MATHEMATICA, Tome 57 (80), N° 1–2, 2015, pp. 104–116

WITT OVERGROUPS FOR UNIPOTENT ELEMENTS IN EXCEPTIONAL ALGEBRAIC GROUPS OF BAD CHARACTERISTIC

IULIAN I. SIMION

Abstract. Let G be a simple exceptional algebraic group defined over an algebraically closed field of bad characteristic. The decompositions as a product of Witt groups of the connected component of the double centralizer $Z(C_G(u))^\circ$ for unipotent elements u is given up to isogeny. For type G_2, F_4 and E_6 minimal dimensional connected overgroups for unipotent elements are constructed in G whenever $u \in C_G(u)^\circ$.

MSC 2010. 20G15, 17B45.

Key words. Exceptional algebraic group of bad characteristic, Witt group, isogeny.

1. INTRODUCTION

Let G be a simple algebraic group defined over an algebraically closed field k of characteristic p and let $\mathfrak{g} = \mathrm{Lie}(G)$ be its Lie algebra. The characteristic is called bad for G if it equals 2 when G is not of type A_n, 3 when G is of exceptional type or 5 when G is of type E_8. In all other cases of positive characteristic we say that p is good for G.

The problems addressed in this article originate from the Jacobson-Morozov theorem, a fundamental result in the theory of complex semisimple Lie algebras. The theorem states that, when $k = \mathbb{C}$, any nilpotent element $e \in \mathfrak{g}$ lies in an $\mathfrak{sl}_2$-subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$. For each e, the subalgebra $\mathfrak{a}$ was shown to be unique up to conjugation by the stabilizer $C_G(e)$ in the adjoint action of G on $\mathfrak{g}$ [5]. The extent to which these results go through when $p > 0$ is described in [18].

In the general case of positive characteristic much work has been done on generalizing different aspects of the aforementioned results. If $p = 0$ then $\mathfrak{a}$ is the Lie algebra of a subgroup $A \subseteq G$ and the maximal torus of A gives a useful grading on $\mathfrak{g}$. These aspects were extended to good positive characteristic by A. Premet [8] (see also [3, §5.1]).

From a different perspective, the correspondence between nilpotent orbits in $\mathfrak{g}$ and unipotent conjugacy classes in G raises the question of the existence

The author was supported by the Swiss National Science Foundation through grants 200021-122267 and 200020-135144 as well as by the grant CPDR131579/13 of Padova University.

and properties of A_1-type overgroups for unipotent elements $u \in G$. These problems were investigated when p is good for G in [17, 10]. The order of u is an obvious obstruction for the existence of such overgroups. If $u^p = 1$ then there exists a subgroup $A \ni u$ of type A_1 [17] and in particular a 1-dimensional unipotent group $u \in U \subseteq A$ (the unipotent radical of a Borel subgroup of A). Moreover A and U have several good properties, e.g. if $u \neq 1$ then $C_G(u) = C_G(U) = C_G(\mathrm{Lie}(U))$ [10].

It is natural to ask if there exist overgroups W for u when $u^p \neq 1$ with similar properties. This led to the analysis of minimal closed connected abelian subgroups $W \subseteq G$ containing u. The structure of commutative connected unipotent groups [12, VII§2] shows that the smallest candidate for such a subgroup containing u is isogenous to a Witt group (definitions given in §2) and has dimension d where the order of u is p^d.

If each non-trivial unipotent element u has order p then the 1-dimensional subgroup containing it is sometimes called 'the saturation' of u. Loosely speaking, there are unipotent elements of order greater than p if 'the group absorbs the additive structure of the ground field'.

R. Proud showed that in good characteristic there exists a subgroup W of G containing u isomorphic to a Witt group of minimal possible dimension [9]. However there are several choices for these overgroups and no canonical one is known. Without the condition of minimality, a canonical choice for W is the connected component of the double centralizer $C_G(C_G(u)) = Z(C_G(u))$. In [11] it is shown that $Z(C_G(u))^\circ$ decomposes as a direct sum of Witt groups where one of the summands contains u. Properties of the double centralizers are obtained in [6] where in particular a dimension formula is deduced.

In bad characteristic the situation changes dramatically since most tools used for the above mentioned results are not available (a short survey on this is given in [15]). Since in this setting there exist unipotent elements u with $u \notin C_G(u)^\circ$, it is clear that there will not always exist connected abelian overgroups for unipotent elements. In this paper we consider the case where G is a simple exceptional algebraic group and p is bad for G. We are interested in the following questions concerning Witt subgroups of G. For a unipotent element $u \in G$:

(1) What is the decomposition of $Z(C_G(u))^\circ$ as a product of Witt groups (up to isogeny)?
(2) Does there exist a Witt subgroup $W \subseteq G$ containing u?

A description of double centralizers of unipotent elements for exceptional algebraic groups in bad characteristic is given in [13, 14]. The methods there allow for an explicit calculation of $C_G(C_G(u))^\circ$. More precisely, for each unipotent conjugacy class represented by an element u, a parametrization of a maximal connected unipotent subgroup of $C_G(u)$ was calculated (with one exception: when G is of type F_4 and u is in the unipotent class $C_3(a_1)$) as well as parametrizations of $Z(C_G(u))^\circ$ in all cases. These parametrizations are topological embeddings which can be used together with Lemma 3.1 (see §3) to give an answer to the first question for all exceptional algebraic groups in bad characteristic – see Theorem 3.1. This extends the analysis in [11].

That the results in [11] do not extend to our setting can be seen for example from the unipotent class A_2 in E_6 if $p = 2$. In this case a representative u is distinguished in an A_2 subsystem subgroup so by [9] there exists a connected abelian overgroup W for u (in H and therefore in G). However since by [13] $u \notin Z(C_G(u))^\circ$ it follows that $W \not\subset Z(C_G(u))^\circ$. So there might exist a Witt subgroup W containing u even if $u \notin Z(C_G(u))^\circ$ and one cannot expect in general that minimal dimensional overgroups W of u have the property $C_G(u) = C_G(W)$. In addition, since Witt groups are connected abelian it is clear that if $u \notin C_G(u)^\circ$ then no Witt overgroup of u exists in G. All cases for which $u \notin C_G(u)^\circ$ are given in [7, Corollary 4] or [13, 14]. For all other unipotent classes, methods similar to the ones in [9] for handling the exceptional type groups can be used to construct Witt overgroups $W \subseteq G$ (up to isogeny) for certain representatives u. We use [1] for such calculations and obtain the desired subgroups when G is of type G_2, F_4 and E_6 in Proposition 4.2.

2. PRELIMINARIES

The ring of Witt vectors was first described in [19]. We will only be concerned with the additive structure of this ring which we recall here.

Definition 2.1. Let p be a prime. For m-tuples $a = (a_0, \dots, a_{m-1})$, $b = (b_0, \dots, b_{m-1}) \in \mathbb{Z}^m$ the *Witt sum* $a \oplus b = (c_0, \dots, c_{m-1})$ is defined as follows

$$c_0 = a_0 + b_0, \quad c_k = a_k + b_k + \frac{1}{p^k}\left[\sum_{i=0}^{k-1} p^i (a_i^{p^{k-i}} + b_i^{p^{k-i}} - c_i^{p^{k-i}})\right].$$

One checks that the c_k's are polynomials in the a_i's and b_i's with coefficients in $\mathbb{Z}$ (see for example [2] or [4]).

Let k be a field of characteristic p. The m-dimensional *Witt group* is the algebraic group $W_m(k) = (k^m, \oplus)$, where the group structure is given by the above polynomials and as algebraic varieties k^m is the affine space $\mathbb{A}_m(k)$. If it is clear what the field k is then we write W_m for $W_m(k)$.

Example 2.1. For W_2 we have

$$(a_1, a_2) \oplus (b_1, b_2) = (a_1 + b_1, a_2 + b_2 + \frac{1}{p}\,[a_1^p + b_1^p - (a_1 + b_1)^p])$$

and it is easy to realize these groups as subgroups of $\mathrm{GL}_{p+1}(k)$. If

$$\phi(a_1,a_2)=\begin{bmatrix} 1 & a_1 & \frac{a_1^2}{2!} & \dots & \frac{a_1^{p-1}}{(p-1)!} & a_2 \\ 0 & 1 & a_1 & \frac{a_1^2}{2!} & \dots & \frac{a_1^{p-1}}{(p-1)!} \\ 0 & 0 & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ \vdots & & & & 1 & a_1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

one checks that $\phi(a_1,a_2)\phi(b_1,b_2)=(a_1+b_1,a_2+b_2+\sum_{i=1}^{p-1}\frac{a_1^i}{i!}\frac{b_1^{p-i}}{(p-i)!})$. Since $(p-1)!=-1$ in k (of characteristic p) we recover the Witt sum.

An exposition on Witt groups (as algebraic groups) can be found in [12, Chapter VII] from where we take the following definitions and result.

Definition 2.2. An *isogeny* is a surjective algebraic group homomorphism with finite kernel. For commutative connected unipotent groups G_1 and G_2 the existence of an isogeny $G_1 \to G_2$ is equivalent to the existence of an isogeny $G_2 \to G_1$ by [12, Theorem VII.10.1]. If these equivalent conditions are satisfied we call the two groups *isogenous.*

Theorem 2.1. [12, Theorem VII.10.1] *Every commutative connected unipotent group is isogenous to a product of Witt groups.*

In what follows we use additive notation for Witt groups and for connected abelian unipotent groups. Following the discussion in [12, pg. 172] we identify the subgroup of elements of order less than or equal to p^i of W_n with $W_i = p^{n-i}W_n$.

Definition 2.3. For a commutative unipotent group H the *period* of H is defined to be the smallest power of p such that $p^nH=1$. It is the exponent of H. We denote it by $\mathrm{per}(H)$. In particular, $\mathrm{per}(W_n)=p^{\dim(W_n)}$.

Since Witt groups are connected and unipotent any subgroup $W \subseteq G$ isogenous to a Witt group lies in U, the unipotent radical of some Borel subgroup B. Moreover, since W is abelian it lies in the unipotent radical of some Borel subgroup of $C_G(u)$ for each $u \in W$. For each unipotent conjugacy class one can choose a representative u in U explicitly in such a way that the unipotent radical of a Borel subgroup of $C_G(u)$ is $C_U(u)^\circ$ [13, 14, Proposition B].

Moreover, fixing a maximal torus T in B (and therefore in G) a parametrization $\phi_u : \mathbb{A}_d \to C_U(u)^\circ$ can be expressed with respect to the root subgroups determined by T. If Φ^+ is the set of positive roots determined by T and B then $U=\prod_{\alpha\in\Phi^+}U_\alpha$ for a fixed (but arbitrary) ordering of the roots and where U_α is the root group corresponding to the root α for which we have an isomorphism of algebraic groups $\mathbf{u}_\alpha : (k,+) \to U_\alpha \subset G$. With this notation, ϕ_u has the form

$$\phi_u : k^d \ni \mathrm{z} = (\mathrm{z}_1,\dots,\mathrm{z}_d) \mapsto \prod_{\alpha\in\Phi^+} \mathbf{u}_\alpha(P_\alpha(\mathrm{z})),$$

where $P_\alpha(z)$ are polynomials in z with coefficients in $\mathbb{Z}$. For our calculations, the ordering of the roots is the one used in [1], and if α is the i-th root, we write $\mathbf{u}_i$ instead of $\mathbf{u}_\alpha$ for brevity as in [13, 14]. Recall also that for U we have Chevalley's commutator formula which is extensively used in *loc.cit.* in order to obtain the maps ϕ_u as well as similar parametrizations for $Z(C_G(u))^\circ$.

3. DECOMPOSITION OF DOUBLE CENTRALIZERS OF UNIPOTENT ELEMENTS

In this section, we treat the decomposition of the connected component of $Z(C_G(u))^\circ$ for a unipotent element u in an exceptional algebraic group G when p is bad for G. The following proposition allows us to make use of the parametrization of $Z(C_G(u))^\circ$ described in [13, 14].

LEMMA 3.1. *Let H be a connected abelian unipotent group with* $\mathrm{per}(H) = p^n$ *and denote by d_i the dimension of $p^{n-i}H$ for $0 \le i \le n$. Then H is isogenous to the product $\prod_{i=1}^n W_{n-i+1}^{a_i}$ where $a_i = d_i + d_{i-2} - 2d_{i-1}$.*

Proof. Suppose that $H = \prod_{j=1}^n W_{n-j+1}^{b_j}$. Under this assumption the decomposition of H is determined by the b_j so it is equivalent to

$$\forall 0 \le i \le n,\ d_i = \dim p^{n-i}H = \dim \prod_{j=1}^n (p^{n-i}W_{n-j+1})^{b_j} = \sum_{j=1}^n b_j \dim(W_{i-j+1})$$

with the convention that $W_l = 0$ if $l \le 0$. The non-zero terms in the above sum are those for which $j < i+1$ so $d_i = \sum_{j=1}^i (i-j+1)b_j$. It is not difficult to see that this is equivalent to $b_j = d_j + d_{j-2} - 2d_{j-1}$ with $d_l = 0$ for all $l \le 0$.

Now, if H is isogenous to a product of Witt groups $W = \prod_{j=1}^n W_{n-j+1}^{a_j}$, choose an isogeny $\phi : H \to W$. Since $\phi(p^m H) = p^m\phi(H)$ and ϕ is surjective the restriction of ϕ to $p^m H$ is itself an isogeny from $p^m H$ to $p^m W$. In particular $\dim p^m H = \dim p^m W$ for all $m \in \mathbb{N}$. Since these dimensions determine W uniquely they determine H uniquely up to isogeny. □

EXAMPLE 3.1. In order to compute $p^i Z(C_U(\tilde{u}))^\circ$ we can proceed as follows. Suppose $\mathrm{char}(k) = 2$ and consider the case of the unipotent class D_5 in E_6. A representative is $\tilde{u} = \mathbf{u}_9(-1)\mathbf{u}_2(1)\mathbf{u}_{10}(1)\mathbf{u}_7(1)\mathbf{u}_6(1)$ [13, Table 6] and a calculation shows that $Z(C_G(\tilde{u}))^\circ$ has dimension 4 and that it has a parametrization $\phi : k^4 \to U$ given by

$$\begin{aligned}\phi(\mathrm{z}) \quad = \quad & \mathbf{u}_{13}(\mathrm{z}_1)\mathbf{u}_{14}(\mathrm{z}_1)\mathbf{u}_{16}(\mathrm{z}_1)\mathbf{u}_{18}(\mathrm{z}_1)\mathbf{u}_{23}(\mathrm{z}_1)\mathbf{u}_{27}(\mathrm{z}_1+\mathrm{z}_1^2)\mathbf{u}_{28}(\mathrm{z}_2)\\ & \cdot\mathbf{u}_{29}(\mathrm{z}_2+\mathrm{z}_1+\mathrm{z}_1^2)\mathbf{u}_{32}(\mathrm{z}_3)\mathbf{u}_{33}(\mathrm{z}_3+\mathrm{z}_1+\mathrm{z}_1^2)\mathbf{u}_{35}(\mathrm{z}_3+\mathrm{z}_1+\mathrm{z}_1^2)\mathbf{u}_{36}(\mathrm{z}_4).\end{aligned}$$

Calculating, we obtain

$$\phi(z)^2 = \mathbf{u}_{28}(z_1^2)\mathbf{u}_{29}(z_1^2)\mathbf{u}_{32}(z_1^2)\mathbf{u}_{33}(z_1^2)\mathbf{u}_{35}(z_1^2)\mathbf{u}_{36}(z_1^2)$$

and $\phi(z)^{2^2} = 1$ which shows that $2Z(C_G(\tilde{u}))^\circ$ is 1-dimensional and that $2^2Z(C_G(\tilde{u}))^\circ = 1$. So, in this case, by Lemma 3.1, $Z(C_G(\tilde{u}))^\circ$ is isogenous to $W_2W_1^2$.

Treating all unipotent conjugacy classes case-by-case we obtain:

THEOREM 3.1. *Let G be an exceptional algebraic group and suppose that the characteristic p of k is bad for G. For a unipotent element $u \in G$, the connected component of the double centralizer $Z(C_G(u))^\circ$ is isogenous to the product of Witt groups given in Tables* 1 *to* 5.

Class	$p=2$	$p=3$
E_7	$W_3W_2W_1^2$	$W_2^2W_1^3$
$E_7(a_1)$	$W_2^2W_1^2$	$W_2W_1^4$
$E_7(a_2)$	$W_2^2W_1$	$W_2W_1^3$
$E_7(a_3)$	$W_2W_1^2$	$W_2W_1^2$
E_6	W_2^2	$W_2W_1^2$
$E_6(a_1)$	W_2W_1	W_1^3
D_6	$W_2W_1^2$	$W_2W_1^2$
$E_7(a_4)$	W_1^3	W_1^3
$D_6(a_1)$	W_1^4	W_1^4
D_5A_1	W_1^3	W_1^3
A_6	W_1^2	W_1^2
$E_7(a_5)$	W_1^2	W_1^2
D_5	W_1^3	W_1^3
$E_6(a_3)$	W_1^2	W_1^2
$D_6(a_2)$	W_1^3	W_1^3
$D_5(a_1)A_1$	W_1^2	W_1^2
A_5A_1	W_1^2	W_1^2
A_5'	W_1^2	W_1^2
A_4A_2	W_1	W_1
$D_5(a_1)$	W_1^2	W_1^2
A_4A_1	W_1	W_1
D_4A_1	W_2	W_1^2
A_5''	W_2W_1	W_2W_1

Class	$p=2$	$p=3$
$A_3A_2A_1$	W_1	W_1
A_4	W_2	W_2
$A_3A_2^{(2)}$	W_1^2	–
A_3A_2	W_1	W_2
$D_4(a_1)A_1$	W_1^2	W_2
D_4	W_2	W_1^2
$A_3A_1^2$	W_1^2	W_2
$D_4(a_1)$	W_1	W_1
$(A_3A_1)'$	W_1^2	W_2
$A_2^2A_1$	W_1	W_1
$(A_3A_1)''$	W_1^2	W_2
$A_2A_1^3$	W_1	W_1
A_2^2	W_1	W_1
A_3	W_1^2	W_2
$A_2A_1^2$	W_1	W_1
A_2A_1	W_1	W_1
A_1^4	W_1	W_1
A_2	W_1	W_1
$(A_1^3)'$	W_1	W_1
$(A_1^3)''$	W_1	W_1
A_1^2	W_1	W_1
A_1	W_1	W_1

Table 1 – Witt decomposition E_7.

Class	$p=2$	$p=3$
F_4	W_2^2	$W_1^2W_2$
$F_4(a_1)$	W_1^4	W_1^3
$F_4(a_2)$	W_1^2	W_1^2
C_3	W_1W_2	W_1^2
B_3	W_1W_2	W_1^2
$F_4(a_3)$	W_1^2	W_1
$C_3(a_1)^{(2)}$	W_1^4	–
$C_3(a_1)$	W_1^3	W_2
$\tilde{A}_2A_1^{(2)}$	W_1^2	–
$\tilde{A}_2A_1$	W_1	W_1
$B_2^{(2)}$	W_1^3	–
B_2	W_1^2	W_2
$A_2\tilde{A}_1$	W_1	W_1
$\tilde{A}_2$	W_1	W_1
A_2	W_1	W_1
$A_1\tilde{A}_1$	W_1^2	W_1
$\tilde{A}_1^{(2)}$	W_1^2	–
$\tilde{A}_1$	W_1	W_1
A_1	W_1	W_1

Class	$p=2$	$p=3$
E_6	$W_1W_2W_3$	$W_1^4W_2$
$E_6(a_1)$	W_2W_3	W_1^5
D_5	$W_1^2W_2$	W_1^4
$E_6(a_3)$	W_1W_2	W_1^3
$D_5(a_1)$	W_1W_2	W_1^3
A_5	W_1W_2	W_1^3
A_4A_1	W_2	W_1^2
D_4	W_2	W_1^2
A_4	W_1W_2	W_1W_2
$D_4(a_1)$	W_1	W_1
A_3A_1	W_1^2	W_2
$A_2^2A_1$	W_1	W_1
A_3	W_1^2	W_2
$A_2A_1^2$	W_1	W_1
A_2^2	W_2	W_1^2
A_2A_1	W_2	W_1^2
A_2	W_1	W_1
A_1^3	W_1	W_1
A_1^2	W_1	W_1
A_1	W_1	W_1

Table 2 – Witt decomposition F_4 and E_6.

Class	$p=2$	$p=3$	$p=5$
E_8	$W_3W_2^2W_1$	$W_3W_1^5$	$W_2W_1^6$
$E_8(a_1)$	$W_3W_1^4$	$W_2W_1^5$	W_1^7
$E_8(a_2)$	$W_3W_1^3$	$W_2W_1^4$	W_1^6
$E_8(a_3)$	$W_3W_1^2$	$W_2W_1^3$	W_1^5
$E_8(a_4)$	$W_2W_1^2$	$W_2W_1^2$	W_1^4
E_7	$W_3W_1^2$	$W_2W_1^3$	W_1^5
$E_8(b_4)$	$W_2W_1^2$	$W_2W_1^2$	W_1^4
$E_8(a_5)$	W_2W_1	W_2W_1	W_1^3
$E_7(a_1)$	$W_2W_1^3$	$W_2W_1^3$	W_1^5
$E_8(b_5)$	W_2W_1	W_2W_1	W_1^3
D_7	W_2W_1	W_2W_1	W_1^3
$E_8(a_6)$	W_2	W_2	W_1^2

Table 3 – Center of centralizer E_8.

Class	$p=2$	$p=3$	$p=5$
$E_7(a_2)$	$W_2W_1^2$	$W_2W_1^2$	W_1^4
E_6A_1	W_2W_1	W_2W_1	W_1^3
$(D_7(a_1))^{(2)}$	W_2W_1	–	–
$D_7(a_1)$	W_2W_1	W_2W_1	W_1^3
$E_8(b_6)$	W_2	W_1^2	W_1^2
$E_7(a_3)$	W_2W_1	W_2W_1	W_1^3
$E_6(a_1)A_1$	W_2	W_1^2	W_1^2
$A_7^{(3)}$	–	W_1^2	–
A_7	W_1^2	W_1	W_1^2
$D_7(a_2)$	W_1^2	W_1^2	W_1^2
E_6	W_2^2	$W_2W_1^2$	$W_2W_1^2$
D_6	W_2W_1	W_2W_1	W_1^3
$(D_5A_2)^{(2)}$	W_1^2	–	–
D_5A_2	W_1^2	W_1^2	W_1^2
$E_6(a_1)$	W_2W_1	W_1^3	W_2W_1
$E_7(a_4)$	W_1^3	W_1^3	W_2W_1
A_6A_1	W_1	W_1	W_1
$D_6(a_1)$	W_1^3	W_1^3	W_2W_1
A_6	W_1^2	W_1^2	W_2
$E_8(a_7)$	W_1	W_1	W_1
D_5A_1	W_1^3	W_1^3	W_2W_1
$E_7(a_5)$	W_1^2	W_1^2	W_2
$E_6(a_3)A_1$	W_1^2	W_1^2	W_2
$D_6(a_2)$	W_1^2	W_1^2	W_2
$D_5(a_1)A_2$	W_1^2	W_1^2	W_2
A_5A_1	W_1^2	W_1^2	W_2
A_4A_3	W_1	W_1	W_1
D_5	W_1^3	W_1^3	W_2W_1
$E_6(a_3)$	W_1^2	W_1^2	W_2
$(D_4A_2)^{(2)}$	W_1	–	–
D_4A_2	W_1	W_1^2	W_2

Class	$p=2$	$p=3$	$p=5$
$A_4A_2A_1$	W_1	W_1	W_1
$D_5(a_1)A_1$	W_1^2	W_1^2	W_2
A_5	W_1^2	W_1^2	W_2
A_4A_2	W_1	W_1	W_1
$A_4A_1^2$	W_1	W_1	W_1
$D_5(a_1)$	W_1^2	W_1^2	W_2
A_3^2	W_1	W_1	W_1
A_4A_1	W_1	W_1	W_1
$D_4(a_1)A_2$	W_1	W_1	W_1
D_4A_1	W_2	W_1^2	W_2
$A_3A_2A_1$	W_1	W_1	W_1
A_4	W_2	W_2	W_1^2
$(A_3A_2)^{(2)}$	W_1	–	–
A_3A_2	W_1	W_2	W_1^2
$D_4(a_1)A_1$	W_1	W_1	W_1
$A_3A_1^2$	W_1^2	W_2	W_1^2
$A_2^2A_1^2$	W_1	W_1	W_1
D_4	W_2	W_1^2	W_2
$D_4(a_1)$	W_1	W_1	W_1
A_3A_1	W_1^2	W_2	W_1^2
$A_2^2A_1$	W_1	W_1	W_1
A_2^2	W_1	W_1	W_1
$A_2A_1^3$	W_1	W_1	W_1
A_3	W_1^2	W_2	W_1^2
$A_2A_1^2$	W_1	W_1	W_1
A_2A_1	W_1	W_1	W_1
A_1^4	W_1	W_1	W_1
A_2	W_1	W_1	W_1
A_1^3	W_1	W_1	W_1
A_1^2	W_1	W_1	W_1
A_1	W_1	W_1	W_1

Table 4 – Witt decomposition E_8 (continued).

Class	$p=2$	$p=3$
G_2	W_2	W_1^2
$G_2(a_1)$	W_1	W_1^2
$\tilde{A}_1^{(3)}$	–	W_1^2
$\tilde{A}_1$	W_1	W_1
A_1	W_1	W_1

Table 5 – Witt decomposition G_2.

4. CONSTRUCTING WITT OVERGROUPS

In this section, we are interested in the existence of minimal dimensional connected abelian overgroups W for unipotent elements $u \in G$. As before, U is the unipotent radical of a Borel subgroup B and $u \in U$. Let p^d be the order of u. If there is a subgroup W of G containing u and isogeneous to a Witt group then it lies in $C_G(u)^\circ$. In particular a necessary condition for the existence of such an overgroup is $u \in C_G(u)^\circ$. But is it sufficient, i.e., can we choose W to be isogenous to W_d?

We can replace u by a conjugate $\tilde{u}$ such that $C_B(\tilde{u})^\circ$ is a Borel subgroup of $C_G(u)$ [13, 14]. Since by assumption W is connected and unipotent, it will lie in some Borel subgroup of $C_G(\tilde{u})$ which is conjugate (in $C_G(\tilde{u})$) to $C_B(\tilde{u})^\circ$ so the existence of W in G reduces to the existance of W in $C_U(\tilde{u})^\circ$. This shows that one can restrict to the case where u is distinguished. Indeed, W will have to be a subgroup of $C_G(T_u)$ for some maximal torus T_u of $C_G(u)$. Suppose this is not the case, then since $C_W(T_u)$ is a connected group containing u if $[W, T_u] \neq 1$ we get a contradiction with the minimality of W.

Below we treat those cases where $u \in G$ of type G_2, F_4 and E_6 and we consider all classes not just the distinguished ones. Before doing so we give one property which all connected abelian unipotent overgroups of certain u share. Recall that, for a parabolic subgroup $P = L \rtimes Q$ with unipotent radical Q, an element $u \in Q$ is called a *Richardson element* if the P-conjugacy class of u is dense in Q.

PROPOSITION 4.1. *Let $P = L \rtimes Q$ be a parabolic subgroup and suppose $u \in Q$ is a Richardson element of P. If $V \subseteq Q$ is a connected group containing u then $u^G \cap V$ is dense in V.*

Proof. There are a finite number of unipotent classes in G and V is irreducible so there exists a unipotent class v^G such that $\overline{v^G \cap V} = V$. By assumption $u \in V$ so $u \in \overline{v^G}$. We may assume that $v \in V$ otherwise $v^G \cap V$ is empty and contradicting $\overline{v^G \cap V} \neq \emptyset$.

Now, u is a Richardson element so $\overline{u^G \cap Q} = Q \supseteq V \ni v$ and therefore $v \in \overline{u^G}$. It follows that $\overline{v^G} = \overline{u^G}$ so $v^G = u^G$ since both v^G and u^G are open and dense in their closure [16, Lemma 2.3.3] $\square$

We make use of the parametrizations of $C_U(\tilde{u})^\circ$ obtained with the method in [13]. The order of u is p^d. The existence of a connected abelian unipotent group of period p^d is equivalent to the existence of morphism $w : W_d \to C_G(u)^\circ$ with $u \in \mathrm{Im}(w)$ and such that the following diagram commutes

$$\begin{array}{ccc} W_d \times W_d & \xrightarrow{\oplus} & W_d \cong \mathbb{A}_d \\ \downarrow{\scriptstyle w \times w} & & \downarrow{\scriptstyle w} \\ C_U(\tilde{u})^\circ \times C_U(\tilde{u})^\circ & \xrightarrow[\mu]{} & C_U(\tilde{u})^\circ \cong \mathbb{A}_n, \end{array}$$

where μ denotes the multiplication in G. Note that the commutativity of the above diagram implies the commutativity of the diagrams obtained by restricting to the subgroups $p^i W_d$ for all $i \le d$.

In addition, we aim at constructing w such that $w(1,0,\ldots,0) = u$. The isomorphism $C_U(\tilde{u})^\circ \cong \mathbb{A}_n$ allows us to identify the affine algebra of $C_U(\tilde{u})^\circ$ with $k[\mathrm{y}_1,\ldots,\mathrm{y}_n]$. In this setting, the existence of w is equivalent to the existence of a solution for the system

$$\begin{cases} (\oplus \circ w)^* \, \mathrm{y}_i = [(w \times w) \circ \mu]^* \, \mathrm{y}_i \\ \text{with `boundary condition' } w(1,0,\ldots,0) = u, \end{cases}$$

where, for a morphism of algebraic verieties ϕ, ϕ^* is the comorphism. Finding all solutions to such a system can be difficult however we are only interested in finding one solution.

Example 4.1. The $C_3(a_1)$ class in F_4 for $p = 2$ has to be treated separately because this is the only class where we do not have a parametrization of $C_U(\tilde{u})^\circ$. The representative of the class is as in [13, §5.6] where it is shown that the ideal of $C_U(\tilde{u})^\circ$ is $(I, \mathrm{x}_9^2 + \mathrm{x}_{11}^2 + \mathrm{x}_9)$ with

$$I = (\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_5, \mathrm{x}_6, \mathrm{x}_8, \mathrm{x}_{11} + \mathrm{x}_{12}, \mathrm{x}_9 + \mathrm{x}_{15}, \mathrm{x}_{13} + \mathrm{x}_{18}, \mathrm{x}_{16} + \mathrm{x}_{20}).$$

and where the x_i are coordinates on U w.r.t the positive roots. Consider the ideal $J = (I, \mathrm{x}_9^2 + \mathrm{x}_{11}^2 + \mathrm{x}_9, \mathrm{x}_3, \mathrm{x}_4, \mathrm{x}_7, \mathrm{x}_{14}, \mathrm{x}_{17}, \mathrm{x}_{18}, \mathrm{x}_{19}, \mathrm{x}_{20}, \mathrm{x}_{22}, \mathrm{x}_{23}, \mathrm{x}_9 + \mathrm{x}_{10})$ which defines a subvariety V of U and the map $w : W_2 \to V$ given by

$$\begin{aligned} w(\mathrm{t}_1, \mathrm{t}_2) &= \mathbf{u}_9(\mathrm{t}_1^2)\mathbf{u}_{10}(\mathrm{t}_1^2)\mathbf{u}_{11}(\mathrm{t}_1^2 + \mathrm{t}_1)\mathbf{u}_{12}(\mathrm{t}_1^2 + \mathrm{t}_1)\mathbf{u}_{15}(\mathrm{t}_1^2) \cdot \\ &\quad \mathbf{u}_{21}(\mathrm{t}_1^3 + \mathrm{t}_2)\mathbf{u}_{24}(\mathrm{t}_1^6 + \mathrm{t}_2^2). \end{aligned}$$

One checks that

$$\begin{aligned} w(\mathrm{t}_1, \mathrm{t}_2) w(\mathrm{s}_1, \mathrm{s}_2) &= \mathbf{u}_9(\mathrm{s}_1^2 + \mathrm{t}_1^2)\mathbf{u}_{10}(\mathrm{s}_1^2 + \mathrm{t}_1^2)\mathbf{u}_{11}(\mathrm{s}_1 + \mathrm{t}_1 + \mathrm{s}_1^2 + \mathrm{t}_1^2) \cdot \\ &\quad \mathbf{u}_{12}(\mathrm{s}_1 + \mathrm{t}_1 + \mathrm{s}_1^2 + \mathrm{t}_1^2)\mathbf{u}_{15}(\mathrm{s}_1^2 + \mathrm{t}_1^2) \cdot \\ &\quad \mathbf{u}_{21}(\mathrm{s}_2 + \mathrm{t}_2 + \mathrm{t}_1\,\mathrm{s}_1 + \mathrm{s}_1^3 + \mathrm{t}_1\,\mathrm{s}_1^2 + \mathrm{t}_1^2\,\mathrm{s}_1 + \mathrm{t}_1^3) \cdot \\ &\quad \mathbf{u}_{24}(\mathrm{s}_2^2 + \mathrm{t}_2^2 + \mathrm{t}_1^2\,\mathrm{s}_1^2 + \mathrm{s}_1^6 + \mathrm{t}_1^2\,\mathrm{s}_1^4 + \mathrm{t}_1^4\,\mathrm{s}_1^2 + \mathrm{t}_1^6) \\ &= w((\mathrm{t}_1, \mathrm{t}_2) \oplus (\mathrm{s}_1, \mathrm{s}_2)). \end{aligned}$$

So w is a homomorphism with 2-dimensional image. By [12, Proposition VII.9], the image of w is isogenous to a Witt group of dimension 2.

We now turn to the main result of this section.

Proposition 4.2. *Let G be of type G_2, F_4 or E_6. A unipotent element u of order p^d lies in $C_G(u)^\circ$ if and only if there exists a d-dimensional connected abelian unipotent group W containing it.*

Proof. If the characteristic is good, the result is given in [9]. If there is a d-dimensional connected unipotent group, then clearly $u \in C_G(u)^\circ$. For the other implication we consider the unipotent classes of G case-by-case using the representatives $\tilde{u}$ in [13, 14] and the method therein to obtain parametrizations of $C_U(\tilde{u})^\circ$. All cases where $\tilde{u}$ lies in $C_G(\tilde{u})^\circ$ are listed below and we give explicitly a homomorphism $w : W_d \to G$ such that $w(1, 0, \ldots, 0) = \tilde{u}$. The unipotent class $C_3(a_1)$ in F_4 when the characteristic is 2 was treated separately above. For the class $E_6(a_3)$ in characteristic 3 we have chosen a different representative $w(1, 0, \ldots, 0)$ given below. For the $D_4(a_1)$ and $D_5(a_1)$ classes it is not difficult to deduce representatives of the subregular elements in a Levi subgroup of type D_4 and D_5 respectively. For these two classes we give the construction with respect to the root system of these Levi subgroups (the numbering of the positive roots is, as before, the one used in [1]).

$\mathbf{G_2(a_1)}$

$\mathbf{p=2}$ $W = \mathbf{u}_2(\mathrm{t}_1)\mathbf{u}_5(\mathrm{t}_1)\mathbf{u}_6(\mathrm{t}_2)$

$\mathbf{p=3}$ $W = \mathbf{u}_2(\mathrm{t}_1)\mathbf{u}_5(\mathrm{t}_1)\mathbf{u}_6(2\,\mathrm{t}_1^2)$

$(\mathbf{\tilde{A}_1})^{(\mathbf{3})}$

$\mathbf{p=3}$ $W = \mathbf{u}_4(\mathrm{t}_1)\mathbf{u}_6(\mathrm{t}_1)$ diagonally embedded in $Z(C_G(\tilde{u}))^\circ$

$\mathbf{F_4(a_1)}$

$\mathbf{p=3}$ $W = \mathbf{u}_2(\mathrm{t}_1)\mathbf{u}_4(\mathrm{t}_1)\mathbf{u}_7(\mathrm{t}_1)\mathbf{u}_6(\mathrm{t}_1 + 2\,\mathrm{t}_1^2)\mathbf{u}_5(\mathrm{t}_1)\mathbf{u}_8(2\,\mathrm{t}_1 + \mathrm{t}_1^2)\mathbf{u}_9(\mathrm{t}_1 + 2\,\mathrm{t}_1^2)\mathbf{u}_{11}(2\,\mathrm{t}_2 +$
$+ 2\,\mathrm{t}_1^2 + \mathrm{t}_1^3)\mathbf{u}_{12}(2\,\mathrm{t}_1 + \mathrm{t}_1^2)\mathbf{u}_{14}(2\,\mathrm{t}_2)\mathbf{u}_{15}(2\,\mathrm{t}_2 + \mathrm{t}_1 + 2\,\mathrm{t}_1^2)\mathbf{u}_{16}(2\,\mathrm{t}_2 + 2\,\mathrm{t}_1 + 2\,\mathrm{t}_1^2 + 2\,\mathrm{t}_1^3)$
$\mathbf{u}_{17}(\mathrm{t}_1^2 + \mathrm{t}_1^3 + \mathrm{t}_1^4)\mathbf{u}_{18}(\mathrm{t}_2 + 2\,\mathrm{t}_1 + 2\,\mathrm{t}_1^2 + 2\,\mathrm{t}_1^4)\mathbf{u}_{19}(2\,\mathrm{t}_2 + 2\,\mathrm{t}_1 + \mathrm{t}_1^2 + 2\,\mathrm{t}_1^3 + \mathrm{t}_1^4)$
$\mathbf{u}_{20}(2\,\mathrm{t}_2 + \mathrm{t}_1 + 2\,\mathrm{t}_1^2 + \mathrm{t}_1^3 + 2\,\mathrm{t}_1^5)\mathbf{u}_{21}(2\,\mathrm{t}_2 + \mathrm{t}_1 + \mathrm{t}_1\,\mathrm{t}_2 + 2\,\mathrm{t}_1^2 + 2\,\mathrm{t}_1^2\,\mathrm{t}_2 + 2\,\mathrm{t}_1^4 + \mathrm{t}_1^5)$
$\mathbf{u}_{22}(\mathrm{t}_2 + 2\,\mathrm{t}_1 + \mathrm{t}_1^2 + 2\,\mathrm{t}_1^4 + \mathrm{t}_1^5)\mathbf{u}_{23}(\mathrm{t}_2^2 + 2\,\mathrm{t}_1\,\mathrm{t}_2 + \mathrm{t}_1^2 + 2\,\mathrm{t}_1^2\,\mathrm{t}_2 + 2\,\mathrm{t}_1^3 + 2\,\mathrm{t}_1^3\,\mathrm{t}_2 + \mathrm{t}_1^4 + 2\,\mathrm{t}_1^6)$
$\mathbf{u}_{24}(\mathrm{t}_2 + 2\,\mathrm{t}_1 + \mathrm{t}_1\,\mathrm{t}_2 + \mathrm{t}_1^2 + 2\,\mathrm{t}_1^2\,\mathrm{t}_2 + \mathrm{t}_1^3\,\mathrm{t}_2 + 2\,\mathrm{t}_1^4\,\mathrm{t}_2 + 2\,\mathrm{t}_1^5 + \mathrm{t}_1^7)$

$\mathbf{F_4(a_2)}$

$\mathbf{p=3}$ $W = \mathbf{u}_1(\mathrm{t}_1)\mathbf{u}_4(\mathrm{t}_1)\mathbf{u}_9(\mathrm{t}_1)\mathbf{u}_{10}(\mathrm{t}_1)\mathbf{u}_{11}(\mathrm{t}_1 + 2\,\mathrm{t}_1^2)\mathbf{u}_{12}(\mathrm{t}_1 + 2\,\mathrm{t}_1^2)\mathbf{u}_{15}(2\,\mathrm{t}_2 + 2\,\mathrm{t}_1 + \mathrm{t}_1^2)$
$\mathbf{u}_{19}(2\,\mathrm{t}_2 + 2\,\mathrm{t}_1 + 2\,\mathrm{t}_1^2 + 2\,\mathrm{t}_1^3)\mathbf{u}_{21}(2\,\mathrm{t}_2 + 2\,\mathrm{t}_1 + 2\,\mathrm{t}_1^2 + \mathrm{t}_1^3 + \mathrm{t}_1^4)\mathbf{u}_{24}(\mathrm{t}_2 + 2\,\mathrm{t}_1^4 + \mathrm{t}_1^5)$

$\mathbf{F_4(a_3)}$

$\mathbf{p=2}$ $W = \mathbf{u}_4(\mathrm{t}_1)\mathbf{u}_6(\mathrm{t}_1)\mathbf{u}_{10}(\mathrm{t}_1)\mathbf{u}_{16}(\mathrm{t}_2 + \mathrm{t}_1 + \mathrm{t}_1^2)\mathbf{u}_{18}(\mathrm{t}_1)\mathbf{u}_{20}(\mathrm{t}_2)\mathbf{u}_{22}(\mathrm{t}_2 + \mathrm{t}_1 + \mathrm{t}_1^2)$
$\mathbf{u}_{23}(\mathrm{t}_2 + \mathrm{t}_1 + \mathrm{t}_1^3)\mathbf{u}_{24}(\mathrm{t}_1\,\mathrm{t}_2)$

$\mathbf{p=3}$ $W = \mathbf{u}_4(\mathrm{t}_1)\mathbf{u}_6(\mathrm{t}_1)\mathbf{u}_{10}(\mathrm{t}_1)\mathbf{u}_{16}(2\,\mathrm{t}_1 + \mathrm{t}_1^2)\mathbf{u}_{18}(\mathrm{t}_1)\mathbf{u}_{20}(\mathrm{t}_1 + 2\,\mathrm{t}_1^2)\mathbf{u}_{22}(\mathrm{t}_1 + 2\,\mathrm{t}_1^2)$
$\mathbf{u}_{23}(\mathrm{t}_2 + \mathrm{t}_1 + 2\,\mathrm{t}_1^2)\mathbf{u}_{24}(\mathrm{t}_2 + \mathrm{t}_1^2 + 2\,\mathrm{t}_1^3)$

$\mathbf{C_3(a_1)^{(2)}}$

$\mathbf{p=2}$ $W = \mathbf{u}_9(\mathrm{t}_1)\mathbf{u}_{10}(\mathrm{t}_1)\mathbf{u}_{13}(\mathrm{t}_1 + \mathrm{t}_1^2)\mathbf{u}_{15}(\mathrm{t}_1)\mathbf{u}_{16}(\mathrm{t}_1)\mathbf{u}_{18}(\mathrm{t}_1 + \mathrm{t}_1^2)\mathbf{u}_{21}(\mathrm{t}_2)\mathbf{u}_{22}(\mathrm{t}_1 + \mathrm{t}_1^3)$
$\mathbf{u}_{23}(\mathrm{t}_2)\mathbf{u}_{24}(\mathrm{t}_2 + \mathrm{t}_1 + \mathrm{t}_1^3)$

$(\mathbf{\tilde{A}_2 A_1})^{(\mathbf{2})}$

$\mathbf{p=2}$ $W = \mathbf{u}_{11}(\mathrm{t}_1)\mathbf{u}_{12}(\mathrm{t}_1)\mathbf{u}_{16}(\mathrm{t}_1)\mathbf{u}_{18}(\mathrm{t}_1)\mathbf{u}_{21}(\mathrm{t}_2 + \mathrm{t}_1^2)\mathbf{u}_{24}(\mathrm{t}_2)$

$\mathbf{B_2^{(2)}}$

$\mathbf{p=2}$ $W = \mathbf{u}_9(\mathrm{t}_1)\mathbf{u}_{15}(\mathrm{t}_1)\mathbf{u}_{16}(\mathrm{t}_1)\mathbf{u}_{18}(\mathrm{t}_1 + \mathrm{t}_1^2)\mathbf{u}_{21}(\mathrm{t}_2 + \mathrm{t}_1^2)\mathbf{u}_{23}(\mathrm{t}_2)\mathbf{u}_{24}(\mathrm{t}_2 + \mathrm{t}_1^3)$

$\mathbf{\tilde{A}_1^{(2)}}$

$\mathbf{p=2}$ $W = \mathbf{u}_1(\mathrm{t}_1)\mathbf{u}_{20}(\mathrm{t}_1)$ diagonally embedded in $Z(C_G(\tilde{u}))^\circ$

$\mathbf{D_4(a_1)}$

$\mathbf{p=2}$ $W = \mathbf{u}_4(t_1)\mathbf{u}_1(t_1)\mathbf{u}_7(t_1)\mathbf{u}_6(t_1)\mathbf{u}_8(t_2)\mathbf{u}_9(t_2+t_1+t_1^2)\mathbf{u}_{10}(t_2+t_1+t_1^2)$
$\mathbf{u}_{11}(t_2+t_1+t_1^3)\mathbf{u}_{12}(t_2)$

$\mathbf{D_5(a_1)}$

$\mathbf{p=2}$ $W = \mathbf{u}_1(t_1)\mathbf{u}_9(t_1)\mathbf{u}_7(t_1)\mathbf{u}_6(t_1+t_1^2)\mathbf{u}_5(t_1)\mathbf{u}_4(t_1)\mathbf{u}_{10}(t_2+t_1+t_1^2)$
$\mathbf{u}_{11}(t_2+t_1+t_1^2)\mathbf{u}_{12}(t_2)\mathbf{u}_{13}(t_2)\mathbf{u}_{14}(t_1+t_1^3)\mathbf{u}_{15}(t_1^2+t_1^3)\mathbf{u}_{16}(t_2)\mathbf{u}_{17}(t_2+t_2^2)$
$\mathbf{u}_{18}(t_2+t_1^2+t_1^3)\mathbf{u}_{19}(t_3+t_2+t_1+t_2^2+t_1^4)\mathbf{u}_{20}(t_3+t_1 t_2+t_1^2 t_2+t_1^3+t_1^4)$

$\mathbf{E_6(a_1)}$

$\mathbf{p=2}$ $W = \mathbf{u}_1(t_1)\mathbf{u}_2(t_1)\mathbf{u}_9(t_1)\mathbf{u}_{10}(t_1)\mathbf{u}_7(t_1+t_1^2)\mathbf{u}_5(t_1)\mathbf{u}_6(t_1)\mathbf{u}_{11}(t_2)\mathbf{u}_{12}(t_2+t_1+t_1^2)$
$\mathbf{u}_{13}(t_2)\mathbf{u}_{14}(t_2+t_1+t_1^2)\mathbf{u}_{15}(t_2)\mathbf{u}_{16}(t_2+t_1+t_1^2)\mathbf{u}_{17}(t_2+t_1^2+t_1^3)\mathbf{u}_{18}(t_1^2+t_1^3)$
$\mathbf{u}_{19}(t_1^2+t_1^3)\mathbf{u}_{20}(t_1+t_1^3)\mathbf{u}_{21}(t_1^2+t_1^3)\mathbf{u}_{22}(t_2+t_2^2+t_1^3+t_1^4)\mathbf{u}_{23}(t_3+t_2+t_1 t_2+$
$+t_1^2 t_2+t_1^3+t_1^4)\mathbf{u}_{24}(t_2)\mathbf{u}_{25}(t_3+t_2^2+t_1^2+t_1^4)\mathbf{u}_{26}(t_3+t_2+t_2^2+t_1 t_2+t_1^2 t_2)$
$\mathbf{u}_{27}(t_3+t_2+t_1^2 t_2+t_1^3+t_1^3 t_2+t_1^5)\mathbf{u}_{28}(t_3+t_2+t_2^2+t_1 t_2+t_1^2 t_2+t_1^3+t_1^4)$
$\mathbf{u}_{29}(t_3+t_2^2+t_1^2 t_2+t_1^3+t_1^3 t_2+t_1^4)\mathbf{u}_{30}(t_3+t_2+t_1 t_2+t_1^2 t_2+t_1^3+t_1^5)$
$\mathbf{u}_{31}(t_3+t_2^2+t_1 t_2+t_1^3 t_2+t_1^4+t_1^5)\mathbf{u}_{32}(t_3+t_2^2+t_1 t_2^2+t_1^3+t_1^2 t_2^2+t_1^6)$
$\mathbf{u}_{33}(t_3+t_2^3+t_1 t_2^2+t_1^2 t_2+t_1^3+t_1^2 t_2^2+t_1^4 t_2+t_1^6)\mathbf{u}_{34}(t_3+t_2^3+t_1^2 t_2^2+t_1^3 t_2+$
$+t_1^3 t_2^2+t_1^4 t_2+t_1^6+t_1^7)\mathbf{u}_{35}(t_2 t_3+t_2^3+t_1 t_2^2+t_1^2 t_2^2+t_1^5+t_1^7)$
$\mathbf{u}_{36}(t_4+t_3^2+t_2 t_3+t_2^2+t_2^2 t_3+t_1 t_2 t_3+t_2^4+t_1 t_2^3+t_1^2 t_2 t_3+t_1^2 t_2^3+t_1^7+t_1^8)$

$\mathbf{p=3}$ $W = \mathbf{u}_1(t_1)\mathbf{u}_2(t_1)\mathbf{u}_9(t_1)\mathbf{u}_{10}(t_1)\mathbf{u}_5(t_1)\mathbf{u}_6(t_1)\mathbf{u}_{11}(t_1+2t_1^2)\mathbf{u}_{12}(t_1+2t_1^2)$
$\mathbf{u}_{13}(t_1+2t_1^2)\mathbf{u}_{14}(t_1+2t_1^2)\mathbf{u}_{15}(t_1+2t_1^2)\mathbf{u}_{16}(t_1+2t_1^2)\mathbf{u}_{17}(2t_2+t_1^2+2t_1^3)$
$\mathbf{u}_{18}(t_2+t_1+2t_1^3)\mathbf{u}_{19}(t_2+t_1+2t_1^3)\mathbf{u}_{20}(t_2+2t_1+t_1^3)\mathbf{u}_{21}(t_2+t_1+2t_1^3)$
$\mathbf{u}_{22}(t_2+t_1^2+2t_1^4)\mathbf{u}_{23}(t_1^2+2t_1^4)\mathbf{u}_{24}(t_2+t_1^2+2t_1^3)\mathbf{u}_{25}(2t_1+t_1^2+t_1^3+2t_1^4)$
$\mathbf{u}_{26}(t_2+2t_1^2+t_1^3)\mathbf{u}_{27}(t_1+t_1^2+t_1^3+2t_1^4+t_1^5)\mathbf{u}_{28}(t_2+2t_1^2+2t_1^3+2t_1^4)$
$\mathbf{u}_{29}(2t_2+2t_1+t_1^2+2t_1^3+2t_1^4+2t_1^5)\mathbf{u}_{30}(2t_1+t_1^2+2t_1^4+t_1^5)\mathbf{u}_{31}(2t_2+t_1^2+$
$+t_1^3+2t_1^4+2t_1^5)\mathbf{u}_{32}(t_2+t_1+2t_2^2+t_1 t_2+t_1^2+2t_1^3+2t_1^3 t_2+2t_1^4)$
$\mathbf{u}_{33}(t_2+2t_2^2+2t_1 t_2+t_1^2+t_1^3+t_1^3 t_2+t_1^4+2t_1^5+t_1^6)\mathbf{u}_{34}(2t_2+t_1+2t_2^2+$
$2t_1 t_2+t_1^2+t_1^3 t_2+t_1^4+t_1^6+2t_1^7)\mathbf{u}_{35}(t_2^2+t_1^2 t_2+t_1^4+2t_1^4 t_2+2t_1^5+2t_1^6+t_1^7)$
$\mathbf{u}_{36}(t_2+2t_1+t_2^2+t_1^2+t_1^2 t_2+t_1^3 t_2+t_1^4+t_1^4 t_2+t_1^6+t_1^7)$

$\mathbf{E_6(a_3)}$

$\mathbf{p=2}$ $W = \mathbf{u}_1(t_1)\mathbf{u}_8(t_1)\mathbf{u}_9(t_1)\mathbf{u}_{11}(t_1)\mathbf{u}_{12}(t_2)\mathbf{u}_{14}(t_1)\mathbf{u}_{18}(t_1+t_1^2)\mathbf{u}_{19}(t_1)\mathbf{u}_{20}(t_2+t_1+$
$+t_1^2)\mathbf{u}_{21}(t_2)\mathbf{u}_{22}(t_2+t_1+t_1^2)\mathbf{u}_{23}(t_1+t_1^3)\mathbf{u}_{24}(t_2+t_1+t_1^2)\mathbf{u}_{26}(t_1+t_1 t_2+t_1^3)$
$\mathbf{u}_{28}(t_2+t_1+t_1^3)\mathbf{u}_{29}(t_2+t_1+t_1 t_2+t_1^2)\mathbf{u}_{30}(t_3+t_1+t_1 t_2+t_1^2 t_2+t_1^3)\mathbf{u}_{31}(t_2)$
$\mathbf{u}_{33}(t_2+t_2^2+t_1^2+t_1^4)\mathbf{u}_{34}(t_3+t_1+t_2^2+t_1 t_2+t_1^2 t_2+t_1^4)\mathbf{u}_{35}(t_3+t_1+t_1 t_2+$
$+t_1 t_2^2+t_1^3 t_2+t_1^5)\mathbf{u}_{36}(t_2^2+t_1 t_2^2)$

$\mathbf{p=3}$ $v = \mathbf{u}_{13}(1)\mathbf{u}_1(1)\mathbf{u}_{15}(1)\mathbf{u}_6(1)\mathbf{u}_{14}(1)\mathbf{u}_4(1)\mathbf{u}_{17}(1)\mathbf{u}_{18}(2)\mathbf{u}_{20}(1)\mathbf{u}_{21}(2)\mathbf{u}_{36}(1)$
$W = \mathbf{u}_{13}(t_1)\mathbf{u}_1(t_1)\mathbf{u}_{15}(t_1)\mathbf{u}_6(t_1)\mathbf{u}_{14}(t_1)\mathbf{u}_4(t_1)\mathbf{u}_{17}(t_1^2)\mathbf{u}_{18}(2t_1^2)\mathbf{u}_{20}(t_1^2)$
$\mathbf{u}_{21}(2t_1^2)\mathbf{u}_{23}(t_2)\mathbf{u}_{29}(t_2)\mathbf{u}_{31}(2t_2)\mathbf{u}_{36}(t_1^5)$

□

REMARK 4.1. It appears reasonable to expect that, for all connected reductive algebraic groups G, a unipotent element $u \in G$ of order p^d lies in a d-dimensional connected (abelian) subgroup if and only if it lies in $C_G(u)^\circ$.

REFERENCES

[1] THE GAP GROUP, *GAP – Groups, Algorithms, and Programming*, **4.7.8**, 2015.
[2] JACOBSON, N., *Basic algebra. II*, W. H. Freeman and Co., San Francisco, Calif., 1980.

[3] Jantzen, J.C., *Nilpotent orbits in representation theory*, Progr. Math. **228**, Birkhäuser Boston, Boston, MA, 2004.

[4] Karpilovsky, G., *Topics in field theory*, North-Holland Mathematics Studies, **155**, North-Holland Publishing Co., Amsterdam, 1989.

[5] Kostant, B., *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math., **81** (1959), 973–1032.

[6] Lawther, R. and Testerman, D.M., *Centres of centralizers of unipotent elements in simple algebraic groups*, Mem. Amer. Math. Soc., **210** (2011), vi+188.

[7] Liebeck, M.W. and Seitz, G.M., *Unipotent and nilpotent classes in simple algebraic groups and Lie algebras*, Mathematical Surveys and Monographs, **180**, American Mathematical Society, Providence, RI, 2012.

[8] Premet, A., *Nilpotent orbits in good characteristic and the Kempf-Rousseau theory.*, J. Algebra, **260** (2003), 338–366.

[9] Proud, R., *Witt groups and unipotent elements in algebraic groups*, Proc. London Math. Soc. (3), **82** (2001), 647–675.

[10] Seitz, G.M., *Unipotent elements, tilting modules, and saturation*, Invent. Math., **141** (2000), 467–502.

[11] Seitz, G.M., *Unipotent centralizers in algebraic groups*, J. Algebra, **279** (2004), 226–259.

[12] Serre, J.-P., *Algebraic groups and class fields*, Graduate Texts in Mathematics **117**, Springer-Verlag, New York, 1988.

[13] Simion, I.I., *Double centralizers of unipotent elements in simple algebraic groups of type G_2, F_4 and E_6*, J. Algebra, **382** (2013), 335–367.

[14] Simion, I.I., *Double centralizers of unipotent elements in simple algebraic groups of type E_7 and E_8*, J. Pure Appl. Algebra, **219** (2015), 930–977.

[15] Simion, I.I. and Testerman, D.M., *Unipotent overgroups in simple algebraic groups*, In N.S. Narasimha Sastry, editor, *Groups of Exceptional Type, Coxeter Groups and Related Geometries*, Springer Proceedings in Mathematics & Statistics, Springer, 2014.

[16] Springer, T.A., *Linear algebraic groups*, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, Second Edition, 2009.

[17] Testerman, D.M., *A_1-type overgroups of elements of order p in semisimple algebraic groups and the associated finite groups*, J. Algebra, **177** (1995), 34–76.

[18] Thomas, R.A. and Stewart, I.D., *The Jacobson-Morozov Theorem and Complete Reducibility of Lie Subalgebras.* arXiv:1507.06234, 2015.

[19] Witt, E., *Zyklische Körper und Algebren der Charakteristik p vom Grad p^n. Struktur diskret bewerteter Körper mit vollkommenem Restklassenkörper der Charakteristic p*, J. Reine Angew. Math., **176** (1937), 126–140.

Received September 22, 2015
Accepted November 14, 2015

University of Padova
Department of Mathematics
Via Trieste 63
35121 Padova, Italy
E-mail: iulian.simion@math.unipd.it