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SEMIREGULAR MODULES RELATIVE TO A PRERADICAL

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Abstract. Let τ_M be a preradical on the category $\sigma[M]$ for some module M. A module $N \in \sigma[M]$ is called τ_M-semiregular in $\sigma[M]$ if for all $n \in N$, there exists a decomposition $N = A \oplus B$ such that A is a projective submodule of nR and $nR \cap B \subseteq \tau_M(N)$. We prove that if $N \in \sigma[M]$ is a projective module, then N is τ_M-semiregular if and only if N is finitely τ_M-supplemented and that $\tau_M(N)$ is quasi finitely strongly lifting (for short QFSL) if and only if every finitely generated submodule of $N/\tau_M(N)$ is a direct summand and $\tau_M(N)$ is QFSL. Furthermore, it is shown that if $N \in \sigma[M]$ is a τ_M-semiregular module, then N is finitely refinable if and only if every submodule of $\tau_M(N)$ is QFSL in N if and only if every finitely generated submodule of $\tau_M(N)$ is DM in N.

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1. INTRODUCTION

Throughout this paper R will denote an associative ring with identity, M a unitary right R-module. The notation $N \leq^{\oplus} M$ denotes that N is a direct summand of M. For a module N, $\mathrm{Rad}(N)$, $\mathrm{Soc}(N)$, and $Z_M(N)$ are the radical, the socle and the sum of the M-singular submodules of N, respectively.

Let $M \in$ Mod-R. By $\sigma[M]$ we mean the full subcategory of Mod-R whose objects are submodules of M-generated modules.

A functor τ_M from $\sigma[M]$ to itself is called a *preradical* on $\sigma[M]$ if it satisfies the following properties:

(i) $\tau_M(N)$ is a submodule of N, for every $N \in \sigma[M]$;

(ii) If $f : N' \to N$ is a homomorphism in $\sigma[M]$, then $f(\tau_M(N')) \leq \tau_M(N)$ and $\tau_M(f)$ is the restriction of f to $\tau_M(N')$.

Throughout the paper τ_M will be a preradical on $\sigma[M]$. In case $M = R$, we write $\tau(N)$ instead of $\tau_M(N)$. Note that if K is a summand of $N \in \sigma[M]$, then $K \cap \tau_M(N) = \tau_M(K)$. Rad, Soc, and Z_M are preradicals on $\sigma[M]$.

Many authors work with various extensions of semiregular modules (see [2, 3, 9, 11, 14]). In [10], Nicholson and Yousif introduced I-semiregular rings for an ideal I of a ring R. Alkan and Özcan generalized in [2] this concept to modules and defined F-semiregular modules for a submodule F of a module N. In this paper we define τ_M-semiregular modules in $\sigma[M]$ for any preradical τ_M, by taking $\tau_M(N)$ as a fully invariant submodule of N. A module $N \in \sigma[M]$ is called τ_M*-semiregular* in $\sigma[M]$ if for all $n \in N$, there exists a decomposition $N = A \oplus B$ such that A is a projective submodule of nR and $nR \cap B \subseteq \tau_M(N)$.

If $\sigma[M]$ =Mod-R, then it is said that N is τ-semiregular. In this note, we investigate some properties of such modules. We show that if $M = A \oplus B$ is a module over a local ring R, then M is τ-semiregular if and only if both of A and B are τ-semiregular. We prove that if $N \in \sigma[M]$ is a projective module, then N is τ_M-semiregular if and only if N is finitely τ_M-supplemented and that $\tau_M(N)$ is QFSL if and only if every finitely generated submodule of $N/\tau_M(N)$ is a direct summand and $\tau_M(N)$ is QFSL. It is shown that if $N \in \sigma[M]$ is a τ_M-semiregular module, then N is finitely refinable if and only if every submodule of $\tau_M(N)$ is QFSL in N if and only if every finitely generated submodule of $\tau_M(N)$ is DM in N.

2. τ_M-SEMIREGULAR MODULES

A module $N \in \sigma[M]$ is called *τ_M-semiregular* in $\sigma[M]$ if for all $n \in N$, there exists a decomposition $N = A \oplus B$ such that A is a projective submodule of nR and $nR \cap B \subseteq \tau_M(N)$.

LEMMA 2.1. *The following conditions are equivalent for a module $N \in \sigma[M]$:*

(1) *N is τ_M-semiregular.*

(2) *For any finitely generated submodule K of N, there exists a homomorphism g from N to K such that $g^2 = g$, gM is projective and $(1-g)K \leq \tau_M(N)$.*

(3) *For any finitely generated submodule K of N, there exists a decomposition $N = A \oplus B$ such that A is a projective submodule of K and $K \cap B \subseteq \tau_M(N)$.*

(4) *For any finitely generated submodule K of N, K can be written as $K = A \oplus S$ where A is a projective summand of N and $S \subseteq \tau_M(N)$.*

Proof. The assertions follow from [2, Theorem 2.3]. □

PROPOSITION 2.1. *Every direct summand of a τ_M-semiregular module in $\sigma[M]$ is τ_M-semiregular.*

Proof. Let $N \in \sigma[M]$ be a τ_M-semiregular module and $K \leq^{\oplus} N$. If $k \in K$, then N has a decomposition $N = A \oplus B$ such that A is a projective submodule of kR and $kR \cap B \subseteq \tau_M(N)$. It follows that $K = A \oplus (K \cap B)$ and $kR \cap (K \cap B) \leq kR \cap B \subseteq \tau_M(N)$. Thus $kR \cap (K \cap B) \subseteq \tau_M(K)$. Therefore K is τ_M-semiregular. □

Like in [4], a submodule $K \subseteq N \in \sigma[M]$ is called *τ_M-supplement* provided there exists some $U \subseteq N$ such that $N = U + K$ and $U \cap K \subseteq \tau_M(K)$.

PROPOSITION 2.2. *Let $N \in \sigma[M]$ be a τ_M-semiregular module. If $N = X+Y$ such that $Y \leq^{\oplus} N$ and $X \cap Y$ is cyclic, then Y contains a τ_M-supplement of X in N.*

Proof. Since N is τ_M-semiregular, and $X \cap Y$ is cyclic, we have, by Lemma 2.1, $X \cap Y = A \oplus B$, where A is a projective summand of N and $B \subseteq \tau_M(N)$. Since $Y \leq^{\oplus} N$, we have $B \subseteq \tau_M(Y)$. Write $Y = A \oplus A'$. It follows that $X \cap Y = A \oplus (X \cap Y \cap A') = A \oplus (X \cap A')$. Let $\pi : A \oplus A' \to A'$ be the natural projection. It follows that $X \cap A' = \pi(A \oplus (X \cap A')) = \pi(X \cap Y) = \pi(A \oplus B) = \pi(B)$ hence $X \cap A' \subseteq \tau_M(A')$, and that $N = X + Y = X + A + A' = X + A'$. Therefore A' is a τ_M-supplement of X in N that is contained in Y. □

Lemma 2.2. *Let $N \in \sigma[M]$ be a τ_M-semiregular module. Then every indecomposable cyclic submodule C of N is either contained in $\tau_M(N)$ or a projective summand of N.*

Proof. By Lemma 2.1, we have $C = A \oplus B$ such that A is a projective summand of N in $\sigma[M]$ and $B \subseteq \tau_M(N)$. Since C is indecomposable, we have either $C = A$ or $C = B$. □

If M has a largest submodule, i.e. a proper submodule which contains all other proper submodules, then M is called a *local* module. A ring is a *local ring* if and only if R_R (or $_RR$) is a local module.

Corollary 2.1. *Let M be a module over a local ring R. If M is τ-semiregular, then every cyclic submodule is either contained in $\tau(M)$ or a projective summand of M.*

Proof. The proof follows from Lemma 2.2, and the fact that every cyclic module over a local ring is a local module. □

Theorem 2.1. *Let $M = A \oplus B$ be a module over a local ring R. Then M is τ-semiregular if and only if both of A and B are τ-semiregular.*

Proof. Let C be an arbitrary cyclic submodule of M. Then $C = (a + b)R$, where $a \in A$ and $b \in B$. Since A and B are τ-semiregular, then we have nothing to prove whenever $a = 0$ or $b = 0$. Now to avoid trivially we may consider C is not contained in $\tau(M)$. Since $(a + b)R \leq aR + bR$, we have aR or bR is not contained in $\tau(M)$. Without loss of generality we may assume aR is not contained in $\tau(M)$, hence it is not contained in $\tau(A)$. By Corollary 2.1, aR is a projective summand of A. Since $aR \oplus bR = (a + b)R + bR$, we have, by [8, 4.47], that there exists $K \leq (a + b)R$ such that $aR \oplus bR = K \oplus bR$. It follows that $(a + b)R = K \oplus [(a + b)R \cap bR]$. Since C is a local module, and C is not contained in bR, we have that $C = K$. Now we show that K is a projective summand of M. It is clear that $aR \oplus B = K + B$, and so $K \cap B = K \cap (K \oplus bR) \cap B = (aR \oplus bR) \cap B \cap K = bR \cap K = 0$. As $aR \leq^{\oplus} A$, we have $K \oplus B = aR \oplus B \leq^{\oplus} M$ and so $C = K \leq^{\oplus} M$. On the other hand, since $aR \oplus bR = K \oplus bR$, $aR \cong K$. Thus K is projective. The converse follows from Proposition 2.1. □

Let M be a module. A preradical τ_M on $\sigma[M]$ is called a *left exact preradical* if for any submodule K of $N \in \sigma[M]$, $\tau_M(K) = K \cap \tau_M(N)$ (see [13]). For example, Soc and Z_M are left exact preradicals on $\sigma[M]$.

Lemma 2.3. *Let τ_M be a left exact preradical on $\sigma[M]$. Then the following are equivalent:*
(1) *Every injective module is τ_M-semiregular in $\sigma[M]$;*
(2) *Every module is τ_M-semiregular in $\sigma[M]$.*

Proof. (1) $\Rightarrow$ (2) Let N be a module in $\sigma[M]$ and K a finitely generated submodule of N. Since $\widehat{N}$, the M-injective hull of N, is τ_M-semiregular by (1), there is a decomposition $K = A \oplus B$ such that A is a projective summand of $\widehat{N}$ in $\sigma[M]$ and $B \leq \tau_M(\widehat{N})$. Then A is a projective summand of N in $\sigma[M]$ and $B \leq N \cap \tau_M(\widehat{N}) = \tau_M(N)$. Thus N is τ_M-semiregular in $\sigma[M]$.
(2) $\Rightarrow$ (1) is clear. □

We mention that the proof of Lemma 2.3 is similar to [12, Lemma 3.1].
If M is a Noetherian injective cogenerator in $\sigma[M]$, then it is called a *Noetherian Quasi-Frobenius* or *QF-module* [14].

Corollary 2.2. *Let M be a finitely generated self-projective module which is a selfgenerator in $\sigma[M]$. If M is a Noetherian QF-module, then every module in $\sigma[M]$ is Z_M-semiregular in $\sigma[M]$.*

Proof. Let $N \in \sigma[M]$ be injective in $\sigma[M]$. Then N is projective in $\sigma[M]$ from [14, 48.14]. By [7, Theorem 3.11], $Z_M(M) = \mathrm{Rad}(M)$. Since N is M-generated and projective in $\sigma[M]$, N is isomorphic to a summand of $M^{(\Lambda)}$ for an index set Λ. This implies that $Z_M(N) = \mathrm{Rad}(N)$. Again by [14, 48.14], M is perfect in $\sigma[M]$. Then N is semiperfect in $\sigma[M]$ by [14, 43.8]. Hence N is Z_M-semiregular. By Lemma 2.3, every module in $\sigma[M]$ is Z_M-semiregular in $\sigma[M]$. □

A module $N \in \sigma[M]$ is called (*finitely*) *τ_M-supplemented* if each of its (finitely generated) submodules has a τ_M-supplement in N. It is clear that a τ_M-semiregular module is finitely τ_M-supplemented.

Proposition 2.3. *Let $N \in \sigma[M]$ be a projective finitely τ_M-supplemented module and assume that every τ_M-supplement submodule is a direct summand of N. Then N is τ_M-semiregular.*

Proof. Let U be a finitely generated submodule of N. Then there exists a submodule K of N such that $U \cap K \subseteq \tau_M(K)$ and $N = K + U$. Hence K is a direct summand of N and since $N = K + U$ and N is projective, it follows that $N = K \oplus A$ such that $A \subseteq U$. Then $U = A \oplus (K \cap U)$ and so N is τ_M-semiregular. □

Let U be a submodule of a module $N \in \sigma[M]$. U is called *(finitely) strongly lifting* in N if for every (finitely generated) submodule A of N whenever $N/U = (A + U)/U \oplus (B + U)/U$, then N has a decomposition $N = P \oplus Q$ such that $P \subseteq A$, $(A + U)/U = (P + U)/U$ and $(B + U)/U = (Q + U)/U$ [11]. The submodule U is called *quasi (finitely) strongly lifting* in N or briefly QSL (QFSL) if whenever $(A + U)/U$ is a direct summand of N/U (and A is a finitely generated submodule of N), then N has a direct summand P such that $P \subseteq A$ and $P + U = A + U$ [1].

Proposition 2.4. *Consider the following conditions for a module* $N \in \sigma[M]$.

(1) N *is* τ_M*-semiregular.*

(2) (i) *Every finitely generated submodule of* $N/\tau_M(N)$ *is a direct summand.*

(ii) $\tau_M(N)$ *is finitely strongly lifting.*

Then (1) $\Rightarrow$ (2)(i). *If* N *is projective, then* (1) $\Rightarrow$ (2)(ii). *If* N *is projective and, for every summand* K *of* N, *there exists a decomposition* $N = A \oplus B$ *such that* $A \subseteq K \cap \tau_M(N)$ *and* $B \cap K \cap \tau_M(N) + L \neq N$ *for any proper submodule* L *of* N/L *singular, then* (2) $\Rightarrow$ (1).

Proof. The assertions follow from [2, Theorem 2.12]. □

Proposition 2.5. *Let* $N \in \sigma[M]$ *be a* τ_M*-semiregular module. Then* $\tau_M(N)$ *is QFSL in* N.

Proof. Let $N/\tau_M(N) = [K + \tau_M(N)/\tau_M(N)] \oplus L/\tau_M(N)$ for a finitely generated submodule K of N and a submodule L of N. Since N is τ_M-semiregular, there exists a decomposition $N = A \oplus B$ where A is a projective submodule of K and $B \cap K \subseteq \tau_M(N)$. Thus $K = A \oplus (B \cap K)$ and so $A + \tau_M(N) = K + \tau_M(N)$. □

Theorem 2.2. *Let* $N \in \sigma[M]$ *be a projective module. Then the following statements are equivalent:*

(1) N *is* τ_M*-semiregular;*

(2) N *is finitely* τ_M*-supplemented and* $\tau_M(N)$ *is QFSL;*

(3) *Every finitely generated submodule of* $N/\tau_M(N)$ *is a direct summand and* $\tau_M(N)$ *is QFSL.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1) Let U be a finitely generated submodule of N. By (3), we have $N/\tau_M(N) = [U+\tau_M(N)/\tau_M(N)]\oplus[K/\tau_M(N)]$ for a submodule K and so there exists a decomposition $N = A\oplus B$ such that $A \subseteq U$, $A+\tau_M(N) = U+\tau_M(N)$. Since $\tau_M(N) = \tau_M(A) \oplus \tau_M(B)$, it follows that $U \cap B \subseteq (U + \tau_M(N)) \cap (B + \tau_M(N)) = (A+\tau_M(N))\cap(B+\tau_M(N)) = [(A+\tau_M(B))\cap B]+\tau_M(A) = \tau_M(N)$. Hence $U \cap B \subseteq \tau_M(N) \cap B \subseteq \tau_M(B)$ and so N is τ_M-semiregular. □

In [3], a proper submodule K of N is called *DM* in N if there exists a direct summand S of N such that $S \subseteq K$ and $N = S + X$ whenever $N = K + X$ for a submodule X of N. A module $N \in \sigma[M]$ is said to be *finitely refinable* if, whenever $N = A + B$ for a finitely generated submodule A and a submodule B, there exists a direct summand C of N such that $C \subseteq A$ and $N = C + B$ (see [6]).

Theorem 2.3. *Let* $N \in \sigma[M]$ *be a module. Consider the following conditions:*

(1) *N is finitely refinable.*
(2) *Every submodule of $\tau_M(N)$ is QFSL in N.*
(3) *Every finitely generated submodule of $\tau_M(N)$ is DM in N.*
Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). If N is τ_M-semiregular, then (3) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) Let H be a submodule of $\tau_M(N)$ and $(L+H)/H \oplus K/H = N/H$ for a finitely generated submodule L and a submodule K of N. Then $L+K = N$ and so there exists a direct summand S of N such that $S+K = N$ and $S \subseteq L$. Hence $(S+H)/H \oplus K/H = (L+H)/H \oplus K/H$ and so $S+H = L+H$.

(2) $\Rightarrow$ (3) Let K be a finitely generated submodule of $\tau_M(N)$ such that $N = K + L$ for a submodule L. Then $K/(K \cap L)$ is a direct summand of $N/(K \cap L)$. Thus there exists a direct summand S of N such that $S \subseteq K$ and $S + (K \cap L) = K$. Then $S + L = N$ and so K is DM in N.

(3) $\Rightarrow$ (1) Assume that every finitely generated submodule of $\tau_M(N)$ is DM in N. Let K be a finitely generated submodule of N and $N = K + L$ for a submodule L. Then $K = A \oplus (K \cap B)$ such that $N = A \oplus B$ and $K \cap B \subseteq \tau_M(B)$, since N is τ_M-semiregular. It follows that $N = A+(K\cap B)+L$ and so $B = (K \cap B) + [(A+L) \cap B]$. Since every finitely generated submodule of $\tau_M(B)$ is DM in B by [3, Lemma 3.2], there exists a direct summand C of B such that $B = [(A + L) \cap B] + C$ and $C \subseteq K \cap B$ and so $A \oplus C$ is a direct summand of N and $N = (A + C) + L$. Then K is DM in N. □

Proposition 2.6. *Let τ and ρ be preradicals and $N \in \sigma[M]$ be a τ_M-semiregular module such that $\tau_M(N) + L = N$ and $\tau_M(N) \cap L \subseteq \rho_M(L)$ for a finitely generated submodule L of N. Then there is a decomposition $N = A \oplus B$ such that A is ρ_M-semiregular and $B \subseteq \tau_M(N)$.*

Proof. Let N be τ_M-semiregular. Then there exists a decomposition $N = A \oplus B$ such that A is projective, $L = A \oplus (B \cap L)$ and $B \cap L \subseteq \tau_M(B)$ hence $B \cap L \subseteq \tau_M(N) \cap L \subseteq \rho_M(L)$. Now we show that A is ρ_M-semiregular and $B \subseteq \tau_M(N)$. Let K be a finitely generated submodule of A. Since A is a direct summand of N, it is also τ_M-semiregular. Thus there exists a decomposition $A = X \oplus Y$ such that X is a projective submodule of K and $Y \cap K \subseteq \tau_M(Y)$. Also $Y \cap K \subseteq \tau_M(Y) \cap L \subseteq \rho_M(N) \cap Y = \rho_M(Y)$ because Y is a direct summand of N. Thus A is ρ_M-semiregular. Since $\tau_M(N) = \tau_M(A) + \tau_M(B)$, we have $N = \tau_M(N) + L = \tau_M(A) + \tau_M(B) + A + (B \cap L) = A \oplus \tau_M(B)$ and so $\tau_M(B) = B \subseteq \tau_M(N)$. □

Theorem 2.4. *Let R be a local ring. Then the following statements are equivalent:*
(1) *R_R is τ-semiregular;*
(2) *Every finitely generated free R-module is τ-semiregular;*
(3) *Every finitely generated projective R-module is τ-semiregular;*
(4) *If F is a finitely generated free R-module and N is a finitely generated fully invariant submodule of F, then F/N is τ-semiregular.*

Proof. (1) ⇒ (2) Let R be τ-semiregular. Then by Theorem 2.1, a finitely generated free module is τ-semiregular.

(2) ⇒ (3) ⇒ (1) and (4) ⇒ (1) They are clear.

(2) ⇒ (4) Let K/N be a finitely generated submodule of F/N. Then there exists a decomposition $F = A \oplus B$ where A is a projective submodule of K and $B \cap K \subseteq \tau(B)$. Then $F/N = (A+N)/N \oplus (B+N)/N$ and $(A+N)/N$ is a projective submodule of K/N. Moreover, $(B+N)/N \cap K/N = (B \cap K + N)/N \subseteq \tau(B + N/N)$. Hence F/N is τ-semiregular. □

3. $\delta(M)$ AND SOC(M)

According to Zhou [15], a submodule N of a module M is called *δ-small* in M, denoted by $N \ll_\delta M$, if $N + K \neq M$ for any proper submodule K of M/K singular. Moreover, Zhou introduced the following fully invariant submodule of a module M

$$\delta(M) = \bigcap \{N \leq M : M/N \text{ is singular simple}\}.$$

Then $\delta(M)$ is the sum of all $\delta(M)$-small submodules of M by [15, Lemma 1.5].

A pair (P, ρ) is called a *projective δ-cover* of the module M if P is projective and ρ is an epimorphism of P onto M with $\mathrm{Ker}(\rho) \ll_\delta P$ (see [15]).

LEMMA 3.1 ([15, Lemma 2.4]). *Let P be a projective module and N a submodule of P. Then the following are equivalent:*

(1) *P/N has a projective δ-cover.*

(2) *$P = P_1 \oplus P_2$ for some P_1 and P_2 with $P_1 \subseteq N$ and $P_2 \cap N \ll_\delta P$.*

THEOREM 3.1. *Let R be a local ring and M be a projective R-module. Then the following are equivalent:*

(1) *M is δ-semiregular;*

(2) *For every $N \cong \frac{M^n}{K}$ for some $n \in \mathbb{N}$ and finitely generated $K \subseteq M^n$, N has a projective δ-cover.*

(3) *For every finitely generated submodule K of M, M/K has a projective δ-cover.*

Proof. (1) ⇒ (2) Let $N \cong \frac{M^n}{K}$ for some $n \in \mathbb{N}$ and finitely generated $K \subseteq M^n$. Since M is δ-semiregular, M^n is δ-semiregular by Theorem 2.1. Thus, by Lemma 3.1, N has a projective δ-cover.

(2) ⇒ (3) is clear.

(3) ⇒ (1) By Lemma 3.1, M is δ-semiregular. □

COROLLARY 3.1. *Let R be a local ring. Then the following are equivalent:*

(1) *R is δ-semiregular;*

(2) *For every finitely generated right ideal I of R, R/I has a projective δ-cover.*

(3) *Every finitely presented right R-module has a projective δ-cover.*

Lemma 3.2 (See [12, Lemma 2.22]). *Let P be a projective module and $N \leq P$. Then the following are equivalent:*

(1) *P/N has a projective Soc-cover;*

(2) *$P = P_1 \oplus P_2$ for some P_1 and P_2 with $P_1 \leq N$ and $P_2 \cap N \subseteq Soc(P)$.*

Theorem 3.2. *Let R be a local ring and let M be a projective R-module. Then the following are equivalent:*

(1) *M is Soc-semiregular;*

(2) *For every $N \cong \frac{M^n}{K}$ for some $n \in \mathbb{N}$ and finitely generated $K \subseteq M^n$, N has a projective Soc-cover.*

(3) *For every finitely generated submodule K of M, M/K has a projective Soc-cover.*

Proof. (1) $\Rightarrow$ (2) Let $N \cong \frac{M^n}{K}$ for some $n \in \mathbb{N}$ and finitely generated $K \subseteq M^n$. Since M is Soc-semiregular, M^n is Soc-semiregular by Theorem 2.1. Thus, by Lemma 3.2, N has a projective Soc-cover.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) By Lemma 3.2, M is Soc-semiregular. □

Corollary 3.2. *Let R be a local ring. Then the following are equivalent:*

(1) *R is Soc-semiregular;*

(2) *For every finitely generated right ideal I of R, R/I has a projective Soc-cover.*

(3) *Every finitely presented right R-module has a projective Soc-cover.*

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