) *For every closed set A and each $x \notin A$, there exist disjoint α-open sets U and V such that $x \in U$ and $A \subseteq V$.*
(3) *For every open set V of X and $x \in V$, there exists an α-open set U such that $x \in U \subseteq Cl_\alpha(U) \subseteq V$.*

Proof. Let $m_X = \tau^\alpha$ in Theorem 3.11. Since $\tau \subseteq \tau^\alpha$, the corollary easily follows from Theorem 3.11. □

DEFINITION 3.13. A function $f : (X, \tau, m_X) \to (Y, \sigma, n_Y)$ is said to be *mg*-closed (resp. *mg*-open) if for every closed (resp. open) set F of X, $f(F)$ is n_g-closed (resp. n_g-open) in Y.

THEOREM 3.14. *Let $f : (X, \tau, m_X) \to (Y, \sigma, n_Y)$ be a continuous, mg-open and mg-closed surjection. If X is regular, then Y is mixed-regular.*

Proof. Let $y \in Y$ and V be any open set containing y. Take a point $x \in f^{-1}(y)$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is open in X. By the regularity of X, there exists an open set U of X such that $x \in U \subseteq Cl(U) \subseteq f^{-1}(V)$. Then $y \in f(U) \subseteq f(Cl(U)) \subseteq V$ and $f(U)$ is n_g-open and $f(Cl(U))$ is n_g-closed in Y. Therefore, we obtain $y \in f(U) \subseteq nCl(f(U)) \subseteq nCl(f(Cl(U))) \subseteq V$. It follows from Theorem 3.4 that Y is mixed-regular. □

4. GENERALIZATIONS OF NORMAL SPACES

DEFINITION 4.1. A mixed space (X, τ, m_X) is said to be mixed-normal if for every pair of disjoint closed sets A and B, there exist disjoint m_g-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

THEOREM 4.2. *For a mixed space (X, τ, m_X) such that m_X has property $\mathcal{B}$, the following properties are equivalent:*

(1) *X is mixed-normal.*
(2) *For every pair of disjoint closed sets A and B, there exist disjoint m-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
(3) *For any closed set A and an open set V containing A, there exists an m-open set U such that $A \subseteq U \subseteq mCl(U) \subseteq V$.*
(4) *For any closed set A and an open set V containing A, there exists an m_g-open set U such that $A \subseteq U \subseteq mCl(U) \subseteq V$.*

Proof. (1) ⇒ (2): For every pair of disjoint closed sets A and B, there exist disjoint m_g-open sets U and V such that $A \subseteq U$ and $B \subseteq V$. By Proposition 3.2, $A \subseteq mInt(U)$ and $B \subseteq mInt(V)$. Since m_X has property $\mathcal{B}$, mInt(U) and mInt(V) are m-open and disjoint.

(2) ⇒ (3): Let A be a closed set and V be an open set containing A. Since A and $X - V$ are disjoint closed sets, there exist disjoint m-open sets U and W such that $A \subseteq U$ and $X - V \subseteq W$. Again, $U \cap W = \emptyset$ implies that $mCl(U) \cap W = \emptyset$ and hence $mCl(U) \subseteq X - W$. Thus, we have $A \subseteq U \subseteq mCl(U) \subseteq X - W \subseteq V$.

(3) ⇒ (4): The proof is obvious.

(4) ⇒ (1): Let A and B be two disjoint closed subset of X. By hypothesis, there exists an m_g-open set U such that $A \subseteq U \subseteq mCl(U) \subseteq X - B$. Let $W = X - mCl(U)$. Since every m-open set is m_g-open, U and W are the required disjoint m_g-open sets containing A and B, respectively. Therefore, (X, τ, m_X) is mixed-normal. □

A subset A of a topological space (X, τ) is said to be semi-open [9] if $A \subseteq Cl(Int(A))$. Let $SO(X, \tau)$ be the family of all semi-oppen sets of (X, τ). Then $SO(X, \tau)$ is an m-structure with property $\mathcal{B}$ and by setting $m_X = SO(X, \tau)$, we obtain the following characterizations due to Arya and Nour [8]. A topological space (X, τ) is said to be s-normal [11] if for any pair of disjoint closed sets A and B of X, there exist disjoint semi-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Corollary 4.3 ([8]). *For a topological space (X, τ), the following properties are equivalent:*

(1) *X is s-normal.*
(2) *For every pair of disjoint closed sets A and B, there exist disjoint gs-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
(3) *For any closed set A and an open set V containing A, there exists an gs-open set U such that $A \subseteq U \subseteq sCl(U) \subseteq V$.*
(4) *For any closed set A and a g-open set B containing A, there exists a semi-open set U such that $A \subseteq U \subseteq sCl(U) \subseteq Int(B)$.*
(5) *For any g-closed set A and an open set B containing A, there exists a semi-open set U such that $A \subseteq sCl(A) \subseteq U \subseteq sCl(U) \subseteq B$.*

Proof. Let $m_X = SO(X, \tau)$, then an m_g-closed set is gs-closed. Therefore, the proof follows from Proposition 3.2 and Theorem 4.2. □

Theorem 4.4. *For a mixed space (X, τ, m_X) such that m_X has property $\mathcal{B}$ and $\tau \subseteq m_X \subseteq \tau^\alpha$, the following properties are equivalent:*

(1) *X is normal.*
(2) *X is mixed-normal.*

Proof. (1) ⇒ (2): Since $\tau \subseteq m_X$, the proof follows from Theorem 4.2.

(2) ⇒ (1): Let A and B be two disjoint closed subsets of X. By Theorem 4.2, there exist disjoint m-open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Since $m_X \subseteq \tau^\alpha$, U and V are τ^α-open sets. Hence $A \subseteq U \subseteq Int(Cl(Int(U))) = G$ and $B \subseteq V \subseteq Int(Cl(Int(V))) = H$. Then G and H are the required disjoint open sets containing A and B, respectively. Hence X is normal. □

Corollary 4.5. *For a topological space (X, τ), the following properties are equivalent:*

(1) *X is normal.*
(2) *For any disjoint closed sets A and B, there exist disjoint g-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

(3) *For any closed set A and an open set V containing A, there exist an g-open set U such that $A \subseteq U \subseteq Cl(U) \subseteq V$.*

Proof. In Theorem 4.4, set $m_X = \tau$. Then the proof is obvious by Theorem 4.2. □

COROLLARY 4.6. *For a mixed space (X, τ, m_X) such that m_X has property $\mathcal{B}$, the following properties hold:*
(1) *If $m_X \subseteq \tau^\alpha$, then every mixed-normal space is normal.*
(2) *If $\tau \subseteq m_X$, then every normal space is mixed-normal.*

THEOREM 4.7. *Let a mixed space (X, τ, m_X) be mixed-normal. If F is closed and A is a g-closed set such that $A \cap F = \emptyset$, then there exist disjoint m_g-open sets U and V such that $A \subseteq U$ and $F \subseteq V$.*

Proof. Since $A \cap F = \emptyset$, $A \subseteq X - F$ where $X - F$ is open. Therefore, by hypothesis, $Cl(A) \subseteq X - F$. Since $Cl(A) \cap F = \emptyset$ and X is mixed-normal, there exist disjoint m_g-open sets U and V such that $A \subseteq Cl(A) \subseteq U$ and $F \subseteq V$. □

THEOREM 4.8. *For a mixed space (X, τ, m_X), mixed-normality implies the following equivalent properties:*
(1) *For every closed set A and every g-open set B containing A, there exists an m_g-open set U such that $A \subseteq mInt(U) \subseteq U \subseteq B$.*
(2) *For every g-closed set A and every open set B containing A, there exists an m_g-closed set U such that $A \subseteq U \subseteq mCl(U) \subseteq B$.*

Proof. First, we show that mixed-normailty implies (1). Let A be a closed set and B be a g-open set containing A. Then $A \cap (X - B) = \emptyset$, where A is closed and $X - B$ is g-closed. By Theorem 4.7, there exist disjoint m_g-open sets U and V such that $A \subseteq U$ and $X - B \subseteq V$. Since $U \cap V = \emptyset$, we have $U \subseteq X - V$. By Proposition 3.2, $A \subseteq mInt(U)$. Therefore, $A \subseteq mInt(U) \subseteq U \subseteq X - V \subseteq B$.

(1) ⇒ (2): Let A be a g-closed set and B be an open set containing A. Then $X - B$ is a closed set contained in the g-open set $X - A$. By (1), there exists an m_g-open set V such that $X - B \subseteq mInt(V) \subseteq V \subseteq X - A$. Therefore, $A \subseteq X - V \subseteq mCl(X - V) \subseteq B$. Let $U = X - V$, then $A \subseteq U \subseteq mCl(U) \subseteq B$ and U is the required m_g-closed set.

(2) ⇒ (1): Let A be any closed set and B be any g-open set containing A. Then $X - B$ is a g-closed set contained in the open set $X - A$. By (2), there exists an m_g-closed set V such that $X - B \subseteq V \subseteq mCl(V) \subseteq X - A$. Therefore, $A \subseteq X - mCl(V) \subseteq X - V \subseteq B$. Let $U = X - V$, then $A \subseteq mInt(U) \subseteq U \subseteq B$ and U is the required m_g-open set. □

DEFINITION 4.9. A function $f : (X, \tau, m_X) \to (Y, \sigma, n_Y)$ is said to be m_g-closed if for every m_g-closed set F of X, $f(F)$ is closed in Y.

THEOREM 4.10. *If a function $f : (X, \tau, m_X) \to (Y, \sigma, n_Y)$ is a continuous m_g-closed surjection and X is mixed-normal, then Y is normal.*

Proof. Let A and B be any disjoint closed sets of Y. Since f is continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of X. Since X is mixed-normal, there exist disjoint m_g-open sets U and V of X such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Put $H = Y - f(X - U)$ and $G = Y - f(X - V)$, then H and G are open sets in Y, $A \subseteq H$ and $B \subseteq G$. Since $U \cap V = \emptyset$ and f is surjective, we have $H \cap G = \emptyset$. This shows that Y is normal. □

DEFINITION 4.11. A function $f : (X, \tau, m_X) \to (Y, \sigma, n_Y)$ is said to be m_g-continuous if $f^{-1}(A)$ is m_g-open in X for every n_g-open in Y.

THEOREM 4.12. *If a function $f : (X, \tau, m_X) \to (Y, \sigma, n_Y)$ is closed m_g-continuous injection. If Y is mixed-normal, then X is mixed-normal.*

Proof. Let A and B be any disjoint closed sets of X. Since f is a closed injection, $f(A)$ and $f(B)$ are disjoint closed sets of Y. By mixed-normality of Y, there exist disjoint n_g-open sets U and V of Y such that $f(A) \subseteq U$ and $f(B) \subseteq V$. Since f is m_g-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint m_g-open sets in X and $A \subseteq f^{-1}(U)$ and $B \subseteq f^{-1}(V)$. This shows that X is mixed-normal. □

REFERENCES

[1] A. Al-Omari and T. Noiri, *On operators in ideal minimal spaces*, Mathematica, **58** (81) (2016), 3–13.

[2] A. Al-Omari and H. Al-saadi, *A topology via ω-local functions in ideal spaces*, Mathematica, **60** (83) (2018), 103–110.

[3] A. Al-Omari and T. Noiri, *Generalizations of regular and normal spaces*, Annales Univ. Sci. Budapest., **61** (2018), 121–135.

[4] A. Al-Omari and T. Noiri, *Operators in minimal spaces with hereditary classes*, Mathematica, **61** (84) (2019), 101–110.

[5] A. Al-Omari, H. Al-saadi and T. Noiri, *On extremally disconnected spaces via m-structures*, Commun. Korean Math. Soc., **34** (2019), 351–359.

[6] H. Al-Saadi and A. Al-Omari, *Some operators in ideal topological spaces*, Missouri J. Math. Sci., **30** (2018), 59–71.

[7] H. Al-saadi, A. Al-Omari and T. Noiri, *On hyperconnected spaces via m-structures*, Ital. J. Pure Appl. Math., **42**, (2019), 290–300.

[8] S.P. Arya and T.M. Nour, *Characterizations of s-normal spaces*, Indian J. Pure Appl. Math., **21** (1990), 717–719.

[9] N. Levine, *Semi-open sets and semi-continuity in topological spaces,* Amer. Math. Monthly, **70** (1963), 36–41.

[10] N. Levine, *Generalized closed sets in topology,* Rend. Circ. Mat. Palermo (2), **19** (1970), 89–96.

[11] S.N. Maheshwari and R. Prasad, *On s-normal spaces*, Bull. Math. Soc. Sci. Math. R. S. Roumanie, **20**(**70**) (1978), 27–29.

[12] H. Maki, K.C. Rao and A. Nagoor Gani, *On generalizing semi-open and preopen sets*, Pure Appl. Math. Sci., **49** (1999), 17–29.

[13] M. Navaneethakrishnan, J. Paulraj Joseph and D. Sivaraj, *I_g-normal and I_g-regular spaces*, Acta Math. Hungar., **125** (2009), 327–340.

[14] T. Noiri, *Almost αg-closed functions and separation axioms*, Acta Math. Hungar., **82** (1999), 193–205.
[15] T. Noiri, *A unified theory for modifications of g-closed sets*, Rend. Circ. Mat. Palermo (2), **56** (2007), 171–184.
[16] T. Noiri, *A unified theory for certain modifications of generalized closed sets*, Int. J. Gen. Top., **1** (2008), 87–99.
[17] V. Popa and T. Noiri, *On M-continuous functions*, An. Univ. "Dunărea de Jos" Galaţi, Ser. Mat. Fiz. Mec. Teor. (2), **43** (**23**) (2000), 31–41.
[18] J. Sanabria, E. Rosas, C. Carpintero and M. Salas-Brown, *On the further unified theory of ideal generalized closed sets*, J. Adv. Math. Stud., **4** (2011), 83–96.

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