MATHEMATICA, Tome 57 (80), N$^{\circ}$ 1-2, 2015, pp. 90–103

ON BESSEL-MAITLAND MATRIX FUNCTION

AYMAN SHEHATA and SUBUHI KHAN

Abstract. The main object of this paper is to consider the Bessel-Maitland matrix function in the following form:

$$\phi(A,B;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}\Gamma^{-1}(kA+B).$$

A different approach is adopted to study the radius of regularity, order and type of this function. Certain properties including integral representation and differential recurrence relations are also derived. Further, the composite Bessel-Maitland matrix function is introduced and its properties are discussed.

MSC 2010. 15A60, 33C05, 33C45, 34A05.

Key words. Hypergeometric matrix function, Bessel-Maitland matrix function, integral representation.

1. INTRODUCTION

An important generalization of special functions [7, 15] is special matrix functions and polynomials. The study of special matrix functions is important due to their applications in certain areas of statistics, physics and engineering, and often new perspectives in special functions are motivated by such connections, see for example [9] (for applications in statistics). Several special functions, called recently special functions of fractional calculus, play a very important and interesting role as solutions of fractional order differential equations [1, 16, 27], such as the Bessel-Maitland function with its auxiliary functions [5, 6, 8, 14]. It has been established that there is a close link between scalar polynomials satisfying higher order recurrence relations and orthogonal matrix polynomials. Keeping in view this fact, the matrix-valued counterparts of special functions such as special functions of matrix arguments and special functions with matrix parameters, have gained increasing interest. Constantine and Muirhead [3] studied the hypergeometric functions of two argument matrices. The special polynomials with matrix parameters provide the solutions of the corresponding matrix differential equations. These matrix differential equations are the systems of differential equations, each of

The author Ayman Shehata expresses his sincere appreciation to Mrs. Shimaa Ibrahim Moustafa Abdal-Rahman (Department of Mathematics, Faculty of Science, Assiut University), for her kind interest, encouragements, help, suggestions, and comments. The authors would like to thank the referees for their valuable comments and suggestions which have led to a better presentation in this study.

which is satisfied by the corresponding scalar special polynomial. In the same way the other results for special matrix polynomials like generating functions, series definitions, recurrence relations *etc.* can be viewed as the systems of equations, which are satisfied by the corresponding scalar special polynomials. Recently, the theory of special functions with matrix parameters is developed and their properties are also studied by various authors, see for example [10, 12, 13, 17, 18, 19, 20, 21, 22, 24].

Our main motivation to write this paper is to complement the results of shehata [23] and their interesting and useful properties in the future. In this paper, the matrix form of the function $\phi(\rho, \beta; z)$ is considered, which we shall call as Bessel-Maitland matrix function. The radius of regularity, order and type of this function are studied by adopting a different approach. Certain properties inluding integral representation and differential recurrence relations are also established. Further, the composite Bessel-Maitland matrix function is introduced and its properties are discussed.

1.1. **Preliminaries.** We review certain definitions and concepts related to elementary matrix functions.

Throughout this paper, for a matrix A in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A. Its two-norm will be denoted by $||A||$, and is defined by

$$||A|| = \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2}$$

where for a vector y in $\mathbb{C}^N$, $||y||_2 = (y^T y)^{\frac{1}{2}}$ is the Euclidean norm of y. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and if A, B are matrices in $\mathbb{C}^{N\times N}$ such that $\sigma(A) \subset \Omega$, $\sigma(B) \subset \Omega$ and $AB = BA$, then from the properties of the Riesz-Dunford functional calculus ([4], p.558), it follows that

$$f(A)g(B) = g(B)f(A). \tag{1.1}$$

The reciprocal gamma function denoted by $\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}$ is an entire function of the complex variable z. Then the image of A under the action of Γ^{-1} denoted by $\Gamma^{-1}(A)$ is a well-defined matrix. Further, if

(1.2) $A + nI$ is an invertible matrix for every non-negative integer n,

then $\Gamma(A)$ is an invertible matrix and its inverse coincides with $\Gamma^{-1}(A)$. Thus the formula

$$(A)(A+I)\dots(A+(n-1)I)\Gamma^{-1}(A+nI) = \Gamma^{-1}(A), \tag{1.3}$$

is well defined. From equation (1.1) and under condition (1.2), equation (1.3) takes the form

$$(A)(A+I)\cdots(A+(n-1)I) = \Gamma(A+nI)\,\Gamma^{-1}(A),\ n \geq 1. \tag{1.4}$$

Thus, for any matrix A in $\mathbb{C}^{N\times N}$, the Pochhammer symbol or shifted factorial is defined as:

$$\begin{aligned}(A)_n =& A(A+I)\cdots(A+(n-1)I)\\ =&\Gamma(A+nI)\Gamma^{-1}(A),\ n\geq 1;\\ (A)_0 =& I.\end{aligned}\tag{1.5}$$

A matrix A in $\mathbb{C}^{N\times N}$ is said to be positive stable matrix, if

$$\mathrm{Re}(\mu)\nleq 0,\ \mu\in\sigma(A),\ \sigma(A):=\ \textit{spectrum of}\ A.\tag{1.6}$$

For a positive stable matrix A in $\mathbb{C}^{N\times N}$ and an integer $n\geq 1$, we note that [11]:

$$\Gamma(A)=\lim_{n\longrightarrow\infty}(n-1)![(A)_n]^{-1}n^A\tag{1.7}$$

and also in accordance with [11], if A and B are positive stable matrices in $\mathbb{C}^{N\times N}$, then the gamma matrix function $\Gamma(A)$ and the beta matrix function $\mathbf{B}(A,B)$ are defined as:

$$\Gamma(A)=\int_0^\infty \mathrm{e}^{-t}t^{A-I}\mathrm{d}t;\quad t^{A-I}=\exp\Big((A-I)\ln t\Big)\tag{1.8}$$

and

$$\mathbf{B}(A,B)=\int_0^1 t^{A-I}(1-t)^{B-I}\mathrm{d}t,\tag{1.9}$$

respectively. Thus, if A and B are commuting positive stable matrices then

$$\mathbf{B}(A,B)=\mathbf{B}(B,A)$$

and commutativity is a necessary condition for the symmetry of the beta matrix function. Also, we have

$$\mathbf{B}(A,B)=\Gamma(A)\Gamma(B)\Gamma^{-1}(A+B),\tag{1.10}$$

where A and B are commuting matrices in $\mathbb{C}^{N\times N}$ such that A, B and $A+B$ are positive stable matrices.

In 1933, E. Maitland Wright introduced a generalization of the Bessel function $J_\nu(z)$ [15] in the form ([28], p.72 (1.3), [5, 6]):

$$\phi(z)=\phi(\rho,\beta;z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)\Gamma(\rho k+\beta)},\rho>-1,\beta\in\mathbb{C}.\tag{1.11}$$

Some important properties of this integral function of z, in particular its asymptotic expansion for large z are studied in [28]. These functions have the following relation with the Bessel function $J_\nu(z)$:

$$J_\nu(z)=\left(\frac{1}{2}z\right)^\nu\phi\left(1,\nu+1;-\frac{1}{4}z^2\right).\tag{1.12}$$

Thus, apart from a trivial factor, the Bessel function is a particular case of the ϕ-function.

Stirling's formula is an approximation for large factorials, precisely, (see [2])

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \tag{1.13}$$

in the sense that the ratio of the two sides tends to 1 as $n \to \infty$. It is easy to show that

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1. \tag{1.14}$$

Note that, for an integer n, the functional equation becomes $\Gamma(n) = (n-1)!$.

2. BESSEL-MAITLAND MATRIX FUNCTION

We consider the Bessel-Maitland matrix function denoted by $\phi(A, B; z)$, in the following form:

$$\phi(A, B; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}\Gamma^{-1}(kA + B) \tag{2.1}$$

where A and B are matrices in $\mathbb{C}^{N\times N}$ satisfying the conditions $\mathrm{Re}(a) > -1$ for all eigenvalues $a \in \sigma(A)$ and $\mathrm{Re}(b) > 0$ for all eigenvalues $b \in \sigma(B)$, and $kA + B$ is matrix in $\mathbb{C}^{N\times N}$ such that $kA + B$ is an invertible matrix for every integer $k \geq 0$.

Motivation here is a study of the order and type of the Bessel-Maitland matrix function is the link between the subjects of special functions and complex analysis. We establish an important property of the Bessel-Maitland matrix function by proving the following result:

Theorem 2.1. *Let A and B be matrices in $\mathbb{C}^{N\times N}$ satisfying the condition $\mathrm{Re}(a) > -1$ for all eigenvalues $a \in \sigma(A)$ and $\mathrm{Re}(b) > 0$ for all eigenvalues $b \in \sigma(B)$, and $kA + B$ is matrix in $\mathbb{C}^{N\times N}$ such that $kA + B$ is an invertible matrix for every integer $k \geq 0$. Then the Bessel-Maitland matrix function is an entire function of order $\rho \leq 1$ and type $\tau \leq 0$.*

Proof. Let

$$\phi(A, B; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}\Gamma^{-1}(kA + B) = \sum_{k=0}^{\infty} U_k z^k. \tag{2.2}$$

First, we show that the matrix power series (2.2) converges uniformly in any bounded domain of the complex variable z. From (2.2), we have the following inequality

$$\|\phi(A, B; z)\| \leq \left\| \sum_{k=0}^{\infty} \frac{z^k}{k!}\Gamma^{-1}(kA + B)\right\| \leq \sum_{k=0}^{\infty} \left\|\frac{\Gamma^{-1}(kA+B)z^k}{k!}\right\|.$$

Thus, the matrix power series (2.2) converges uniformly in any bounded domain of the complex variable z.

Now, the radius of regularity of the function $\phi(A,B;z)$ is given as:

$$\frac{1}{R}= \limsup_{k\to\infty}\left\|\left(U_k\right)^{\frac{1}{k}}\right\| = \limsup_{k\to\infty}\left\|\left(\frac{1}{k!}\Gamma^{-1}(kA+B)\right)^{\frac{1}{k}}\right\|,$$

that is,

$$\begin{aligned}\frac{1}{R} &= \limsup_{k\to\infty}\left\|\left(\frac{1}{\sqrt{2\pi k}(\frac{k}{\mathrm{e}})^k\ \sqrt{2\pi(kA+B-I)}(\frac{kA+B-I}{\mathrm{e}})^{kA+B-I}}\right)^{\frac{1}{k}}\right\| \\ &= \limsup_{k\to\infty}\left\|\left(\frac{1}{(2\pi\ k)^{\frac{1}{2k}}(\frac{k}{\mathrm{e}})\ (2\pi(kA+B-I))^{\frac{1}{2k}}(\frac{kA+B-I}{\mathrm{e}})^{\frac{kA+B-I}{k}}}\right)\right\| \le 0.\end{aligned} \tag{2.3}$$

Thus, the Bessel-Maitland matrix function is an entire function.

Next, we calculate the order of the Bessel-Maitland matrix function of the complex variable as follows it is shown in the following:

$$\rho= \limsup_{k\to\infty}\left\|\frac{k\ln(k)}{\ln(\frac{1}{U_k})}\right\| = \limsup_{k\to\infty}\left\|\frac{k\ln(k)}{\ln(k!\ \Gamma(kA+B))}\right\|,$$

which gives

$$\begin{aligned}\rho &= \limsup_{k\to\infty}\left\|\frac{k\ln(k)}{\ln(\sqrt{2\pi k}(\frac{k}{e})^k\ \sqrt{2\pi(kA+B-I)}(\frac{kA+B-I}{e})^{kA+B-I})}\right\| \\ &= \limsup_{k\to\infty}\left\|\frac{k\ln(k)}{E_1}\right\| = \limsup_{k\to\infty}\left\|\frac{1}{E_2}\right\|,\end{aligned}$$

where

$$\begin{aligned}E_1 &= \ln(\sqrt{2\pi k})I + \ln\left(\frac{k}{e}\right)^k I + \ln(\sqrt{2\pi(kA+B-I)}) \\ &\quad + \ln\left(\left(\frac{kA+B-I}{e}\right)^{kA+B-I}\right) \\ &= \frac{1}{2}\ln(2\pi k)I + k\ln\left(\frac{k}{e}\right)I + \frac{1}{2}\ln(2\pi(kA+B-I)) \\ &\quad + (kA+B-I)\ln\left(\frac{kA+B-I}{e}\right),\end{aligned}$$

$$\begin{aligned}E_2 &= \frac{\ln(2\pi k)}{2k\ln(k)}I + \frac{k\ln(k)}{k\ln(k)}I - \frac{k\ln(e)}{k\ln(k)}I + \frac{\ln(2\pi(kA+B-I))}{2k\ln(k)} \\ &\quad + \frac{(kA+B-I)\ln(kA+B-I)}{k\ln(k)} - \frac{(kA+B-I)\ln(e)}{k\ln(k)} \\ &= \frac{\ln(2\pi k)}{2k\ln(k)}I + \frac{k\ln(k)}{k\ln(k)}I - \frac{\ln(e)}{\ln(k)}I + \frac{\ln(2\pi(kA+B-I))}{2k\ln(k)} \\ &\quad + \frac{(kA+B-I)\ln(kA+B-I)}{k\ln(k)} - \frac{(kA+B-I)\ln(e)}{k\ln(k)}\end{aligned}$$

or, equivalently

$$\rho \leq \limsup_{k\to\infty} \left\| \frac{1}{I + \frac{(kA+B-I)\ln(kA+B-I)}{k\ln(k)}} \right\| \leq 1. \tag{2.4}$$

Further, the type of the Bessel-Maitland matrix function is formulated as follows:

$$\tau = \frac{1}{e\rho} \limsup_{k\to\infty} \left\| k\Big(U_k\Big)^{\frac{\rho}{k}} \right\| = \frac{1}{e\rho} \limsup_{k\to\infty} \left\| k\left(\frac{1}{k!\,\Gamma(kA+B))}\right)^{\frac{\rho}{k}} \right\|,$$

which gives

$$\begin{aligned}
\tau =& \frac{1}{e\rho} \limsup_{k\to\infty} \left\| k\left(\frac{1}{\sqrt{2\pi k}(\frac{k}{e})^k\ \sqrt{2\pi(kA+B-I)}(\frac{kA+B-I}{e})^{kA+B-I}}\right)^{\frac{\rho}{k}} \right\| \\
=& \frac{1}{e\rho} \limsup_{k\to\infty} \left\| \frac{k}{(2\pi k)^{\frac{\rho}{2k}}(\frac{k}{e})^{\frac{\rho}{k}}\ (2\pi(kA+B-I))^{\frac{\rho}{2k}}(\frac{kA+B-I}{e})^{\frac{\rho(kA+B-I)}{k}}} \right\| \\
=& \frac{1}{\rho} \limsup_{k\to\infty} \left\| \frac{1}{(2\pi k)^{\frac{\rho}{2k}}(\frac{k}{e})^{\frac{\rho}{k}-1}\ (2\pi(kA+B-I))^{\frac{\rho}{2k}}(\frac{kA+B-I}{e})^{\frac{\rho(kA+B-I)}{k}}} \right\|,
\end{aligned}$$

that is,

$$\tau \leq \frac{1}{\rho} \limsup_{k\to\infty} \left\| \frac{\frac{k}{e}}{(\frac{kA+B-I}{e})^{\frac{\rho(kA+B-I)}{k}}} \right\| \leq 0. \tag{2.5}$$

□

Next, we derive an integral representation for the Bessel-Maitland matrix function by proving the following result:

THEOREM 2.2. *Let A and B be matrices in $\mathbb{C}^{N\times N}$ satisfying the condition* $\mathrm{Re}(a) > -1$ *for all eigenvalues $a \in \sigma(A)$ and* $\mathrm{Re}(b) > 0$ *for all eigenvalues $b \in \sigma(B)$, and $kA+B$ is matrix in $\mathbb{C}^{N\times N}$ such that $kA+B$ is an invertible matrix for every integer $k \geq 0$. Then the Bessel-Maitland matrix function $\phi(A,B;z)$ defined by (2.1) for complex variable z has the following integral representation:*

$$\phi(A,B;z) = \frac{1}{2\pi \mathrm{i}} \int_C \exp(sI + zs^{-A})s^{-B}\,\mathrm{d}s. \tag{2.6}$$

Proof. Since the contour integral representation for the reciprocal gamma function is given as: (see [15], p. 115, No. (5.10.5))

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{e}^t t^{-z}\mathrm{d}t, \tag{2.7}$$

where C is the path encircling the origin in the positive direction, beginning at and returning to positive infinity with respect for the branch cut along the positive real axis.

Thus, from (2.7), we have

$$\Gamma^{-1}(kA+B)=\frac{1}{2\pi \mathrm{i}}\int_C \exp(s)s^{-(kA+B)}\mathrm{d}s. \tag{2.8}$$

Making use of equation (2.8) in series definition (2.1), we have

$$\phi(A,B;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}\frac{1}{2\pi \mathrm{i}}\int_C \exp(s)s^{-(kA+B)}\ \mathrm{d}s, \tag{2.9}$$

Now, interchanging the order of the integration and summation in the r.h.s. of the equation (2.9), we find

$$\phi(A,B;z)=\frac{1}{2\pi \mathrm{i}}\int_C \exp(s)s^{-B}\sum_{k=0}^{\infty}\frac{(zs^{-A})^k}{k!}\mathrm{d}s,$$

which on using exponential property yields assertion (2.6). □

In the following theorem, we derive another integral expression involving Bessel-Maitland matrix function $\phi(A,B;z)$.

THEOREM 2.3. *For* $\mathrm{Re}(a)$, $\mathrm{Re}(b)$, $\mathrm{Re}(c)$ *and* $\mu>0$ *the following integral expression involving Bessel-Maitland matrix function* $\phi(A,B;z)$ *holds true:*

$$\begin{aligned}&\int_0^{\infty}\mathrm{e}^{-az^{\mu}}z^{c-1}\ \phi(A,B;b\ z)\ \mathrm{d}z\\ &=\frac{1}{2\mu\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{b^k}{k!}\left(\frac{2}{a}\right)^{\frac{k+c}{\mu}}\Gamma\left(\frac{k+c}{2\mu}\right)\Gamma\left(\frac{k+c+\mu}{2\mu}\right)\Gamma^{-1}(kA+B).\end{aligned} \tag{2.10}$$

Proof. Denoting the l.h.s. of assersion (2.10) by $\mathbf{I}$ and using definition (2.1), we have

$$\begin{aligned}\mathbf{I}&=\int_0^{\infty}\mathrm{e}^{-az^{\mu}}\ z^{c-1}\phi(A,B;bz)\mathrm{d}z\\ &=\sum_{k=0}^{\infty}\frac{b^k}{k!}\Gamma^{-1}(kA+B)\int_0^{\infty}\mathrm{e}^{-az^{\mu}}\ z^{k+c-1}\mathrm{d}z,\end{aligned} \tag{2.11}$$

which on using integral (see [26])

$$\begin{aligned}\int_0^{\infty}\mathrm{e}^{-at^{\mu}}\ t^{n-1}\mathrm{d}t=&\frac{1}{\mu}\Gamma\left(\frac{n}{\mu}\right)\frac{1}{a^{\frac{n}{\mu}}},\ \mathrm{Re}(a)>0;\ \mathrm{Re}(n)>0;\\ &\left|\arg\left(\frac{n}{\mu}\right)\right|<\pi;\ \mu>0,\end{aligned} \tag{2.12}$$

takes the form

$$\mathbf{I}=\ \textstyle\sum_{k=0}^{\infty}\frac{b^k}{k!}\Gamma^{-1}(kA+B)\frac{1}{\mu}\Gamma\left(\frac{k+c}{\mu}\right)\frac{1}{a^{\frac{k+c}{\mu}}}\ .$$

Further, making use of the following well-known duplication formula for the gamma function [26],

$$\Gamma\Big(\frac{k+c}{\mu}\Big) = \frac{2^{\frac{k+c}{\mu}-1}}{\sqrt{\pi}}\Gamma\Big(\frac{k+c}{2\mu}\Big)\Gamma\Big(\frac{k+c+\mu}{2\mu}\Big), \tag{2.13}$$

we get assertion (2.10). □

Further, in order to derive the differential properties of the Bessel-Maitland matrix function $\phi(A,B;z)$, we prove in the following result:

THEOREM 2.4. *For* $|z| < \infty$; $|\arg z| < \pi$ *and* $n \in \mathbb{N}$, *the Bessel-Maitland matrix function* $\phi(A,B;z)$ *defined by* (2.1) *satisfies the following differential relations:*

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\,\phi(A,B;z) = \phi(A,B+nA;z), \tag{2.14}$$

$$\theta^{\underline{n}}\,\phi(A,B;z) = z^n\,\phi(A,B+nA;z) \tag{2.15}$$

and

$$\Big[z\frac{\mathrm{d}}{\mathrm{d}z}\Big(z\frac{\mathrm{d}}{\mathrm{d}z}-1\Big)\Big(z\frac{\mathrm{d}}{\mathrm{d}z}-2\Big)\ldots\Big(z\frac{\mathrm{d}}{\mathrm{d}z}-n+1\Big) - z^n\frac{\mathrm{d}^n}{\mathrm{d}z^n}\Big]\phi(A,B;z) = \boldsymbol{0}, \tag{2.16}$$

where $\theta = z\frac{\mathrm{d}}{\mathrm{d}z}$ *and* $\theta^{\underline{n}}$ *denotes the falling factorial defined by*

$$\theta^{\underline{n}} = \theta(\theta-1)(\theta-2)\cdots(\theta-(n-1)). \tag{2.17}$$

Proof. From definition (2.1), we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\,\phi(A,B;z) = \frac{\mathrm{d}}{\mathrm{d}z}\Big[\textstyle\sum_{k=0}^{\infty}\frac{z^k}{k!}\Gamma^{-1}(kA+B)\Big].$$

Differentiating the r.h.s. of the above equation, we find

$$\frac{\mathrm{d}}{\mathrm{d}z}\,\phi(A,B;z) = \sum_{k=1}^{\infty}\frac{z^{k-1}}{(k-1)!}\Gamma^{-1}(kA+B).$$

Now, setting $k = r+1$, in the r.h.s. of the above equation, we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\,\phi(A,B;z) = \sum_{r=0}^{\infty}\frac{z^r}{r!}\Gamma^{-1}(rA+A+B),$$

which, in view of definition (2.1), gives

$$\frac{\mathrm{d}}{\mathrm{d}z}\,\phi(A,B;z) = \phi(A,B+A;z). \tag{2.18}$$

Again, differentiating (2.18) with respect to z, we find

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\,\phi(A,B;z) = \frac{\mathrm{d}}{\mathrm{d}z}\Big[\,\phi(A,B+A;z)\Big].$$

Further, making use of (2.18) in the r.h.s. of the above equation, we get

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\,\phi(A,B;z) = \phi(A,B+2A;z).$$

Repeating the above process or by using mathematical induction, we get assertion (2.14).

Next, we prove assertion (2.15) by using mathematical induction.

In view of equations (2.17) and (2.18), we can write

$$\theta^{\underline{1}}\,\phi(A,B;z) = z\,\phi(A,B+A;z), \tag{2.19}$$

which shows that, the result is true for $n = 1$.

Let the result be true for some $n = k$, so that, we have

$$\theta^{\underline{k}}\,\phi(A,B;z) = z^k\,\phi(A,B+kA;z). \tag{2.20}$$

Since, in view of definition (2.17), we have

$$\theta^{\underline{k+1}} = \theta^{\underline{k}}\,(\theta - k).$$

Therefore, it follows that

$$\theta^{\underline{k+1}}\,\phi(A,B;z) = \theta^{\underline{k}}\,(\theta - k)\,\phi(A,B;z),$$

which on using equation (2.20) in the r.h.s. gives

$$\theta^{\underline{k+1}}\,\phi(A,B;z) = (\theta - k)z^k\,\phi(A,B+kA;z).$$

Now, in view of the fact that $\theta = z\frac{\mathrm{d}}{\mathrm{d}z}$ and using equation (2.18) in the r.h.s. of the above equation, we get

$$\begin{aligned}\theta^{\underline{k+1}}\,\phi(A,B;z) &= kz^k\,\phi(A,B+kA;z) + z^{k+1}\,\phi(A,B+(k+1)A;z)\\ &\quad -kz^k\,\phi(A,B+kA;z),\end{aligned}$$

which shows that the result is true for $n = k+1$. This proves assertion (2.15).

Finally, to prove assertion (2.16), we express equation (2.15) as:

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\underline{n}}\phi(A,B;z) = z^n\phi(A,B+nA;z), \tag{2.21}$$

which on using equation (2.14), in the r.h.s. gives

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\underline{n}}\phi(A,B;z) = z^n\frac{\mathrm{d}^n}{\mathrm{d}z^n}\phi(A,B;z). \tag{2.22}$$

Simplifying the above equation we get assertion (2.16). □

Theorem 2.5. *Let A and B be matrices in $\mathbb{C}^{N\times N}$ satisfying the conditions* $\mathrm{Re}(a) > -1$ *for all eigenvalues* $a \in \sigma(A)$ *and* $\mathrm{Re}(b) > 0$ *for all eigenvalues* $b \in \sigma(B)$, *and $kA+B$ is matrix in $\mathbb{C}^{N\times N}$ such that $kA+B$ is an invertible matrix for every integer $k \geq 0$. The derivative of the BesselMaitland matrix functions holds true:*

$$\phi(A,B;z) = B\phi(A,B+I;z) + Az\frac{\mathrm{d}}{\mathrm{d}z}\phi(A,B+I;z). \tag{2.23}$$

Proof. Using definition (2.1), we have

$$\begin{aligned}
&B\phi(A, B+I; z) + Az\frac{\mathrm{d}}{\mathrm{d}z}\phi(A, B+I; z) \\
&= B\sum_{k=0}^{\infty}\frac{1}{k!}\Gamma^{-1}(kA+B+I)z^k + Az\frac{\mathrm{d}}{\mathrm{d}z}\sum_{k=0}^{\infty}\frac{1}{k!}\Gamma^{-1}(kA+B+I)z^k \\
&= B\sum_{k=0}^{\infty}\frac{1}{k!}\Gamma^{-1}(kA+B+I)z^k + A\sum_{k=0}^{\infty}\frac{1}{k!}\Gamma^{-1}(kA+B+I)kz^k \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\Gamma^{-1}(kA+B+I)(kA+B)z^k = \phi(A, B; z).
\end{aligned}$$

Thus the proof is completed. □

THEOREM 2.6. *Let A, B and $B-I$ be matrices in $\mathbb{C}^{N\times N}$ satisfying the conditions* $\mathrm{Re}(a) > -1$ *for all eigenvalues $a \in \sigma(A)$ and* $\mathrm{Re}(b) > 0$ *for all eigenvalues $b \in \sigma(B)$, and $kA+B$ is matrix in $\mathbb{C}^{N\times N}$ such that $kA+B$ is an invertible matrix for every integer $k \geq 0$. Then the Bessel-Maitland matrix function satisfying the following pure matrix recurrence relations:*

$$B\phi(A, B; z) = \phi(A, B-I; z) - zA\phi(A, A+B; z). \tag{2.24}$$

Proof. From (2.19) and using the relation (2.23), the equation (2.24) follows directly. □

In the next section, we introduce the composite Bessel-Maitland matrix function and study some properties of this function.

3. COMPOSITE BESSEL-MAITLAND MATRIX FUNCTION

Suppose that

$$\phi(A_i, B_i; z_i) = \sum_{k_i=0}^{\infty}\frac{z_i^{k_i}\Gamma^{-1}(k_iA_i+B_i)}{k_i!},\ i = 1, 2, \ldots, s \tag{3.1}$$

are s Bessel-Maitland matrix functions, where A_1, A_2, $\cdots$, A_s and B_1, B_2, $\ldots$, B_s are matrices in $\mathbb{C}^{N\times N}$ satisfying the conditions $\mathrm{Re}(a_i) > -1$ for all eigenvalues $a_i \in \sigma(A_i)$ and $\mathrm{Re}(b_i) > 0$ for all eigenvalues $b_i \in \sigma(B_i)$ $(i = 1, 2, \ldots, s)$ and $k_1A_1+B_1$, $k_2A_2+B_2$, $\ldots$, $k_sA_s+B_s$ are matrices in $\mathbb{C}^{N\times N}$ with the condition that $k_iA_i+B_i$ are invertible matrices for all integers $k_i \geq 0$ $(i = 1, 2, \ldots, s)$. In order to introduce the composite Bessel-Maitland matrix function, which we shall denote by $\underline{\phi}(\underline{A}, \underline{B}; \underline{z})$, we use the following notation in accordance with [20]:

$$\underline{k} = (k_1, k_2, \ldots, k_s), \tag{3.2}$$

$$(\underline{k})! = k_1!k_2!\cdots k_s!, \tag{3.3}$$

$$(\underline{k}) = k_1 + k_2 + \cdots + k_s, \tag{3.4}$$

$$\underline{z}^{\underline{k}} = z_1^{k_1} z_2^{k_2} \cdots z_s^{k_s}, \tag{3.5}$$

$$\Gamma^{-1}(\underline{kA+B}) = \Gamma^{-1}(k_1A_1+B_1)\Gamma^{-1}(k_2A_2+B_2)\cdots\Gamma^{-1}(k_sA_s+B_s), \tag{3.6}$$

$$\underline{\phi(A,B;z)} = \big(\phi(A_1,B_1;z_1),\ \phi(A_2,B_2;z_2),\dots,\phi(A_s,B_s;z_s)\big). \tag{3.7}$$

In view of definition (2.1) and using notations (3.2)-(3.7), we construct the following matrix function:

$$\underline{\phi}(\underline{A},\underline{B};\underline{z}) = \sum_{\underline{k}=0}^{\infty} \frac{\underline{z}^{\underline{k}}\Gamma^{-1}(\underline{kA+B})}{(\underline{k})!}. \tag{3.8}$$

This matrix function is called the composite Bessel-Maitland matrix function of complex variables $z_1, z_2, \dots, z_s$.

We begin the study of this matrix function by calculating its radius of convergence R. For this purpose, let $U_{\underline{k}} := \frac{\Gamma^{-1}(\underline{kA+B})}{(\underline{k})!}$, keeping in mind that $\sigma_{\underline{k}} \geq 1$. Hence

$$\begin{aligned}\frac{1}{R} &= \limsup_{(\underline{k})\to\infty}\left(\left\|\left(\frac{U_{\underline{k}}}{\sigma_{\underline{k}}}\right)^{\frac{1}{(\underline{k})}}\right\|\right) = \limsup_{(\underline{k})\to\infty}\left\|\left(\frac{\Gamma^{-1}(\underline{kA+B})}{(\underline{k})!}\right)^{\frac{1}{(\underline{k})}}\right\|\left(\frac{1}{\sigma_{\underline{k}}}\right)^{\frac{1}{(\underline{k})}}\\ &= \limsup_{(\underline{k})\to\infty}\left\|\left(\frac{\Gamma^{-1}(k_1A_1+B_1)\Gamma^{-1}(k_2A_2+B_2)\cdots\Gamma^{-1}(k_sA_s+B_s)}{k_1!k_2!\cdots k_s!}\right)^{\frac{1}{(\underline{k})}}\right\|\left(\frac{1}{\sigma_{\underline{k}}}\right)^{\frac{1}{(\underline{k})}},\end{aligned}$$

where

$$\sigma_{\underline{k}} = \begin{cases} \left(\frac{k_1+k_2+\dots+k_s}{k_1}\right)^{\frac{k_1}{2}}\left(\frac{k_1+k_2+\dots+k_s}{k_2}\right)^{\frac{k_2}{2}}\cdots\left(\frac{k_1+k_2+\dots+k_s}{k_s}\right)^{\frac{k_s}{2}}, & \underline{k}\neq 0;\\ 1, & \underline{k}=0,\end{cases}$$

that is,

$$\begin{aligned}\frac{1}{R} &\leq \limsup_{(\underline{k})\to\infty}\left\|\left(\frac{\Gamma^{-1}(k_1A_1+B_1)\Gamma^{-1}(k_2A_2+B_2)\cdots\Gamma^{-1}(k_sA_s+B_s)}{k_1!k_2!\cdots k_s!}\right)^{\frac{1}{(\underline{k})}}\right\|\\ &\leq \limsup_{(\underline{k})\to\infty}\left\|\left(\frac{1}{\sqrt{2\pi k_1}(\frac{k_1}{e})^{k_1}\sqrt{2\pi k_2}(\frac{k_2}{e})^{k_2}\cdots\sqrt{2\pi k_s}(\frac{k_s}{e})^{k_s}}\right.\right.\\ &\quad\times\frac{1}{\sqrt{2\pi(k_1A_1+B_1-I)}(\frac{k_1A_1+B_1-I}{e})^{k_1A_1+B_1-I}}\\ &\quad\times\frac{1}{\sqrt{2\pi(k_2A_2+B_2-I)}(\frac{k_2A_2+B_2-I}{e})^{k_2A_2+B_2-I}}\cdots\\ &\quad\left.\left.\times\frac{1}{\sqrt{2\pi(k_sA_s+B_s-I)}(\frac{k_sA_s+B_s-I}{e})^{k_sA_s+B_s-I}}\right)^{\frac{1}{(\underline{k})}}\right\|.\end{aligned} \tag{3.9}$$

For positive numbers μ_i and positive integer k, we can write

$$k_i = \mu_i k, \quad i = 1,2,3,\dots,s. \tag{3.10}$$

Making use of equation (3.10) in inequality (3.9), we find

$$\begin{aligned}\frac{1}{R} \leq \limsup_{k(\mu_1+\mu_2+\cdots+\mu_s)\to\infty} \Bigg\| \Bigg(& \frac{1}{\sqrt{2\pi\mu_1 k}(\frac{\mu_1 k}{e})^{\mu_1 k}\ \sqrt{2\pi\mu_2 k}(\frac{\mu_2 k}{e})^{\mu_2 k} \cdots \sqrt{2\pi\mu_s k}(\frac{\mu_s k}{e})^{\mu_s k}} \\ & \times \frac{1}{\sqrt{2\pi(\mu_1 k A_1 + B_1 - I)}(\frac{\mu_1 k A_1 + B_1 - I}{e})^{\mu_1 k A_1 + B_1 - I}} \cdots \\ & \times \frac{1}{\sqrt{2\pi(\mu_2 k A_2 + B_2 - I)}(\frac{\mu_2 k A_2 + B_2 - I}{e})^{\mu_2 k A_2 + B_2 - I}} \cdots \\ & \times \frac{1}{\sqrt{2\pi(\mu_s k A_s + B_s - I)}(\frac{\mu_s k A_s + B_s - I}{e})^{\mu_s k A_s + B_s - I}} \Bigg)^{\frac{1}{k(\mu_1+\mu_2+\cdots+\mu_s)}} \Bigg\| = 0. \end{aligned} \tag{3.11}$$

Thus the radius of convergence of the composite Bessel-Maitland matrix function is infinity, which means that the composite Bessel-Maitland matrix function is an entire function.

Next, we derive an integral formula for the composite Bessel-Maitland matrix function. In view of equation (2.8), we have

$$\begin{aligned} \Gamma^{-1}(k_1 A_1 + B_1) &= \frac{1}{2\pi \mathrm{i}} \int_{C_1} \exp(r_1) r_1^{-(k_1 A_1 + B_1)}\, \mathrm{d}r_1, \\ \Gamma^{-1}(k_2 A_2 + B_2) &= \frac{1}{2\pi \mathrm{i}} \int_{C_2} \exp(r_2) r_2^{-(k_2 A_2 + B_2)}\, \mathrm{d}r_2, \\ &\vdots\ \vdots \\ \Gamma^{-1}(k_s A_s + B_s) &= \frac{1}{2\pi \mathrm{i}} \int_{C_s} \exp(r_s) r_s^{-(k_s A_s + B_s)}\, \mathrm{d}r_s. \end{aligned} \tag{3.12}$$

Substituting integral expressions (3.12) in the r.h.s. of series definition (3.8), we find

$$\begin{aligned} \underline{\phi}(\underline{A}, \underline{B}; \underline{z}) = \sum_{\underline{k}=0}^{\infty} \frac{\underline{z}^{\underline{k}}}{(\underline{k})!\ (2\pi \mathrm{i})^s} & \int_{C_1} \exp(r_1) r_1^{-(k_1 A_1 + B_1)}\, \mathrm{d}r_1 \\ \times \int_{C_2} \exp(r_2) r_2^{-(k_2 A_2 + B_2)}\, \mathrm{d}r_2 \cdots & \int_{C_s} \exp(r_s) r_1^{-(k_s A_s + B_s)}\, \mathrm{d}r_s. \end{aligned} \tag{3.13}$$

Now, interchanging the order of the integration and summation in equation (3.13) and using notations (3.4) and (3.5), we get

$$\underline{\phi}(\underline{A}, \underline{B}; \underline{z}) = \frac{1}{(2\pi \mathrm{i})^s} \int_{C_1} \int_{C_2} \cdots \int_{C_s} \exp((\underline{r}) I) \sum_{\underline{k}=0}^{\infty} \frac{\underline{z}^{\underline{k}}}{(\underline{k})!}\ (\underline{r}^{-\underline{A}})^{\underline{k}}\ \underline{r}^{-\underline{B}}\, \mathrm{d}r_1 \mathrm{d}r_2 \cdots \mathrm{d}r_s\ ,$$

which on simplification gives the following integral representation for the composite Bessel-Maitland matrix function $\underline{\phi}(\underline{A}, \underline{B}; \underline{z})$:

$$\underline{\phi}(\underline{A}, \underline{B}; \underline{z}) = \frac{1}{(2\pi \mathrm{i})^s} \left(\int_{C_1} \int_{C_2} \cdots \int_{C_s} \exp\left((\underline{r}) I + \underline{z}\ \underline{r}^{-\underline{A}}\right) \underline{r}^{-\underline{B}}\, \mathrm{d}r_1 \mathrm{d}r_2 \cdots \mathrm{d}r_s \right).$$

4. CONCLUDING REMARKS

We have established several important properties of the Bessel-Maitland matrix function $\phi(A, B; z)$ and composite Bessel-Maitland matrix function $\underline{\phi}(\underline{A}, \underline{B}; \underline{z})$ under suit spectral conditions.

The scalar form (1.11) of the Bessel-Maitland matrix function $\phi(A, B; z)$ is obtained by replacing the matrix A by a real number $\rho > -1$ and the matrix B by a complex number β. Further, we express relation (1.12) in the matrix form as:

$$J_A(z) = \left(\frac{1}{2}z\right)^A \phi\left(I, A+I; -\frac{1}{4}z^2\right),$$

which may be regarded as a kind of generalized Bessel matrix function.

The results established in this paper are significant. Other important properties of these new families of Bessel-Maitland matrix functions will be explored in future investigations and applications of the fractional calculus.

REFERENCES

[1] Annaby, M.H. and Mansour, Z.S., *q-Fractional Calculus and Equations*, Lecture Notes in Mathematics, **2056**, Springer, 2012.

[2] Coleman, A.J., *A simple proof of Stirling's formula*, Amer. Math. Monthly, **58** (1951), 334–336.

[3] Constantine, A.G. and Muirhead, R.J., *Partial differential equations for hypergeometric function of two argument matrices*, J. Multivariate Anal., **2** (1972), 332–338.

[4] Dunford, N. and Schwartz, J.T., *Linear Operators, Part I. General Theory*, Interscience Publishers, INC. New York, 1957.

[5] El-Shahed, M. and Salem, A., *q-analogue of Wright function*, Abstr. Appl. Anal., **2008** (2008), Art. ID 962849, p. 11.

[6] El-Shahed, M. and Salem, A., *An extension of Wright function and its properties*, J. Math., **2015** (2015) , Article ID 950728, p. 11.

[7] Erdelyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G., *Higher Transcendental Functions*, Vol. I, II, III. McGraw-Hill, New York Toronto and London, 1953. Reprinted: Krieger, Melbourne, Florida, 1981.

[8] Goreno, R., Kilbas, A.A., Mainardi, F. and Rogosin, S.V., *Mittag-Leffler Functions. Related Topics and Applications*, Springer, Heidelberg, Germany, 2014.

[9] James, A.T., *Special Functions of Matrix and Single Argument in Statistics, Theory and Applications of Special Functions*, R.A. Askey (Ed.) Academic Press, New York, (1975), 497–520.

[10] Jódar, L., Company, R. and Navarro, E., *Bessel matrix functions: Explict solution of coupled Bessel type equations*, Util. Math., **46** (1994), 129–141.

[11] Jódar, L. and Cortés, J.C., *Some properties of gamma and beta matrix functions*, Appl. Math. Lett., **11** (1998), 89–93.

[12] Jódar, L. and Cortés, J.C., *On the hypergeometric matrix function*, J. Comput. Appl. Math., **99** (1998), 205–217.

[13] Jódar, L. and Cortés, J.C., *Closed form general solution of the hypergeometric matrix differential equation*, Math. Comput. Modelling, **32** (2000), 1017–1028.

[14] Kilbas, A.A., *Fractional calculus of the generalized Wright function*, Fract. Calc. Appli. Anal., **8** (2005), 113–126.

[15] Lebedev, N.N., *Special Functions and Their Applications*, Dover Publications Inc., New York, 1972.

[16] Miller, K.S. and Ross, B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
[17] Mohamed, M.T. and Shehata, A., *A study of Appell's matrix functions of two complex variables and some properties*, Adv. Appl. Math. Sci., **9** (2011), 23–33.
[18] Navarro, E., Company, R. and Jódar, L., *Bessel matrix differential equations: Explict solutions of initial and two-point boundary value problems*, Appl. Math. (Warsaw), **22** (1994/1995), 11–23.
[19] Sastre, J. and Jódar, L., *Asymptotics of the modified Bessel and incomplete gamma matrix functions*, Appl. Math. Lett., **16** (2003), 815–820.
[20] Sayyed, K.A.M., Metwally, M.S. and Mohammed, M.T., *Certain hypergeometric matrix functions*, Sci. Math. Jpn., **69** (2009), 315–321.
[21] Shehata, A., *On p and q-Horn's matrix function of two complex variables*, Appl. Math., **2** (2011), 1437–1442.
[22] Shehata, A., *Certain $pl(m, n)$-Kummer matrix function of two complex variables under differential operator*, Appl. Math., **4** (2013), 91–96.
[23] Shehata, A., *Some relations on Konhauser matrix polynomials*, Miskolc Math. Notes, **17** (2016), 605–633.
[24] Shehata, A., *A new extension of Bessel matrix functions*, Southeast Asian Bull. Math., **40** (2016), 265–288.
[25] Shehata, A., *A new kind of Legendre matrix polynomials*, Gazi Univ. J. Sci., **29** (2016), 535–558.
[26] Olver, F.W.J., Lozier, D.W., Boisvert, R.F. and Clark, C.W., *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2012.
[27] Podlubny, I., *Fractional Differential Equations*, Academic Press, New York, NY, USA, 1999.
[28] Wright, E.M., *On the coefficients of power series having exponential singularities*, Proc. London Math. Soc., **8** (1933), 71–79.

Received July 30, 2015
Accepted July 27, 2016

Assiut University
Department of Mathematics
Faculty of Science
Assiut 71516, Egypt
and
Unaizah, Qassim University
Department of Mathematics
College of Science and Arts
Qassim, Kingdom of Saudi Arabia
E-mail: drshehata2006@yahoo.com

Aligarh Muslim University
Department of Mathematics
Aligarh-202002, India
E-mail: subuhi2006@gmail.com