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LIPSCHITZ APPROXIMATION OF THE n-QUADRATIC FUNCTIONAL EQUATIONS

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Abstract. Stability of functional equations is a classical problem proposed by Ulam. Stability of some functional equations are verified in Lipschitz and L^p spaces. In this paper, we prove stability of the n-quadratic functional equations in Lipschitz spaces.

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1. INTRODUCTION

The study of stability problems for functional equations concerning the stability of group homomorphisms was raised by a question of Ulam [18] and affirmatively was answered for Banach spaces by Hyers [5]. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. A large list of references can be found, for example, in the papers [1, 2, 6, 7, 12, 13]. A stability problem for the following quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1}$$

was solved by Skof [15] for functions $f : E_1 \longrightarrow E_2$, where E_1 is a normed space and E_2 a Banach space. For other type of quadratic mapping we refer the readers to see [11]. Czerwik et al. proved the Hyers–Ulam–Rassias stability of the quadratic functional equation (1) in normed and Lipschitz spaces (cf. [3, 4]). The stability type problems for some functional equations were also studied in Lipschitz spaces [16, 17, 10]. The general solution and the stability of the following 2-variable quadratic functional equation

$$f(x+z,y+w)+f(x-z,y-w)=2f(x,y)+2f(z,w)$$

is established in complete normed spaces [2]. We applied this form of 2-variable quadratic functional equations to establish quartic functional equations in Lipschitz spaces (see [8, 9]). Ravi et al. [14] discussed the general solution and the stability of the 3-variable quadratic functional equation

$$f(x+y,z+w,u+v)+f(x-y,z-w,u-v)=2f(x,z,u)+2f(y,w,v).$$

We introduce the n-variable quadratic functional equation as follows:

$$\begin{aligned} f(x_1+y_1,...,x_n+y_n)+f(x_1-y_1,...,x_n-y_n) \\ =2f(x_1,...,x_n)+2f(y_1,...,y_n). \end{aligned} \tag{2}$$

Lipschitz spaces have a rich algebra structure and various universal properties. These algebras present many opportunities for future research. One may find some of open problems in this area in chapter 7 of [19].

Throughout this paper G is an abelian group and W a vector space. Let $f: G^n \to W$ be a function. We say that f is n-quadratic, if f satisfies (2). Let $\mathbb{T}(W)$ be a collection of subsets of W. We denote by $\mathbb{M}(G^n, \mathbb{T}(W))$ the subset of all functions $f: G^n \longrightarrow W$ such that $\mathtt{Im}f \subset B$ for some $B \in \mathbb{T}(W)$. The family $\mathbb{M}(G^n, \mathbb{T}(W))$ is a vector space and contains all constant functions. We denote by $CB(W)$ the family of all closed balls with center at zero. This family admits a left invariant mean (briefly LIM), if the subset $\mathbb{T}(W)$ is linearly invariant, in the sense that $A+B \in \mathbb{T}(W)$ for all $A, B \in \mathbb{T}(W)$ and $x+\alpha A \in \mathbb{T}(W)$ for all $x \in W$, $\alpha \in \mathbb{R}$, $A \in \mathbb{T}(W)$, and there exists a linear operator $\Omega: \mathbb{M}(G^n, \mathbb{T}(W)) \longrightarrow W$ such that

(i) if $\mathtt{Im}f \subset A$ for some $A \in \mathbb{T}(W)$, then $\Omega[f] \in A$,
(ii) if $f \in \mathbb{M}(G^n, \mathbb{T}(W))$ and $(a_1,...,a_n) \in G^n$, then $\Omega[f^{a_1,\cdots,a_n}] = \Omega[f]$,

where $f^{a_1,\cdots,a_n}(x_1,...,x_n) = f(x_1+a_1,...,x_n+a_n)$.

Let $\mathbf{d}: G^n \times G^n \longrightarrow \mathbb{T}(W)$ be a set-valued function such that

$$\mathbf{d}((x_1+a_1,...,x_n+a_n),(u_1+a_1,...,u_n+a_n)) = \mathbf{d}((x_1,...,x_n),(u_1,...,u_n))$$

for all $(a_1,...,a_n),(x_1,...,x_n),(u_1,...,u_n) \in G^n$. We say that $f: G^n \longrightarrow W$ is $\mathbf{d}$-Lipschitz if

$$f(x_1,...,x_n) - f(u_1,...,u_n) \in \mathbf{d}((x_1,...,x_n),(u_1,...,u_n))$$

for all $(x_1,...,x_n),(u_1,...,u_n) \in G^n$. In particular, when (G^n, d) is a metric group and W a normed space, we define the function $\mu_f: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ to be a module of continuity of the function $f: G^n \longrightarrow W$ if for all $\delta > 0$ and all $(x_1,...,x_n),(u_1,...,u_n) \in G^n$ the condition $d((x_1,...,x_n),(u_1,...,u_n)) \leq \delta$ implies $||f(x_1,...,x_n) - f(u_1,...,u_n)|| \leq \mu_f(\delta)$. A function $f: G^n \longrightarrow W$ is called Lipschitz function if it satisfies the condition

$$||f(x_1,...,x_n) - f(u_1,...,u_n)|| \leq Ld((x_1,...,x_n),(u_1,...,u_n)), \tag{3}$$

where $L > 0$ is a constant and $(x_1,...,x_n),(u_1,...,u_n) \in G^n$. We consider $Lip(G^n, W)$ to be the Lipschitz space consisting of all bounded Lipschitz functions with the norm

$$||f||_{Lip} := \max\{||f||_\infty, \mathtt{P}(f)\},$$

where $||.||_\infty$ is the supremum norm and

$$\texttt{P}(f) = \sup\left\{ \frac{||f(x_1,...,x_n) - f(u_1,...,u_n)||}{d((x_1,...,x_n),(u_1,...,u_n))} : (x_1,...,x_n),(u_1,...,u_n) \in G^n, \right.$$

$$\left. (x_1,...,x_n) \neq (u_1,...,u_n) \right\}. \tag{4}$$

Let $(G^n, +)$ be an abelian group. We say that a metric d on $(G^n, +)$ is invariant under translation if it satisfies the following condition

$$d((x_1 + a_1,...,x_n + a_n),(u_1 + a_1,...,u_n + a_n)) = d((x_1,...,x_n),(u_1,...,u_n))$$

for all $(a_1,...,a_n),(x_1,...,x_n),(u_1,...,u_n) \in G^n$. A metric τ on $G^n \times G^n$ is called a product metric if it is an invariant metric and the following condition holds

$$\begin{aligned}\tau((a_1,...,a_n,x_1,...,x_n),(a_1,...,a_n,u_1,...,u_n)) \\ &= \tau((x_1,...,x_n,a_1,...,a_n),(u_1,...,u_n,a_1,...,a_n)) \\ &= d((x_1,...,x_n),(u_1,...,u_n))\end{aligned}$$

for all $(a_1,...,a_n),(x_1,...,x_n),(u_1,...,u_n) \in G^n$.

In this paper, we establish the stability of the n-quadratic functional equations in Lipschitz spaces.

2. STABILITY IN LIPSCHITZ SPACES

In this section, for the sake of a simplified writing and for a given function $f : G^n \longrightarrow W$, we define $\Pi_n : G^n \times G^n \longrightarrow W$ to be the n-variable quadratic difference of f as

$$\begin{aligned}\Pi_n(x_1,...,x_n;u_1,...,u_n) &:= 2f(x_1,...,x_n) + 2f(u_1,...,u_n) \\ &\quad - f(x_1 + u_1,...,x_n + u_n) - f(x_1 - u_1,...,x_n - u_n)\end{aligned}$$

for all $(x_1,...,x_n),(u_1,...,u_n) \in G^n$. We also define the diagonal set on G as

$$\Delta(G) := \{(x,...,x) \in G^n : x \in G\}.$$

THEOREM 2.1. *Suppose that the family $\mathbb{M}(G^n, \mathbb{T}(W))$ admits LIM. For a function $f : G^n \longrightarrow W$, if $\Pi_n(r_1,...,r_n;\cdot,...,\cdot) : G^n \longrightarrow W$ is $\boldsymbol{d}$-Lipschitz for all $(r_1,...,r_n) \in G^n$, then there exists an n-quadratic function Q such that $f - Q$ is $\frac{1}{2}\boldsymbol{d}$-Lipschitz.*

Proof. The family $\mathbb{M}(G^n, \mathbb{T}(W))$ admits LIM and so there exists a linear operator

$$\Omega : \mathbb{M}(G^n, \mathbb{T}(W)) \longrightarrow W$$

such that

(i) $\Omega[\texttt{F}] \in A$ for some $A \in \mathbb{T}(W)$,

(ii) if for $(u_1, ..., u_n) \in G^n$, $\mathtt{F}^{u_1,...,u_n} : G^n \longrightarrow W$ is defined by

$$\mathtt{F}^{u_1,...,u_n}(r_1, ..., r_n) := \mathtt{F}(r_1 + u_1, ..., r_n + u_n)$$

for every $(r_1, ..., r_n) \in G^n$, then $\Omega[\mathtt{F}] = \Omega[\mathtt{F}^{u_1,...,u_n}]$ and $\mathtt{F}^{u_1,...,u_n} \in \mathbb{M}(G^n, \mathbb{T}(W))$.

Define the function $\mathtt{F}_{a_1,...,a_n} : G^n \longrightarrow W$ by

$$\begin{aligned}\mathtt{F}_{a_1,...,a_n}(x_1, ..., x_n) &:= \frac{1}{2}f(x_1 + a_1, ..., x_n + a_n)\\ &\quad + \frac{1}{2}f(x_1 - a_1, ..., x_n - a_n) - f(x_1, ..., x_n)\end{aligned}$$

for all $(a_1, ..., a_n) \in G^n$. We have

$$\begin{aligned}\mathtt{F}_{a_1,...,a_n}(x_1, ..., x_n) &= \frac{1}{2}f(x_1 + a_1, ..., x_n + a_n) + \frac{1}{2}f(x_1 - a_1, ..., x_n - a_n)\\ &\quad - f(x_1, ..., x_n) - f(a_1, ..., a_n)\\ &\quad - \frac{1}{2}f(x_1, ..., x_n) - \frac{1}{2}f(x_1, ..., x_n) + f(x_1, ..., x_n)\\ &\quad + f(0, ..., 0) + f(a_1, ..., a_n) - f(0, ..., 0)\\ &= \frac{1}{2}\Pi_n(x_1, ..., x_n; 0, ..., 0) - \frac{1}{2}\Pi_n(x_1, ..., x_n; a_1, ..., a_n)\\ &\quad + f(a_1, ..., a_n) - f(0, ..., 0)\end{aligned}$$

for all $(x_1, ..., x_n), (a_1, ..., a_n) \in G^n$. Then, $\mathtt{Im}\ \mathtt{F}_{a_1,...,a_n} \subset B$ and $B \in \mathbb{T}(W)$, where $B := \frac{1}{2}\mathbf{d}((0, ..., 0), (a_1, ..., a_n)) + f(a_1, ..., a_n) - f(0, ..., 0)$. This demonstrate that $\mathtt{F}_{a_1,...,a_n} \in \mathbb{M}(G^n, \mathbb{T}(W))$. Note that $\mathbb{M}(G^n, \mathbb{T}(W))$ contains constant functions. By using property (i) of Ω, we conclude that if $f : G^n \longrightarrow W$ is constant, i.e., $f(x_1, ..., x_n) = C \in W$ for $(x_1, ..., x_n) \in G^n$, then $\Omega[f] = C$.

Consider $Q : G^n \longrightarrow W$ defined by $Q(x_1, ..., x_n) := \Omega[\mathtt{F}_{x_1,...,x_n}]$ for each $(x_1, ..., x_n) \in G^n$. We will prove that $f - Q$ is $\frac{1}{2}\mathbf{d}$-Lipschitz. For this, we define, for every $(x_1, ..., x_n) \in G^n$, the constant function $C_{x_1,...,x_n} : G^n \longrightarrow W$ by $C_{x_1,...,x_n}(u_1, ..., u_n) := f(x_1, ..., x_n)$ for all $(u_1, ..., u_n) \in G^n$. Then,

$$\begin{aligned}(f(x_1, ..., x_n) &- Q(x_1, ..., x_n)) - (f(u_1, ..., u_n) - Q(u_1, ..., u_n))\\ &= (\Omega[C_{x_1,...,x_n}] - \Omega[\mathtt{F}_{x_1,...,x_n}]) - (\Omega[C_{u_1,...,u_n}] - \Omega[\mathtt{F}_{u_1,...,u_n}])\\ &= \Omega[C_{x_1,...,x_n} - \mathtt{F}_{x_1,...,x_n}] - \Omega[C_{u_1,...,u_n} - \mathtt{F}_{u_1,...,u_n}]\\ &= \Omega[\frac{1}{2}\Pi_n(\cdot, ..., \cdot; x_1, ..., x_n) - \frac{1}{2}\Pi_n(\cdot, ..., \cdot; u_1, ..., u_n)]\end{aligned}$$

for all $(x_1, ..., x_n), (u_1, ..., u_n) \in G^n$. By the $\mathbf{d}$-Lipschitz property of Π_n and property (i) of Ω we see that

$$\Omega[\frac{1}{2}\Pi_n(\cdot, ..., \cdot; x_1, ..., x_n) - \frac{1}{2}\Pi_n(\cdot, ..., \cdot; u_1, ..., u_n)] \in \frac{1}{2}\mathbf{d}((x_1, ..., x_n), (u_1, ..., u_n))$$

for all $(x_1,...,x_n),(u_1,...,u_n)\in G^n$. Therefore

$$(f(x_1,...,x_n)-Q(x_1,...,x_n)) \\ -(f(u_1,...,u_n)-Q(u_1,...,u_n))\in \frac{1}{2}\mathbf{d}((x_1,...,x_n),(u_1,...,u_n))$$

for all $(x_1,...,x_n),(u_1,...,u_n)\in G^n$, i.e., $f-Q$ is a $\frac{1}{2}\mathbf{d}$-Lipschitz function. We have

$$2Q(x_1,...,x_n)+2Q(u_1,...,u_n)=2\Omega[\mathtt{F}_{x_1,...,x_n}]+2\Omega[\mathtt{F}_{u_1,...,u_n}].$$

Property (ii) of Ω ensures

$$\Omega[\mathtt{F}_{x_1,...,x_n}]=\Omega[\mathtt{F}^{u_1,...,u_n}_{x_1,...,x_n}],\ \Omega[\mathtt{F}_{x_1,...,x_n}]=\Omega[\mathtt{F}^{-u_1,...,-u_n}_{x_1,...,x_n}]$$

for $(u_1,...,u_n)\in G^n$. Hence

$$\begin{aligned}2Q(x_1,...,x_n)+2Q(u_1,...,u_n)&=2\Omega[\mathtt{F}_{x_1,...,x_n}]+2\Omega[\mathtt{F}_{u_1,...,u_n}]\\&=\Omega[\mathtt{F}^{u_1,...,u_n}_{x_1,...,x_n}]+\Omega[\mathtt{F}^{-u_1,...,-u_n}_{x_1,...,x_n}]+2\Omega[\mathtt{F}_{u_1,...,u_n}].\end{aligned}$$

On the other hand,

$$\begin{aligned}&\Omega[\mathtt{F}^{u_1,...,u_n}_{x_1,...,x_n}(r_1,...,r_n)]+\Omega[\mathtt{F}^{-u_1,...,-u_n}_{x_1,...,x_n}(r_1,...,r_n)]+2\Omega[\mathtt{F}_{u_1,...,u_n}(r_1,...,r_n)]\\&=\Omega[\frac{1}{2}f(r_1+x_1+u_1,...,r_n+x_n+u_n)+\frac{1}{2}f(r_1-x_1+u_1,...,r_n-x_n+u_n)\\&\quad-f(r_1+u_1,...,r_n+u_n)]\\&\quad+\Omega[\frac{1}{2}f(r_1+x_1-u_1,...,r_n+x_n-u_n)+\frac{1}{2}f(r_1-x_1-u_1,...,r_n-x_n-u_n)\\&\quad-f(r_1-u_1,...,r_n-u_n)]\\&\quad+\Omega[f(r_1+u_1,...,r_n+u_n)+f(r_1-u_1,...,r_n-u_n)-2f(r_1,...,r_n)]\\&=Q(x_1+u_1,...,x_n+u_n)+Q(x_1-u_1,...,x_n-u_n).\end{aligned}$$

This shows that Q is n-quadratic. □

REMARK 2.1. Consider $Q_1:G\longrightarrow E$ given by $Q_1(x):=Q(x,...,x)$. The function $f-Q$ is $\frac{1}{2}\mathbf{d}$-Lipschitz and so is $(f-Q)_{|\Delta(G)}$. This implies that the function $f_{|\Delta(G)}-Q_1$ is $\frac{1}{2}\mathbf{d}$-Lipschitz. The following equality exhibits that Q_1 is quadratic.

$$\begin{aligned}Q_1(x+y)+Q_(x-y)&=Q(x+y,...,x+y)+Q(x-y,...,x-y)\\&=2Q(x,...,x)+2Q(y,...,y)\\&=2Q_1(x)+2Q_1(y).\end{aligned}$$

COROLLARY 2.1. *Under the hypotheses of Theorem* 2.1, *if* $\mathtt{Im}\,\Pi_n\subset A$ *for some* $A\in\mathbb{T}(W)$, *then* $\mathtt{Im}\,(f-Q)\subset\frac{1}{2}A$.

Proof. We know that

$$\mathtt{Im}(\frac{1}{2}\Pi_n(x_1,...,x_n;\cdot,...,\cdot))\subset\mathtt{Im}(\frac{1}{2}\Pi_n)\subset\frac{1}{2}A$$

and so $\frac{1}{2}\Pi_n(x_1,...,x_n;\cdot,...,\cdot) \in \mathbb{M}(G^n,\mathbb{T}(W))$ for all $(x_1,...,x_n) \in G^n$. By property (i) of Ω, we have

$$f(x_1,...,x_n) - Q(x_1,...,x_n) = \Omega[\frac{1}{2}\Pi_n(\cdot,...,\cdot;x_1,...,x_n)] \in \frac{1}{2}A$$

for all $(x_1,...,x_n) \in G^n$. Consequently, $\mathtt{Im}(f-Q) \subset \frac{1}{2}A$. □

THEOREM 2.2. *Let $(G^n,+,d,\tau)$ be a product metric and W a normed space. Suppose that the family $\mathbb{M}(G^n,CB(W))$ admits LIM.*

(i) *If $f : G^n \longrightarrow W$ is a function, then there exists an n-quadratic function $Q : G \longrightarrow W$ such that $\mu_{\Pi_n} = 2\mu_{f-Q}$.*

(ii) *If $\Pi_n \in \mathbb{M}(G^n \times G^n, CB(W))$, then there exists an n-quadratic function Q such that*

$$||f-Q||_\infty \leq \frac{1}{2}||\Pi_n||_\infty.$$

(iii) *If $\Pi_n \in Lip(G^n \times G^n, W)$, then there exists an n-quadratic function Q such that*

$$||f-Q||_{Lip} \leq \frac{1}{2}||\Pi_n||_{Lip}.$$

Proof. (i) Let $\varphi : G^n \times G^n \longrightarrow \mathbb{R}^+$ be a positive real–valued function defined by

$$\varphi((x_1,...,x_n),(u_1,...,u_n)) := \inf\{\mu_{\Pi_n}(\delta) : d((x_1,...,x_n),(u_1,...,u_n)) \leq \delta\}$$

for all $(x_1,...,x_n),(u_1,...,u_n) \in G^n$. Define the set–valued function $\mathbf{d} : G^n \times G^n \longrightarrow CB(W)$ by

$$\mathbf{d}((x_1,...,x_n),(u_1,...,u_n)) := \varphi((x_1,...,x_n),(u_1,...,u_n))B(0,1)$$

for all $(x_1,...,x_n),(u_1,...,u_n) \in G^n$, where $B(0,1)$ is the unit closed ball in W. Since μ_{Π_n} is the module of continuity of Π_n, then

$$\begin{aligned}
&||\Pi_n(r_1,...,r_n;x_1,...,x_n) - \Pi_n(r_1,...,r_n;u_1,...,u_n)|| \\
&\quad\leq \inf\{\mu_{\Pi_n}(\delta) : \tau((r_1,...,r_n,x_1,...,x_n),(r_1,...,r_n,u_1,...,u_n)) \leq \delta\} \\
&\quad= \inf\{\mu_{\Pi_n}(\delta) : d((x_1,...,x_n),(u_1,...,u_n)) \leq \delta\} \\
&\quad= \varphi((x_1,...,x_n),(u_1,...,u_n))
\end{aligned}$$

for all $(x_1,...,x_n),(u_1,...,u_n) \in G^n$. Hence $\Pi_n(r_1,...,r_n;\cdot,...,\cdot)$ is $\mathbf{d}$-Lipschitz and so Theorem 2.1 implies there exists an n-quadratic function $Q : G^n \longrightarrow W$ such that $f-Q$ is $\frac{1}{2}\mathbf{d}$-Lipschitz. Thus,

$$\begin{aligned}
&(f(x_1,...,x_n) - Q(x_1,...,x_n)) \\
&\qquad - (f(u_1,...,u_n) - Q(u_1,...,u_n)) \in \frac{1}{2}\mathbf{d}((x_1,...,x_n),(u_1,...,u_n))
\end{aligned}$$

for all $(x_1, ..., x_n), (u_1, ..., u_n) \in G^n$. So,

$$\begin{aligned}||(f(x_1, ..., x_n) - Q(x_1, ..., x_n)) - (f(u_1, ..., u_n) - Q(u_1, ..., u_n))||\\ \leq \frac{1}{2}\varphi((x_1, ..., x_n), (u_1, ..., u_n))\\ = \frac{1}{2}\inf\{\mu_{\Pi_n}(\delta) : d((x_1, ..., x_n), (u_1, ..., u_n)) \leq \delta\}\end{aligned}$$

for all $(x_1, ..., x_n), (u_1, ..., u_n) \in G^n$. Consequently, $\mu_{f-Q} = \frac{1}{2}\mu_{\Pi_n}$.

(ii) Assume that $\Pi_n \in \mathbb{M}(G^n \times G^n, CB(W))$. Then, $\mathtt{Im}\,\Pi_n \subset ||\Pi_n||_\infty B(0, 1)$. In view of Corollary 2.1, we conclude that $\mathtt{Im}\ (f - Q) \subset \frac{1}{2}||\Pi_n||_\infty B(0, 1)$ and hence $||f - Q||_\infty \leq \frac{1}{2}||\Pi_n||_\infty$.

(iii) Define the function $\omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ by $\omega(t) := \mathtt{P}(\Pi_n)t$ for all $t \in \mathbb{R}^+$. Since $\Pi_n \in Lip(G^n \times G^n, W)$, we conclude that

$$\begin{aligned}||\Pi_n(r_1, ..., r_n; a_1, ..., a_n) - \Pi_n(x_1, ..., x_n; u_1, ..., u_n)||\\ \leq \mathtt{P}(\Pi_n)\tau((r_1, ..., r_n, a_1, ..., a_n), (x_1, ..., x_n, u_1, ..., u_n))\end{aligned}$$

for all $(r_1, ..., r_n), (a_1, ..., a_n), (x_1, ..., x_n), (u_1, ..., u_n) \in G^n$. Applying now the definition of ω, we get

$$\begin{aligned}||\Pi_n(r_1, ..., r_n; a_1, ..., a_n) - \Pi_n(x_1, ..., x_n; u_1, ..., u_n)||\\ \leq \omega(\tau((r_1, ..., r_n, a_1, ..., a_n), (x_1, ..., x_n, u_1, ..., u_n)))\end{aligned}$$

for all $(r_1, ..., r_n), (a_1, ..., a_n), (x_1, ..., x_n), (u_1, ..., u_n) \in G^n$. It follows that ω is the module of continuity of Π_n and so by part (i) there exists an n-quadratic function $Q : G^n \longrightarrow W$ such that $\mu_{f-Q} = \frac{1}{2}\omega$. We have

$$\begin{aligned}||(f(x_1, ..., x_n) - Q(x_1, ..., x_n)) - (f(u_1, ..., u_n) - Q(u_1, ..., u_n))||\\ \leq \mu_{f-Q}(d((x_1, ..., x_n), (u_1, ..., u_n)))\\ = \frac{1}{2}\omega(d((x_1, ..., x_n), (u_1, ..., u_n)))\\ = \frac{1}{2}\mathtt{P}(\Pi_n)d((x_1, ..., x_n), (u_1, ..., u_n))\end{aligned}$$

for all $(x_1, ..., x_n), (u_1, ..., u_n) \in G^n$. This entails that $f - Q$ is Lipschitz and

$$\mathtt{P}(f - Q) \leq \frac{1}{2}\mathtt{P}(\Pi_n). \tag{5}$$

Since $\Pi_n \in Lip(G^n \times G^n, W)$, Π_n is bounded and $\mathtt{Im}\Pi_n$ is contained in the closed ball with center at zero and radius for some $r > 0$. Therefore $\Pi_n \in \mathbb{M}(G^n \times G^n, CB(W))$. In view of (ii), the function $f - Q$ is bounded and hence $f - Q \in Lip(G^n, W)$. Using (ii) and (5), we find that

$$||f-Q||_{Lip} = \max\{||f-Q||_\infty, \mathtt{P}(f-Q)\} \leq \frac{1}{2}\max\{||\Pi_n||_\infty, \mathtt{P}(\Pi_n)\} = \frac{1}{2}||\Pi_n||_{Lip}.$$

This ends the proof. □

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