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SOME PROPERTIES OF SOLUTIONS OF THE HOMOGENEOUS NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper we consider the following nonlinear homogeneous second order differential equations, $F(x, y, y', y'') = 0$. We present for the solutions, $y \in C^2[a, b]$, of this equation, extremal principle, Sturm-type, Nicolescu-type and Butlewski-type separation theorems. Some applications and examples are given. Open problems are also presented.

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1. INTRODUCTION

Let $F \in C([a, b] \times \mathbb{R}^3)$. We consider the following implicit differential equation

$$F(x, y, y', y'') = 0. \tag{1.1}$$

By definition this equation is homogeneous if the function F is homogeneous with respect to the last three arguments.

In this paper by a solution of the equation (1.1) we understand a function $y \in C^2[a, b]$ which satisfies (1.1). Moreover by a solution we shall understand a nontrivial solution.

The linear case of (1.1) is the following equation

$$y'' + p(x)y' + q(x)y = 0. \tag{1.2}$$

For the equation (1.2) the following properties of the solution are well known (see [10, 11, 12, 7, 3, 13]).

We suppose that p and $q \in C[a, b]$.

THEOREM 1 (Extremal principle). *Let $q(x) < 0$, for all $x \in]a, b[$ and y be a solution of* (1.2). *Then*

(a) *if* $\max\{y(x) |\ x \in [a, b]\} = y(x_0)$ *and* $y(x_0) > 0$ *then* $x_0 \in \{a, b\}$;
(b) *if* $\min\{y(x) |\ x \in [a, b]\} = y(x_0)$ *and* $y(x_0) < 0$ *then* $x_0 \in \{a, b\}$.

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Theorem 2 (Sturm's separation theorem). *If y_1 and y_2 are two linear independent solutions of* (1.2)*, then the zeros of y_1 and y_2 separate each other.*

Theorem 3 (Nicolescu's theorem, [6]). *We suppose that $q(x) > 0$, for all $x \in [a, b]$. If y is a solution of* (1.2)*, then the zeros of y and y' separate each other.*

Theorem 4 (Butlewski's theorem, [1]). *We suppose that $q(x) > 0$, for all $x \in [a, b]$. If y_1 and y_2 are two linear independent solutions of* (1.2)*, the zeros of y_1' and y_2' separate each other.*

The aim of this paper is to extend the above results to the solutions of (1.1). For some results in this directions see [14], [8] and [9].

2. HOMOGENEOUS NONLINEAR SECOND ORDER DIFFERENTIAL EQUATION: EXAMPLES

Example 1. The equation

$$y''^2 + yy'' + y^2 = 0 \tag{2.1}$$

has the only solution $y = 0$.

Example 2. We consider the equation

$$y''^3 + y^3 = 0. \tag{2.2}$$

This equation is equivalent with the equation $y'' + y = 0$.

Example 3. We consider the equation

$$y''^3 - y^3 = 0. \tag{2.3}$$

This equation is equivalent with the equation $y'' - y = 0$.

Example 4. We consider the equation

$$y''^2 - y^2 = 0. \tag{2.4}$$

A function y is a solution of this equation if and only if y is a solution of $y'' - y = 0$ or of $y'' + y = 0$.

Example 5. (Painlevé (1902), see [4], 6.122). The following equation was studied by Painlevé

$$yy'' - y'^2 + p(x)yy' + q(x)y^2 = 0. \tag{2.5}$$

Example 6. (Tonelli (1927), [14]). The following equation was considered by Tonelli

$$(y^2 + y'^2)y'' + p(x)y^3 = 0. \tag{2.6}$$

Example 7.

$$\prod_{k=1}^{m} (y'' + p_k(x)y' + q_k(x)y) = 0 \tag{2.7}$$

We shall use the above examples to exemplify our general results.

For other examples of such equations see [4] and [12].

3. EXTREMAL PRINCIPLES

We consider the equation (1.1) with $F \in C([a,b] \times \mathbb{R}^3)$ a homogeneous function with respect to the last three arguments. We have the following extremal principle.

THEOREM 5. *We suppose that* $F(x, r_1, 0, r_2) \neq 0$, *for all* $x \in [a,b], r_1 > 0$ *and* $r_2 \leq 0$. *Let* y *be a solution of (1.1). We have:*

(a) *if* $\max\{y(x) |\ x \in [a,b]\} = y(x_0)$ *and* $y(x_0) > 0$ *then* $x_0 \in \{a,b\}$;
(b) *if* $\min\{y(x) |\ x \in [a,b]\} = y(x_0)$ *and* $y(x_0) < 0$ *then* $x_0 \in \{a,b\}$.

Proof. (a) Let $x_0 \in]a,b[$ be such that, $y(x_0) > 0$ is the maximum value of y on $[a,b]$. Since $y \in C^2[a,b]$ we have that $y(x_0) > 0, y'(x_0) = 0, y''(x_0) \leq 0$. From (1.1) we have

$$F(x_0, y(x_0), 0, y''(x_0)) = 0.$$

But in the condition of our theorem the first part of this relation is not equal to zero. So, $x_0 \in \{a,b\}$.

(b) We remark that if y is a solution of (1.1) then, $-y$ is also a solution. We apply (a) for $-y$. □

EXAMPLE 8. We consider the equation

$$y''^{2n+1} + p(x)y'^{2n+1} + q(x)y^{2n+1} = 0,\ x \in [a,b] \text{ with } n \in \mathbb{N}. \tag{3.1}$$

If $q(x) < 0, \forall x \in]a,b[$, then we have for the solution of the equation (3.1) the extremal principle given by Theorem 5.

REMARK 1. If $p_k, q_k \in C[a,b]$ and $q_k(x) < 0,\ \forall x \in [a,b]$ and $k = \overline{1,m}$, then the equation (7) satisfies the conditions of Theorem 5.

Now let us consider the bilocal problem

$$F(x, y, y', y'') = 0, \tag{1.1}$$

$$y(a) = 0, y(b) = 0. \tag{3.2}$$

We have for this problem the following result.

THEOREM 6. *We suppose that* $F(x, r_1, 0, r_2) \neq 0$, *for all* $x \in [a,b], r_1 > 0$ *and* $r_2 \leq 0$. *Then the problem* (1.1)–(3.2) *has the only solution* $y = 0$.

Proof. Follows from Theorem 5. □

4. ZEROS OF THE SOLUTION OF (1.1)

Now we consider the following conditions on (1.1):

(u_0) If y is a solution of (1.1) and for some $x_0 \in [a,b],\ y(x_0) = 0$ and $y'(x_0) = 0$, then $y = 0$.
(u_1) If y_1 and y_2 are solutions of (1.1) and for some $x_0 \in [a,b],\ y_1(x_0) = y_2(x_0) > 0$ and $y_1'(x_0) = y_2'(x_0)$, then $y_1 = y_2$.

(u_2) If y is a solution of (1.1) and for some $x_0 \in [a,b]$, $y'(x_0) = 0$, $y''(x_0) = 0$, then $y = 0$.

(u_3) If y_1 and y_2 are solutions of (1.1) and for some $x_0 \in [a,b]$, $y_1'(x_0) = y_2'(x_0) > 0$ and $y_1''(x_0) = y_2''(x_0)$, then $y_1 = y_2$.

By standard arguments we have

LEMMA 1. *If y is a solution of* (1.1) *then condition* (u_0) *implies that the zeros of y are simple and isolated on* $[a,b]$.

LEMMA 2. *If y is a solution of* (1.1) *then condition* (u_2) *implies that the zeros of y' are simple and isolated on* $[a,b]$.

In what follow we also need the following result (see [14], [10], p. 163 and [5]).

LEMMA 3 (Tonelli's Lemma). *Let $y_1, y_2 \in C^1[a,b]$ be two functions which satisfy the following conditions:*

(i) $y_1(a) = y_1(b) = 0$ *and* $y_1(x) > 0$ *for all* $x \in]a,b[$;

(ii) $y_2(x) > 0$ *for all* $x \in [a,b]$.

Then there exists $\lambda > 0$ and $x_0 \in]a,b[$ such that:

$$y_2(x_0) = \lambda y_1(x_0) \text{ and } y_2'(x_0) = \lambda y_1'(x_0).$$

Using Lemma 3, Tonelli give in [14] the following result.

THEOREM 7 (Sturm-type separation theorem). *For the homogeneous equation* (1.1) *we suppose that it satisfies conditions* (u_0) *and* (u_1). *Then if y_1, y_2 are two linear independent solutions of* (1.1) *and $x_1, x_2 \in [a,b]$ are two consecutive zeros of y_1, then y_2 has at least one zero in* $[x_1, x_2]$.

Our results are the following.

THEOREM 8 (Nicolescu-type separation theorem). *For the homogeneous equation* (1.1), *we suppose that:*

(i) *it satisfies condition* (u_2);

(ii) $F(x, \lambda^2, \lambda, 1) \neq 0$, *for all* $\lambda \in \mathbb{R}$.

Then, if y is a solution of (1.1), *the zeros of y and y' separate each other.*

Proof. We consider x_1 and x_2 two consecutive zeros of $y'(x)$. We have to prove that $y(x)$ has at least one zero in the interval (x_1, x_2).

We suppose that $y(x) \neq 0$, $x \in [x_1, x_2]$. Applying Tonelli's Lemma 3 there exists $x_0 \in (x_1, x_2)$ and $\lambda > 0$ (or < 0) such that

$$y(x_0) = \lambda y'(x_0), \; y'(x_0) = \lambda y''(x_0).$$

So,

$$y'(x_0) = \tfrac{1}{\lambda} y(x_0), \; y''(x_0) = \tfrac{1}{\lambda^2} y(x_0). \tag{4.1}$$

Using (4.1) in $F(x_0, y(x_0), y'(x_0), y''(x_0)) = 0$ we obtain that ($y(x_0) \neq 0$)

$$F(x, \lambda^2, \lambda, 1) = 0, \text{ for all } \lambda \in \mathbb{R}.$$

□

REMARK 2. Theorem 8 improves Theorem 2 in [8].

REMARK 3. Theorem 8 works for the equation (7) if $p_k, q_k \in C[a,b]$ and $q_k(x) < 0,\ \forall x \in [a,b]$. It also works for the equation (3).

THEOREM 9. *(Butlewski-type separation theorem) For the homogeneous equation* (1.1)*, we suppose that it satisfies conditions* (u$_2$) *and (u$_3$). Then, if y_1 and y_2 are two linear independent solutions of* (1.1) *and $x_1, x_2 \in [a,b]$ are two consecutive zeros of y_1', then y_2' has at least one zero in $[x_1, x_2]$.*

Proof. We consider x_1 and x_2 two consecutive zeros of $y_1'(x)$. We have to prove that $y_2'(x)$ has at least one zero in $[x_1, x_2]$.

We suppose that $y_2'(x) \neq 0,\ x \in [x_1, x_2]$. Applying Tonelli's Lemma 3 there exists $x_0 \in (x_1, x_2)$ and $\lambda > 0$ (or < 0) such that

$$y_2'(x_0) = \lambda y_1'(x_0),\ y_2''(x_0) = \lambda y_1''(x_0).$$

Taking into account (u$_2$) we have that $y_2(x) = \lambda y_1(x)$ and so we have reached a contradiction. □

5. SOME RESEARCH DIRECTIONS

PROBLEM 1. To give sufficient conditions which imply conditions (u$_0$) and (u$_1$).

PROBLEM 2. To give sufficient conditions which imply conditions (u$_2$) and (u$_3$).

PROBLEM 3. In [2] the authors studied the following problem. Let us consider the equation

$$L(y) := y'' + q(x)y = 0$$

with $q \in C[a,b]$. The problem is to study the functional $h : KerL \to \mathbb{R}_+$ defined by $h(y) := \inf\{d(x_1, x_2) |\ x_1$ and x_2 are two consecutive zeros of $y\}$. In [2] the authors prove that

$$\inf\{h(y) |\ y \in KerL\} = \min\{h(y) |\ y \in KerL\} > 0.$$

The problem is to study the above problem for the equation (1.1).

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