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SOME TYPES OF DERIVATIONS ON HILBERT C*-MODULES AND THEIR OPERATOR ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module. In this paper, we show that if $\mathcal{A}$ is commutative and there exist $x_0, y_0 \in \mathcal{M}$ such that $\langle x_0, y_0\rangle = 1_{\mathcal{A}}$, then every Jordan ternary derivation on $\mathcal{M}$ is a ternary derivation. Moreover, motivated by definition of Jordan $*$-derivations, we study innerness of the linear mapping J on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$, the C^*-algebra of adjointable operators in Hilbert C^*-modules satisfying $J(T^2) = J(T)T + T^*J(T)$ for all $T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$. Also, motivated by definition of reverse $*$-derivations, some necessary conditions for mapping D on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ satisfying $D(TS) = D(S)T^* + S^*D(T)$ for $T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ to be inner will be established. At the end of this paper, we characterize the linear mappings on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ which behave like mapping D when acting on pairs of elements with compact product.

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1. INTRODUCTION

The notion of Jordan $*$-derivations were first mentioned in [20]. A linear mapping J of a $*$-ring $\mathcal{R}$ into itself which satisfies $J(x^2) = J(x)x^* + xJ(x)$ for all $x \in \mathcal{R}$ is called a Jordan $*$-derivation. The problem of representing quadratic forms by sesquilinear ones is closely connected with the structure of Jordan $*$-derivations and this was the motivation of this subject, see [21, 22]. The structure of Jordan $*$-derivations on standard operator algebras was described by Šemrl [19]. Šemrl showed that every Jordan $*$-derivation of $\mathrm{B}(\mathcal{H})$, the algebra of all bounded linear operators on a real Hilbert space $\mathcal{H}$ ($\dim\mathcal{H} > 1$), is inner. For more information about this subject, we refer to [3, 4, 6, 17, 18]. In [7] Brešar and Vukman studied some algebraic properties of Jordan $*$-derivations. As a special case of [[7]; Theorem1] we have that every Jordan $*$-derivation of a complex algebra $\mathcal{A}$ with unit element is inner. Also, they introduced the notion of reverse $*$-derivations and studied some of its properties. A linear mapping D of a $*$-algebra $\mathcal{A}$ is called a reverse $*$-derivation, if $D(xy) = D(y)x^* + yD(x)$ for every $x, y \in \mathcal{A}$.

In this paper, motivated by definition of this notions, we define the notion of reverse $**$-derivation and Jordan left $*$-derivation as follows:

A linear mapping D on a $*$-algebra $\mathcal{A}$ is called a reverse $**$-derivation if, for all $a, b \in \mathcal{A}$,

$$D(ab) = D(b)a^* + b^*D(a).$$

Trivially, the mapping $a \to ba^* - a^*b$ is a reverse $**$-derivation, which is called an inner reverse $**$-derivation.

A linear mapping J on a $*$-algebra $\mathcal{A}$ is called a Jordan left $*$-derivation if

$$J(a^2) = J(a)a + a^*J(a),$$

for all $a \in \mathcal{A}$. Clearly, the mapping $a \mapsto ba - a^*b$ is a Jordan left $*$-derivation which is called an inner Jordan left $*$-derivation. By a similar process as in [9], one may prove that if J is a Jordan left $*$-derivation on a $*$-algebra $\mathcal{A}$ then for every $a, b \in \mathcal{A}$

$$J(ab + ba) = J(a)b + a^*J(b) + J(b)a + b^*J(a), \tag{1}$$

and since $2aba = a(ab + ba) + (ab + ba)a - (a^2b + ba^2)$, we have

$$J(aba) = J(a)ba + a^*J(b)a + a^*b^*J(a). \tag{2}$$

Now we recall some preliminaries and elementary properties of Hilbert C^*-modules which will be used in the sequel. Hilbert C^*-modules are generalization of Hilbert spaces, where the field of complex numbers is replaced by a C^*-algebra. This concept was introduced by Kaplansky [11]. A pre-Hilbert C^*-module $\mathcal{M}$ over a C^*-algebra $\mathcal{A}$, or a pre-Hilbert $\mathcal{A}$-module, is a left $\mathcal{A}$-module with an $\mathcal{A}$-valued inner product $\langle ., . \rangle : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{A}$ satisfying the following conditions:

(i) $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$, for all $x, y, z \in \mathcal{M}$ and $\lambda \in \mathbb{C}$;
(ii) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$ for $x \in \mathcal{M}$;
(iii) $\langle ax, y \rangle = a\langle x, y \rangle$, for every $x, y \in \mathcal{M}$ and $a \in \mathcal{A}$;
(iv) $\langle x, y \rangle^* = \langle y, x \rangle$, for each $x, y \in \mathcal{M}$.

It is well known that $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ defines a norm on $\mathcal{M}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{M}$ is called a Hilbert C^*-module over $\mathcal{A}$ if it is complete with respect to this norm. For example every Hilbert space is a Hilbert $\mathbb{C}$-module. The closure of the span of $\{\langle x, y \rangle : x, y \in \mathcal{M}\}$ is denoted by $\langle \mathcal{M}, \mathcal{M} \rangle$ and $\mathcal{M}$ is called full if $\langle \mathcal{M}, \mathcal{M} \rangle = \mathcal{A}$. The concept of an orthogonal basis of a Hilbert C^*-module is introduced by D. Bakić and B. Guljaš in [2]. The Hilbert C^*-module $\mathcal{H}_{\mathcal{A}}$, the direct sum of a countable number of copies of $\mathcal{A}$, is called standard Hilbert module over $\mathcal{A}$. If the C^*-algebra $\mathcal{A}$ is unital then $\mathcal{H}_{\mathcal{A}}$ possesses the standard orthogonal basis $\{e_i : i \in \mathbb{N}\}$, where $e_i = (0, ..., 0, 1, 0, ..., 0, ...)$ with the unite at the i-th place.

Let $\mathcal{M}$ be a Hilbert C^*-module. Following [14], a linear mapping $\delta : \mathcal{M} \to \mathcal{M}$ is called

(i) a ternary derivation if

$$\delta(\langle x, y \rangle z) = \langle \delta(x), y \rangle z + \langle x, \delta(y) \rangle z + \langle x, y \rangle \delta(z) \tag{3}$$

for every $x, y, z \in \mathcal{M}$,

(ii) a Jordan ternary derivation if, for every $x \in \mathcal{M}$,

$$\delta(\langle x, x \rangle x) = \langle \delta(x), x \rangle x + \langle x, \delta(x) \rangle x + \langle x, x \rangle \delta(x). \tag{4}$$

Abbaspour and Skeide in [1] characterized the generators of dynamical systems on Hilbert modules as those generators of one-parameter groups of Banach space isometries which are ternary derivations.

A JB^*-triple is a complex vector space $\mathcal{J}$ with a continuous mapping $\mathcal{J}^3 \longrightarrow \mathcal{J}$ with $(x, y, z) \longmapsto \{x, y, z\}$ is called a Jordan triple product, which is symmetric and bi-linear in the outer variables and conjugate linear in the middle variable and have the following properties:

- for $x, y, z, u, u \in \mathcal{J}$

$$\{x, y, \{z, u, v\}\} = \{\{x, y, z\}, u, v\} - \{z, \{y, x, u\}, v\} + \{z, u, \{x, y, v\}\},$$

- the mapping $z \to \{x, y, z\}$ is hermitian and has non-negative spectrum,
- $\|\{x, x, x\}\| = \|x\|^3$

In [10], Isidro showed that every Hilbert C^*-module is a JB^*-triple with the Jordan triple product $\{x, y, z\} = \frac{1}{2}(\langle x, y\rangle z + \langle z, y\rangle x)$. A well-known lemma of [16] states that for every Jordan derivation D on JB^*-triple $\mathcal{J}$ the equation $D(\{x, y, x\}) = \{D(x), y, x\} + \{x, D(y), x\} + \{x, y, D(x)\}$ holds for all $x, y \in \mathcal{J}$. Hence for every Jordan ternary derivation δ on Hilbert C^*-module $\mathcal{M}$, we have

$$\delta(\langle x, y\rangle x) = \langle \delta(x), y\rangle x + \langle x, \delta(y)\rangle x + \langle x, y\rangle \delta(x). \tag{5}$$

Herštein [9] showed that every Jordan derivation from a 2-torsion free prime ring into itself is a derivation. Brešar [5] proved that Herštein's result is true for 2-torsion free semiprime rings. In the second section by using equation (5) we show that every Jordan ternary derivation δ on Hilbert C^*-module $\mathcal{M}$ with x_0, y_0 which $\langle x_0, y_0\rangle = 1_{\mathcal{A}}$ is a ternary derivation.

We denote the Banach algebra of all bounded linear $\mathcal{A}$-module homomorphism (i.e. $T(ax) = aT(x)$) from $\mathcal{M}$ into itself, which is called operator of $\mathcal{M}$, by $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$. It is well known that there is no natural involution on this algebra. A linear $\mathcal{A}$-module homomorphism $T : \mathcal{M} \to \mathcal{M}$ is called adjointable, if there exists a linear $\mathcal{A}$-module homomorphism $T^* : \mathcal{M} \to \mathcal{M}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$, for all $x, y \in \mathcal{M}$. T^* is called the adjoint of T. It is well-known that in this case T, T^* are bounded. Indeed, the following result holds true.

Lemma 1.1 ([15]). *Let $\mathcal{M}$ be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$, and T, S be two mappings from $\mathcal{M}$ into itself such that $\langle Tx, y\rangle = \langle x, Sy\rangle$, for $x, y \in \mathcal{M}$. Then T,S are both belong to* $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$.

The set of all adjointable operators in $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is denoted by $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ which becomes a C^*-algebra. For any $x, y \in \mathcal{M}$, the operator $\theta_{x,y} : \mathcal{M} \to \mathcal{M}$ defined by $\theta_{x,y}(\mathrm{z}) = \langle z, y\rangle x$ is called elementary operator. Let us review some properties of these operators in the following:

(i) $(\theta_{x,y})^* = \theta_{y,x}$;
(ii) $\theta_{x,y}\theta_{u,v} = \theta_{\langle u,y\rangle x,v} = \theta_{x,\langle y,u\rangle v}$;
(iii) For any $T \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$ and $S \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ we have $T\theta_{x,y} = \theta_{Tx,y}$, and $\theta_{x,y}S = \theta_{x,S^*y}$;

(iv) If $\mathcal{A}$ is commutative then for any $a \in \mathcal{A}$, $a\theta_{x,y} = \theta_{ax,y} = \theta_{x,a^*y}$.

The linear span of $\{\theta_{x,y} \ : x, y \in \mathcal{M}\}$ will be denoted by $\Theta(\mathcal{M})$ and $\mathcal{K}(\mathcal{M})$ is used for the closed linear span of $\Theta(\mathcal{M})$. $\mathcal{K}(\mathcal{M})$ is a closed two sided ideal in $End^*_{\mathcal{A}}(\mathcal{M})$ and elements of $\mathcal{K}(\mathcal{M})$ are often called compact operators. Reader can find some properties of Hilbert C^*-modules in [15]. The following lemma is essentially due to Brown [8] also one may see Lemma 2.4.3 of [15] for a direct proof.

LEMMA 1.2. *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{M}$ be full Hilbert $\mathcal{A}$-module. Then there exist $x_1, x_2, \ldots, x_k$ in $\mathcal{M}$, such that $\sum_{i=1}^{k} \langle x_i, x_i\rangle = 1_{\mathcal{A}}$.*

Identifying algebras on which all derivations are inner is the most important subject in this area. The authors in [12] give some sufficient conditions on which every derivation on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is inner. Also, P.T. Li et al. in [13] prove that if $\mathcal{A}$ is unital and commutative C^*-algebra and $\mathcal{M}$ is a full Hilbert C^*-module over $\mathcal{A}$ then every derivation of C^*-algebra of $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$, is an inner derivation. In this paper, using their ideas on proving innerness of derivations on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$, we investigate innerness of Jordan left $*$-derivations on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ in the third section. In the last section we prove some theorems involving innerness of reverse $**$-derivations on these spaces. Moreover, we prove that every linear mapping on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ which behave like a reverse $**$-derivation at compact product of elements, is a reverse $**$-derivation.

2. TERNARY DERIVATIONS ON HILBERT C^*-MODULES

Throughout this section, for a linear mapping δ, we define $d_\delta : \mathcal{M} \to \mathcal{M}$ by

$$d_\delta(\langle x, y\rangle z) = \langle \delta(x), y\rangle z + \langle x, \delta(y)\rangle z + \langle x, y\rangle \delta(z)$$

for all $x, y, z \in \mathcal{M}$. Clearly, the mapping d_δ is linear and by equation (5), if δ is a Jordan ternary derivation we have $d_\delta(\langle x, y\rangle x) = \delta(\langle x, y\rangle x)$.

THEOREM 2.1. *Let $\mathcal{A}$ be a commutative C^*-algebra, let $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module such that there exist $x_0, y_0 \in \mathcal{M}$ that $\langle x_0, y_0\rangle = 1_{\mathcal{A}}$ and let δ be a Jordan ternary derivation on $\mathcal{M}$. Then δ is a ternary derivation.*

Proof. It is sufficient to show that $\delta(\langle x, y\rangle z) = d_\delta(\langle x, y\rangle z)$. For $x, y, z \in \mathcal{M}$, by replacing x by $x + z$ in equation (5) and by linearity of δ we obtain that

$$\begin{aligned}\delta(\langle (x+z), y\rangle (x+z)) =& \langle \delta(x+z), y\rangle (x+z) + \langle (x+z), \delta(y)\rangle (x+z) \\ &+ \langle (x+z), y\rangle \delta(x+z) \\ =& d_\delta(\langle x, y\rangle x) + d_\delta(\langle x, y\rangle z) + d_\delta(\langle z, y\rangle x) + d_\delta(\langle z, y\rangle z).\end{aligned}$$

On the other hand, by linearity of δ we get

$$\delta(\langle (x+z), y\rangle (x+z)) = \delta(\langle x, y\rangle x) + \delta(\langle x, y\rangle z) + \delta(\langle z, y\rangle x) + \delta(\langle z, y\rangle z).$$

Comparing two expressions, we have

$$\delta(\langle x, y\rangle z) + \delta(\langle z, y\rangle x) = d_\delta(\langle x, y\rangle z) + d_\delta(\langle z, y\rangle x). \tag{6}$$

By using $\langle x,y\rangle z$ for x in equation (6) and by commutativity of $\mathcal{A}$ we obtain that $\delta(\langle z,y\rangle\langle x,y\rangle z)+\delta(\langle x,y\rangle\langle z,y\rangle z)=d_\delta(\langle x,y\rangle\langle z,y\rangle z)+d_\delta(\langle z,y\rangle\langle x,y\rangle z)$. Hence

$$\delta(\langle x,y\rangle\langle z,y\rangle z)=d_\delta(\langle x,y\rangle\langle z,y\rangle z), \tag{7}$$

for all $x,y,z\in\mathcal{M}$. We can take a positive number λ small enough such that $\langle\lambda z+x_0,y_0\rangle$ is invertible in $\mathcal{A}$. Put $a:=\langle\lambda z+x_0,y_0\rangle^{-1}$. Replace z by $\lambda z+x_0$ in equation (7) to get $\delta(\langle x,y_0\rangle\langle\lambda z+x_0,y_0\rangle(\lambda z+x_0))=d_\delta(\langle x,y_0\rangle\langle\lambda z+x_0,y_0\rangle(\lambda z+x_0))$. Substituting ax for x in the last equaliy and using the fact that $\mathcal{A}$ is a commutative C^*-algebra, we obtain for each $x,z\in\mathcal{M}$

$$\delta(\langle x,y_0\rangle\lambda z)+\delta(\langle x,y_0\rangle x_0)=d_\delta(\langle x,y_0\rangle\lambda z)+d_\delta(\langle x,y_0\rangle x_0).$$

Replacing z by $\lambda^{-1}(z-x_0)$ we can conclude that

$$\delta(\langle x,y_0\rangle z)=d_\delta(\langle x,y_0\rangle z). \tag{8}$$

Again, for $y\in\mathcal{M}$, we can take a positive number μ small enough such that $\langle\mu y+y_0,x_0\rangle$ is invertible in $\mathcal{A}$. Denote $b:=\langle\mu y+y_0,x_0\rangle^{-1}$. Then $\langle b(\mu y+y_0),x_0\rangle=e_{\mathcal{A}}$. So we can replace y_0 by $b(\mu y+y_0)$. By the equality (8), we have

$$\delta(\langle x,b(\mu y+y_0)\rangle z)=d_\delta(\langle x,b(\mu y+y_0)\rangle z).$$

By substituting x by $(b^*)^{-1}x$ and since $\delta(\langle x,y_0\rangle z)=d_\delta(\langle x,y_0\rangle z)$, we have $\delta(\langle x,y\rangle z)=d_\delta(\langle x,y\rangle z)$ for all $x,y,z\in\mathcal{M}$. It means that δ is a ternary derivation. ☐

3. JORDAN LEFT $*$-DERIVATIONS ON $\mathrm{END}^*_{\mathcal{A}}(\mathcal{M})$

The main result of this section is stated as follows.

THEOREM 3.1. *Let $\mathcal{A}$ be a unital and commutative C^*-algebra and $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module with two elements z,x such that $\langle z,x\rangle=1_{\mathcal{A}}$. If every $\mathcal{A}$-module homomorphism Jordan left $*$-derivation on $\Theta(\mathcal{M})$ is inner then any $\mathcal{A}$-module homomorphism Jordan left $*$-derivation on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ is inner.*

Proof. Let J be a $\mathcal{A}$-module homomorphism Jordan left $*$-derivation on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$. First we show that J is a Jordan left $*$-derivation on $\Theta(\mathcal{M})$. For this, we show that $J(\theta_{w,y})\in\Theta(\mathcal{M})$, for each $w,y\in\mathcal{M}$. Since $\langle z,x\rangle=1_{\mathcal{A}}$, we have

$$J(\theta_{z,x})=J(\theta_{z,x}\theta_{z,x})=J(\theta_{z,x})\theta_{z,x}+\theta_{x,z}J(\theta_{z,x}).$$

Since $\Theta(\mathcal{M})$ is two sided ideal in $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ we get $J(\theta_{z,x})\in\Theta(\mathcal{M})$. Also, by (1) we have

$$\begin{aligned}J(\theta_{w,y})+J\left(\theta_{\langle w,y\rangle z,x}\right)=&J\left(\theta_{w,y}+\theta_{\langle w,y\rangle z,x}\right)\\=&J(\theta_{w,x}\theta_{z,y}+\theta_{z,y}\theta_{w,x})\\=&J(\theta_{w,x})\theta_{z,y}+\theta_{x,w}J(\theta_{z,y})+J(\theta_{z,y})\theta_{w,x}\\&+\theta_{y,z}J(\theta_{w,x}).\end{aligned}$$

Since J is $\mathcal{A}$-module homomorphism, $J\left(\theta_{\langle w,y\rangle z,x}\right) = \langle w,y\rangle J(\theta_{z,x})$ which together with the last relations imply that $J\left(\theta_{w,y}\right) \in \Theta(\mathcal{M})$ for each $w, y \in \mathcal{M}$. Since $\Theta(\mathcal{M})$ is the linear span of $\{\theta_{w,y} \ : w, y \in \mathcal{M}\}$, thus $J(\Theta(\mathcal{M})) \subseteq \Theta(\mathcal{M})$. By the assumption that every derivation of $\Theta(\mathcal{M})$ is inner, there exists $T \in \Theta(\mathcal{M})$ such that $J\left(\theta_{w,y}\right) = T\theta_{w,y} - \theta_{y,w}T$ for all $w, y \in \mathcal{M}$. Now for each $S \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ and $y \in \mathcal{M}$, by the equation (2) we obtain

$$\begin{aligned}
T\theta_{y,x}S\theta_{y,x} - \theta_{x,y}S^*\theta_{x,y}T &= T\theta_{y,S^*x}\theta_{y,x} - \theta_{x,y}\theta_{S^*x,y}T \\
&= T\theta_{\theta_{y,S^*x}(y),x} - \theta_{x,\theta_{y,S^*x}(y)}T \\
&= J\left(\theta_{\theta_{y,S^*x}(y),x}\right) \\
&= J\left(\theta_{y,x}S\theta_{y,x}\right) \\
&= J\left(\theta_{y,x}\right)S\theta_{y,x} + \theta_{x,y}J\left(S\right)\theta_{y,x} + \theta_{x,y}S^*J\left(\theta_{y,x}\right) \\
&= T\theta_{y,x}S\theta_{y,x} - \theta_{x,y}TS\theta_{y,x} + \theta_{x,y}J\left(S\right)\theta_{y,x} \\
&+ \theta_{x,y}S^*T\theta_{y,x} - \theta_{x,y}S^*\theta_{x,y}T.
\end{aligned}$$

Therefore

$$\theta_{x,y}\left(J\left(S\right) + S^*T - TS\right)\theta_{y,x} = 0.$$

So $\theta_{x,y}\left(J\left(S\right) + S^*T - TS\right)\theta_{y,x}(z) = 0$ and since $\langle z,x\rangle = 1_{\mathcal{A}}$, we obtain that for each $y \in \mathcal{M}$,

$$\theta_{x,y}\left(J\left(S\right) + S^*T - TS\right)(y) = 0.$$

Hence $\langle\left(J\left(S\right) + S^*T - TS\right)(y), y\rangle x = 0$.

So $\langle\left(J\left(S\right) + S^*T - TS\right)(y), y\rangle\langle z,x\rangle = 0$ which implies that for $y \in \mathcal{M}$

$$\langle\left(J\left(S\right) + S^*T - TS\right)(y), y\rangle = 0.$$

Therefore $J(S) = TS - S^*T$. □

Remark 3.1. In the previous theorem we need two elements $z, x \in \mathcal{M}$ such that $\langle z,x\rangle = 1_{\mathcal{A}}$. There exist many examples of such Hilbert $\mathcal{A}$-modules. For instance, unital C^*-algebras, Hilbert spaces, and $\mathcal{H}_{\mathcal{A}}$, where $\mathcal{A}$ is a unital C^*-algebra have this property. Trivially, a Hilbert $\mathcal{A}$-module over a unital C^*-algebra with an orthogonal basis $\{e_i \ : i \in I\}$ has this property, since in this space, by definition, $\langle e_j, e_j\rangle = 1_{\mathcal{A}}$.

Although, there are Hilbert C^*-modules over even a unital C^*-algebra which does not satisfies this property. As an example let $\mathcal{A} = C[0,1]$, the space of all complex valued continuous functions on $[0,1]$ and let $\mathcal{M} = l^2(C_0(0,1])$, over the C^*-algebra $\mathcal{A}$, where $C_0(0,1]$ is the space of all complex valued continuous functions f on $(0,1]$ with $f(0) = 0$. Obviously, for any $(f_j)_{j\in\mathbb{N}}, (g_j)_{j\in\mathbb{N}} \in \mathcal{M}$, $\langle(f_j)_{j\in\mathbb{N}}, (g_j)_{j\in\mathbb{N}}\rangle(0) = 0$ which implies that the mentioned property does not hold on $\mathcal{M}$.

Corollary 3.1. *Let $\mathcal{A}$ be a unital and commutative C^*-algebra and $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module with an orthogonal basis $\{e_i \ : i \in I\}$. If every $\mathcal{A}$-module homomorphism Jordan left $*$-derivation on $\Theta(\mathcal{M})$ is inner then any $\mathcal{A}$-module homomorphism Jordan left $*$-derivation on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ is inner.*

4. REVERSE $**$-DERIVATIONS ON $\mathrm{END}^*_{\mathcal{A}}(\mathcal{M})$

Before the main results of this section, we prove the following theorem.

THEOREM 4.1. *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{M}$ be a full Hilbert $\mathcal{A}$-module, then every reverse $**$-derivation on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ is spatial (i.e. it is inner but T may not belong to $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$).*

Proof. Let D be an arbitrary reverse $**$-derivation of $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$. By Lemma 1.2, there exist $x_1, x_2, \ldots, x_m$ in $\mathcal{M}$, such that $\sum_{i=1}^m \langle x_i, x_i\rangle = 1_{\mathcal{A}}$. Define $T : \mathcal{M} \longrightarrow \mathcal{M}$ by

$$Ty = \sum_{i=1}^{m} D(\theta_{x_i,y})x_i, \quad y \in \mathcal{M}.$$

Clearly, T is a well-defined additive mapping. For each $S \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ and $y \in \mathcal{M}$ we have

$$\begin{aligned} TSy &= \sum_{i=1}^{m} D(\theta_{x_i,y}S^*)x_i \\ &= \sum_{i=1}^{m} D(S^*)\theta_{y,x_i}(x_i) + \sum_{i=1}^{m} SD(\theta_{x_i,y})(x_i) \\ &= \sum_{i=1}^{m} D(S^*)(\langle x_i, x_i\rangle y) + STy. \end{aligned}$$

So $D(S^*) = TS - ST$. Hence for every $S \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ we obtain that

$$D(S) = TS^* - S^*T$$

which completes the proof. □

Now, we assume that $\mathcal{A}$ is commutative and we will prove that every $*$-$\mathcal{A}$-module homomorphism (i.e. a linear mapping with the property that $D(aT) = a^*D(T)$, for $a \in \mathcal{A}$) reverse $**$-derivation of $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ is inner.

THEOREM 4.2. *Let $\mathcal{A}$ be a unital and commutative C^*-algebra and $\mathcal{M}$ be a full Hilbert $\mathcal{A}$-module. Then every $*$-$\mathcal{A}$-module homomorphism reverse $**$-derivation on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ is inner reverse $**$-derivation.*

Proof. Let D be an arbitrary $*$-$\mathcal{A}$-module homomorphism which reverses $**$-derivation of $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$.

By Lemma 1.2 there exist $x_1, x_2, ..., x_m$ in $\mathcal{M}$ such that $\sum_{i=1}^m \langle x_i, x_i\rangle = 1_{\mathcal{A}}$. Define $T : \mathcal{M} \longrightarrow \mathcal{M}$ by

$$Ty = \sum_{i=1}^{m} D(\theta_{x_i,y})x_i, \quad y \in \mathcal{M}.$$

Obviously, T is a well-defined additive mapping. On the other hand, D is $*$-$\mathcal{A}$-module homomorphism. Therefore, for $a \in \mathcal{A}$, we have

$$\begin{aligned} T(ay) &= \sum_{i=1}^{m} D(\theta_{x_i,ay})x_i \\ &= \sum_{i=1}^{m} D(a^*\theta_{x_i,y})x_i = aT(y), \end{aligned}$$

which implies that T is $\mathcal{A}$-module homomorphism.

Now, for each $B \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$, we have

$$\begin{aligned} TBy = \sum_{i=1}^{m} D(\theta_{x_i,By})x_i &= \sum_{i=1}^{m} D(\theta_{x_i,y}B^*)x_i \\ &= \sum_{i=1}^{m} D(B^*)\theta_{y,x_i}(x_i) + \sum_{i=1}^{m} BD(\theta_{x_i,y})x_i \\ &= D(B^*)y + BTy. \end{aligned}$$

Hence $D(B) = TB^* - B^*T$.

We are going to show that T is adjointable. For proving this, define $S : \mathcal{M} \to \mathcal{M}$ by $Sy = -\sum_{i=1}^{m} D(\theta_{y,x_i})^* x_i$. It is enough to show that for each $w, y \in \mathcal{M}$, $\langle w, Sy\rangle = \langle Tw, y\rangle$. First note that

$$\begin{aligned} D(\theta_{y,x_i}) &= D(\sum_{k=1}^{m} \theta_{y,x_k}\theta_{x_k,x_i}) \\ &= \sum_{k=1}^{m}(D(\theta_{x_k,x_i})\theta_{x_k,y} + \theta_{x_i,x_k}D(\theta_{y,x_k})) \\ &= \sum_{k=1}^{m} \theta_{D(\theta_{x_k,x_i})x_k,y} + \sum_{k=1}^{m} \theta_{x_i,D(\theta_{y,x_k})^*x_k} \\ &= \theta_{Tx_i,y} - \theta_{x_i,Sy}. \end{aligned}$$

So $D(\theta_{y,x_i}) = \theta_{Tx_i,y} - \theta_{x_i,Sy}$ for $y \in \mathcal{M}$.

On the other hand,

$$D(\theta_{y,x_i}) = T\theta_{x_i,y} - \theta_{x_i,y}T = \theta_{Tx_i,y} - \theta_{x_i,y}T$$

which implies that $\theta_{x_i,Sy} = \theta_{x_i,y}T$. So for each $w \in \mathcal{M}$ we have $\langle w, Sy\rangle x_i = \langle Tw, y\rangle x_i$. Now using $\sum_{i=1}^{m}\langle x_i, x_i\rangle \ = 1_A$ we get $\langle w, Sy\rangle = \langle Tw, y\rangle$, for each $w, y \in \mathcal{M}$. Therefore $T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ and D is inner. □

THEOREM 4.3. *Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module and $x, z \in \mathcal{M}$ be such that $\langle z, x\rangle = 1_{\mathcal{A}}$. Suppose that D is a linear mapping on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ such that $D(AB) = D(B)A^* + B^*D(A)$, for each pair $A, B \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ with $AB \in \mathcal{K}(\mathcal{M})$. Then D is a reverse $**$-derivation.*

Proof. Since $\mathcal{K}(\mathcal{M})$ is two sided ideal in $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$, we have $\theta_{x,y}A \in \mathcal{K}(\mathcal{M})$, for $A \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ and $y \in \mathcal{M}$. So by hypothesis for $y \in \mathcal{M}$ and $A \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$

$$D(\theta_{x,y}A) = D(A)\theta_{y,x} + A^*D(\theta_{x,y}).$$

Let $A, B \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$. For any $y \in \mathcal{M}$, we obtain

$$D(\theta_{x,y}AB) = D(AB)\theta_{y,x} + B^*A^*D(\theta_{x,y}).$$

On the other hand,

$$\begin{aligned} D(\theta_{x,y}AB) &= D(B)\theta_{A^*y,x} + B^*D(\theta_{x,A^*y}) \\ &= D(B)A^*\theta_{y,x} + B^*D(\theta_{x,y}A) \\ &= D(B)A^*\theta_{y,x} + B^*D(A)\theta_{y,x} + B^*A^*D(\theta_{x,y}). \end{aligned}$$

Hence by comparing two last equations we have

$$D(AB)\theta_{y,x} = D(B)A^*\theta_{y,x} + B^*D(A)\theta_{y,x}.$$

Now, by acting the two side of this equation on z, we get

$$D(AB)(y) = D(B)A^*(y) + B^*D(A)(y)$$

for $y \in \mathcal{M}$ and $A, B \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$. Therefore D is a reverse $**$-derivation. □

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