MATHEMATICA, Tome 56 (79), N° 2, 2014, pp. 182–192

AN ARCLENGTH PROBLEM FOR SOME SUBCLASSES OF m-FOLD SYMMETRIC UNIVALENT FUNCTIONS

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Abstract. For $0 < \beta \leq 1$, let $\mathcal{F}_m(\beta)$ (respectively $\mathcal{G}_m(\beta)$) denote the class of analytic functions f in the unit disk $\mathbb{D}$ with $f(0) = 0$, $f'(0) = 1$ and $f(e^{\frac{2\pi i}{m}} z) = e^{\frac{2\pi i}{m}} f(z)$ satisfying $\operatorname{Re} P_f(z) < \frac{\beta}{2} + 1$ (respectively $\operatorname{Re} P_f(z) > \frac{\beta}{2} - 1$) for $z \in \mathbb{D}$, where

$$P_f(z) = 1 + \frac{zf''(z)}{f'(z)}.$$

For $|\alpha| < \pi/2$, let $\mathcal{S}_\alpha$ denote the class of univalent functions $f(z)$ for which $zf'(z)$ is spirallike functions which has been introduced by M.S. Robertson [18]. The main aim of this paper is to investigate arclength problem

$$L_r(f) = \int_{|z|=r} |f'(z)|\,|dz|, \quad 0 < r < 1$$

for functions f in $\mathcal{F}_m(\beta)$, $\mathcal{G}_m(\beta)$ and $\mathcal{S}_\alpha$. As a consequence, we shall obtain arclength for functions in some subclasses of the class of univalent functions. In each of these subclasses, we shall provide extremal functions to obtain the sharp upper bound for $L_r(f)$.

MSC 2010. 34C45.
Key words. Univalent functions, starlike, convex, close-to-convex, spirallike functions, hypergeometric functions, arclength.

1. AREA OF A SURFACE AND PERIOD MAP

Let $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ be an open disk with center origin and radius r and $\partial\mathbb{D}_r := \{z \in \mathbb{C} : |z| = r\}$ be the circle with center origin and radius r in the complex plane $\mathbb{C}$. In particular, $\mathbb{D} := \mathbb{D}_1$ denotes the unit disk in $\mathbb{C}$. Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\mathbb{D}$. Here we think of $\mathcal{H}$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. Let $\mathcal{A}$ denote the family of functions f in $\mathcal{H}$ normalized by $f(0) = 0$ and $f'(0) = 1$. A function f is said to be univalent in $\mathbb{D}$ if it is one-to-one in $\mathbb{D}$. Let $\mathcal{S}$ denote the class of univalent functions in $\mathcal{A}$. Denote by $\mathcal{S}^*$ the subclass of functions $f \in \mathcal{A}$ such that f maps $\mathbb{D}$ univalently onto a domain $f(\mathbb{D})$ that is starlike with respect to the origin. That is, $tf(z) \in f(\mathbb{D})$ for each $t \in [0, 1]$. It is well-known that $f \in \mathcal{S}^*$

The author thanks IIT Kharagpur for financial support through ISRD project (Grant No. IIT/SRIC/MA/SUA/2012-13/14).

if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}.$$

Functions in $\mathcal{S}^*$ are refereed to as starlike functions. A domain $\Omega \subset \mathbb{C}$ is said to be convex if it is starlike with respect to every point in Ω. A function $f \in \mathcal{A}$ is said to be convex if $f(\mathbb{D})$ is a convex domain. We denote the class of convex functions in $\mathbb{D}$ by $\mathcal{C}$. A function $f \in \mathcal{A}$ is in $\mathcal{C}$ if and only if

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

It is well-known that $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$. A function $f \in \mathcal{A}$ is said to be close-to-convex if there exists a convex univalent function g and a number $\phi \in \mathbb{R}$ such that

$$\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\phi}\frac{f'(z)}{g'(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$

Let $\mathcal{K}$ denote the class of close-to-convex functions f in $\mathcal{A}$. It is well-known that every close-to-convex function is univalent in $\mathbb{D}$ (see [4]).

A domain Ω containing the origin is said to be α-spirallike if for each point $w_0 \neq 0$ in Ω the arc of the α-spiral from w_0 to the origin lies entirely in Ω. A function f analytic and univalent in the unit disk $\mathbb{D}$ with $f(0) = 0$ is said to be α-spirallike if its range is α-spirallike. A function is spirallike if it is α-spirallike for some α. Spirallike functions can be characterized by an analytic condition which is a slight generalization of the condition for starlikeness. A function $f \in \mathcal{A}$ is α-spirallike if for some real constant α ($|\alpha| < \pi/2$),

$$\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\alpha}\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}.$$

We denote by $\mathcal{S}p(\alpha)$ the class of all α-spriallike functions in $\mathbb{D}$. Thus

$$\bigcup_{-\pi/2<\alpha<\pi/2} \mathcal{S}p(\alpha)$$

denotes the class of spirallike functions in $\mathbb{D}$. In particular, $\mathcal{S}p(0)$ denotes the usual class of starlike functions $\mathcal{S}^*$.

We consider another family of functions that includes the class of convex functions as a proper subfamily. For $|\alpha| < \pi/2$, we say that $f \in \mathcal{S}_\alpha$ if $f \in \mathcal{A}$ is locally univalent in $\mathbb{D}$ and $\mathrm{Re}\,\widetilde{P}_f(z) > 0$ in $\mathbb{D}$, where

$$\widetilde{P}_f(z) = \mathrm{e}^{\mathrm{i}\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right).$$

It is easy to see that $f \in \mathcal{S}_\alpha$ if and only if there exists a function $g \in \mathcal{S}^*$ such that

$$f'(z) = \left(\frac{g(z)}{z}\right)^{(\cos\alpha)\exp(-\mathrm{i}\alpha)}. \tag{1}$$

Also, we observe that the above conditions are precisely the conditions for the function $zf'(z)$ to be spirallike. The class $\mathcal{S}_0$ consists of the normalized convex functions. For general values of α ($|\alpha| < \pi/2$), a function in $\mathcal{S}_\alpha$ need not be univalent in $\mathbb{D}$. For example, the function $f(z) = \mathrm{i}(1-z)^{\mathrm{i}} - \mathrm{i}$ belongs to $\mathcal{S}_{\pi/4} \backslash \mathcal{S}$. In 1968, M.S. Robertson [18] proved that $f \in \mathcal{S}_\alpha$ is univalent if $0 < \cos\alpha \leq 0.2315\ldots$ and showed that there are non-univalent functions $f \in \mathcal{S}_\alpha$ for each α with $\frac{1}{2} < \alpha < 1$. In 1972, R. J. Libera and M.R. Zeigler [7] improved the range of α as $0 < \cos\alpha \leq 0.2564\ldots$ for which $f \in \mathcal{S}_\alpha$ is univalent. Further, in 1975, P.N. Chichra [1] has improved the range of α as $0 < \cos\alpha \leq 0.2588\ldots$. In the same year J.A. Pfaltzgraff [12] has shown that $f \in \mathcal{S}_\alpha$ is univalent whenever $0 < \cos\alpha \leq 1/2$. This settles the improvement of range of α for which $f \in \mathcal{S}_\alpha$ is univalent. On the other hand, V. Singh [17] has shown that functions in $\mathcal{S}_\alpha$ which satisfy $f''(0) = 0$ are univalent for all real values of α with $|\alpha| < \pi/2$. For a general reference about many of these special classes we refer to [4, 5, 13].

Let a, b and c be complex numbers with $c \neq 0, -1, -2, -3, \ldots$. Then the function

$$ {}_2F_1(a,b;c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots, $$

is called the Gaussian hypergeometric function which is analytic in $\mathbb{D}$ and satisfies the differential equation

$$ z(1-z)w''(z) + [c - (a+b+1)z]w' - abw = 0. $$

A function $f \in \mathcal{H}$ is said to be m-fold symmetric ($m = 1, 2, 3, \ldots$) if

$$ f\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{m}} z\right) = \mathrm{e}^{\frac{2\pi\mathrm{i}}{m}} f(z). \tag{2} $$

In particular, every analytic function $f(z)$ is 1-fold symmetric and every odd analytic function $f(z)$ is 2-fold symmetric. If $f \in \mathcal{H}$ is an m-fold symmetric function then it is not difficult to see that f has the following representation

$$ f(z) = z + a_{m+1}z^{m+1} + a_{2m+1}z^{2m+1} + \cdots. $$

For $0 < \beta \leq 1$, let $\mathcal{F}_m(\beta)$ be the class of analytic functions $f \in \mathcal{A}$ satisfying (2) and

$$ \operatorname{Re} P_f(z) < \frac{\beta}{2} + 1 \quad \text{for } z \in \mathbb{D}, $$

where

$$ P_f(z) = 1 + \frac{zf''(z)}{f'(z)}. $$

For $m = 1$ the class $\mathcal{F}_m(\beta)$ reduces to the following class

$$ \mathcal{F}_1(\beta) = \left\{ f \in \mathcal{A} : \operatorname{Re} P_f(z) < \frac{\beta}{2} + 1 \quad \text{for } z \in \mathbb{D} \right\}. $$

If $f(z) = z + \sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}(\beta)$ then it has been proved [11] that

$$|a_n| \leq \frac{\beta}{n(n-1)} \quad \text{for } z \in \mathbb{D}.$$

These bounds are best possible. For the sharp Fekete-Szegö inequality and other important characterization for functions in $\mathcal{F}(\beta)$ we refer to [11]. Further, for $m = 1$ and $\beta = 1$ the class $\mathcal{F}_m(\beta)$ reduces to

$$\mathcal{F}_1 := \mathcal{F}_1(1) = \left\{ f \in \mathcal{A} : \operatorname{Re} P_f(z) < \frac{3}{2} \quad \text{for } z \in \mathbb{D} \right\}.$$

Another class of our interest in this paper is $\mathcal{G}_m(\beta)$. More precisely, for $0 < \beta \leq 1$ let $\mathcal{G}_m(\beta)$ be the class of analytic functions $f \in \mathcal{A}$ satisfying (2) and

$$\operatorname{Re} P_f(z) > \frac{\beta}{2} - 1 \quad \text{for } z \in \mathbb{D}.$$

For $m = 1$ the class $\mathcal{G}_m(\beta)$ reduces to

$$\mathcal{G}_1(\beta) = \left\{ f \in \mathcal{A} : \operatorname{Re} P_f(z) > \frac{\beta}{2} - 1 \quad \text{for } z \in \mathbb{D} \right\}.$$

Further, for $m = 1$ and $\beta = 1$ the class $\mathcal{G}_m(\beta)$ reduces to

$$\mathcal{G}_1 := \mathcal{G}_1(1) = \left\{ f \in \mathcal{A} : \operatorname{Re} P_f(z) > -\frac{1}{2} \quad \text{for } z \in \mathbb{D} \right\}.$$

It is known that $\mathcal{F}_1 \subset \mathcal{S}^*$ and $\mathcal{G}_1 \subset \mathcal{K}$ (see [14, Equation (16)] and [15]). The regions of variability for the classes $\mathcal{F}_1$ and $\mathcal{G}_1$ have been studied in [16]. For a detailed discussion about these classes we refer to [14, 15].

For $f \in \mathcal{S}$ and $0 < r < 1$, let

$$\begin{aligned} L_r(f) &= \int_{|z|=r} |f'(z)|\,|\mathrm{d}z| \\ &= \int_0^{2\pi} r|f'(r\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\theta \end{aligned}$$

denote the arclength of the image of the circle $\partial\mathbb{D}_r$. The class $\mathcal{S}$ is a compact subset of the class $\mathcal{A}$ in $\mathbb{D}$ endowed with the topology of uniform convergence on every compact subset of $\mathbb{D}$. Since the functional $\mathcal{S} \ni f \mapsto L_r(f)$ is continuous on $\mathcal{A}$, a solution of the extremal problem

$$\max_{f \in \mathcal{S}} L_r(f)$$

exists and is in $\mathcal{S}$. Although the arclength problem for functions f in $\mathcal{S}$ has been investigated, the sharp upper bound for $L_r(f)$ is still unknown (see [4], p.39). In 1954, Keogh [6] has investigated the extremal problem

$$\max_{f \in \mathcal{C}} L_r(f)$$

and proved that

$$\max_{f\in\mathcal{C}} L_r(f) \le \frac{2\pi r}{1-r^2}$$

with equality if and only if $f(z) = \frac{z}{1-e^{i\theta}z}$ for some $\theta \in \mathbb{R}$. The problem

$$\max_{f\in\mathcal{S}^*} L_r(f)$$

has been solved by Marx [8], who showed that

$$\max_{f\in\mathcal{S}^*} L_r(f) = L_r(k),$$

where k is the Koebe function $k(z) = \frac{z}{(1-z)^2}$. In 1966, Clunie and Duren [2] solved the extremal problem within the class $\mathcal{K}$ of all close-to-convex functions and shown that

$$\max_{f\in\mathcal{K}} L_r(f) = L_r(k).$$

Duren [3] also obtained an evaluation of $L_r(k)$ in terms of standard elliptic integrals. Arclength problems for the m-fold symmetric function of convex, starlike and close-to-convex functions have been investigated by S.S. Miller (see [9]).

The main aim of this paper is to investigate the arclength $L_r(f)$ $(0 < r < 1)$ for functions f in the classes $\mathcal{F}_m(\beta)$, $\mathcal{G}_m(\beta)$ and $\mathcal{S}_\alpha$. As a consequence we obtain the arclength for functions in the subclasses $\mathcal{F}_1(\beta)$, $\mathcal{G}_1(\beta)$, $\mathcal{F}_1$, and $\mathcal{G}_1$. We shall obtain the sharp upper bound for $L_r(f)$ when f ranges over each of these classes. Before stating our results we recall the following result [9] which plays a vital role in proving our results.

LEMMA 1. ([9, Lemma 5]) *Let $\mu(\phi)$ be non-decreasing on $[0, 2\pi]$ and $\mu(2\pi)-\mu(0) = 1$. If $h(\phi)$ is positive and integrable with respect to $\mu(\phi)$ on $[0, 2\pi]$ then*

$$\exp\left(\int_0^{2\pi} \ln h(\phi)\,(\phi)\right) \le \int_0^{2\pi} h(\phi)\,\mathrm{d}\mu(\phi).$$

2. MAIN RESULTS ON ARCLENGTH PROBLEM

The following result is the arclength problem for the class of univalent functions $f(z)$ for which $zf'(z)$ is spirallike functions which has been introduced by M.S. Robertson [18].

THEOREM 1. *If $f \in \mathcal{S}_\alpha$, then for $0 < r < 1$,*

$$L_r(f) \le L_r(\Psi), \tag{3}$$

where $\Psi'(z) = (1-z)^{-2(\cos\alpha)e^{-i\alpha}}$. The inequality in (3) is sharp.

Proof. Let $f \in \mathcal{S}_\alpha$. Then for $|\alpha| < \frac{\pi}{2}$ we have $\mathrm{Re}\,\widetilde{P}_f(z) > 0$ in $\mathbb{D}$, where

$$\widetilde{P}_f(z) = e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right).$$

Clearly $\widetilde{P}_f(0) = \mathrm{e}^{\mathrm{i}\alpha}$. Then by the well-known Herglotz representation for functions with positive real part, there exists a unit positive measure μ on $(0, 2\pi]$ such that

$$\widetilde{P}_f(z) = (\cos\alpha)\int_0^{2\pi} \frac{1+z\mathrm{e}^{-\mathrm{i}t}}{1-z\mathrm{e}^{-\mathrm{i}t}}\,\mathrm{d}\mu(t) + \mathrm{i}\sin\alpha. \tag{4}$$

A simple computation of (4) gives

$$\frac{f''(z)}{f'(z)} = 2\mathrm{e}^{-\mathrm{i}\alpha}(\cos\alpha)\int_0^{2\pi} \frac{\mathrm{e}^{-\mathrm{i}t}}{1-z\mathrm{e}^{-\mathrm{i}t}}\,\mathrm{d}\mu(t),$$

which implies

$$\log f'(z) = 2\mathrm{e}^{-\mathrm{i}\alpha}(\cos\alpha)\int_0^{2\pi} \log(1-z\mathrm{e}^{-\mathrm{i}t})^{-1}\,\mathrm{d}\mu(t)$$

or

$$f'(z) = \exp\int_0^{2\pi} \log(1-z\mathrm{e}^{-\mathrm{i}t})^{-2\mathrm{e}^{-\mathrm{i}\alpha}(\cos\alpha)}\,\mathrm{d}\mu(t). \tag{5}$$

In view of Lemma 1 and (5) for $f \in \mathcal{S}_\alpha$, we obtain

$$\begin{aligned} |zf'(z)| &= \left| z\exp\int_0^{2\pi} \log(1-z\mathrm{e}^{-\mathrm{i}t})^{-2\mathrm{e}^{-\mathrm{i}\alpha}(\cos\alpha)}\,\mathrm{d}\mu(t)\right| \\ &= |z|\exp\int_0^{2\pi} \ln\left|(1-z\mathrm{e}^{-\mathrm{i}t})^{-2(\cos^2\alpha)}\right|\,\mathrm{d}\mu(t) \\ &\le |z|\int_0^{2\pi} |1-z\mathrm{e}^{-\mathrm{i}t}|^{-2(\cos^2\alpha)}\,\mathrm{d}\mu(t). \end{aligned}$$

Consequently,

$$L_r(f) = \int_0^{2\pi} |zf'(z)|\,\mathrm{d}\theta \le |z|\int_0^{2\pi}\int_0^{2\pi} |1-z\mathrm{e}^{-\mathrm{i}t}|^{-2(\cos^2\alpha)}\,\mathrm{d}\mu(t)\,\mathrm{d}\theta,$$

where $z = r\mathrm{e}^{\mathrm{i}\theta}$. By changing the order of integration and using the identity

$$\int_0^{2\pi} |1-z\mathrm{e}^{-\mathrm{i}t}|^{-2(\cos^2\alpha)}\mathrm{d}\theta = \int_0^{2\pi} |1-z|^{-2(\cos^2\alpha)}\mathrm{d}\theta,$$

we obtain

$$L_r(f) \le |z|\int_0^{2\pi} |1-z|^{-2(\cos^2\alpha)}\mathrm{d}\theta.$$

Let $k(z) = \frac{z}{(1-z)^2} \in \mathcal{S}^*$. Then put

$$\Psi'(z) = \left(\frac{k(z)}{z}\right)^{(\cos\alpha)\mathrm{e}^{-\mathrm{i}\alpha}} = (1-z)^{-2(\cos\alpha)\mathrm{e}^{-\mathrm{i}\alpha}}.$$

In view of (1) it is easy to see that $\Psi \in \mathcal{S}_\alpha$. Thus

$$L_r(\Psi) = \int_0^{2\pi} |z\Psi'(z)|\mathrm{d}\theta = |z|\int_0^{2\pi} |1-z|^{-2(\cos^2\alpha)}\mathrm{d}\theta.$$

Hence $\Psi(z)$ is a solution of the extremal problem

$$\max_{f\in\mathcal{S}_\alpha} L_r(f)$$

and the inequality in (3) is sharp. $\square$

THEOREM 2. *If $f\in\mathcal{F}_m(\beta)$, then for $0<r<1$,*

$$L_r(f)\le L_r(h_m), \tag{6}$$

where $h_m(z)=z\ {}_2F_1\left(\frac{1}{m},-\frac{\beta}{m};1+\frac{1}{m};z^m\right)$. The inequality in (6) *is sharp.*

Proof. If $f\in\mathcal{F}_m(\beta)$ then from (2) it is easy to see that

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)=\operatorname{Re}\left(1+\frac{z^mf''(z^m)}{f'(z^m)}\right)<\frac{\beta}{2}+1 \quad \text{for } z\in\mathbb{D}. \tag{7}$$

Equivalently, (7) can be written as

$$(\beta+2)-2\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)=(\beta+2)-2\operatorname{Re}\left(1+\frac{z^mf''(z^m)}{f'(z^m)}\right)>0. \tag{8}$$

Let $g_1(z)=(\beta+2)-2\left(1+\frac{z^mf''(z^m)}{f'(z^m)}\right)$. Then $g_1(0)=\beta$ and $\operatorname{Re} g_1(z)>0$ in $\mathbb{D}$. Hence by Herglotz representation for functions with positive real part, there exists a unique positive unit measure μ on $(0,2\pi]$ such that

$$g_1(z)=\beta\int_0^{2\pi}\frac{1+z^m\mathrm{e}^{-\mathrm{i}t}}{1-z^m\mathrm{e}^{-\mathrm{i}t}}\,\mathrm{d}\mu(t). \tag{9}$$

Therefore from (8) and (9) we have

$$(\beta+2)-2\left(1+\frac{zf''(z)}{f'(z)}\right)=\beta\int_0^{2\pi}\left(\frac{1+z^m\mathrm{e}^{-\mathrm{i}t}}{1-z^m\mathrm{e}^{-\mathrm{i}t}}\right)\mathrm{d}\mu(t),$$

which implies

$$\frac{f''(z)}{f'(z)}=\beta\int_0^{2\pi}\frac{-z^{m-1}\mathrm{e}^{-\mathrm{i}t}}{1-z^m\mathrm{e}^{-\mathrm{i}t}}\,\mathrm{d}\mu(t).$$

By integrating both sides, we obtain

$$f'(z)=\exp\int_0^{2\pi}\log(1-z^m\mathrm{e}^{-\mathrm{i}t})^{\frac{\beta}{m}}\,\mathrm{d}\mu(t).$$

In view of Lemma 1 and for $f\in\mathcal{F}_m(\beta)$, we see that

$$\begin{aligned}|zf'(z)| &= |z|\exp\int_0^{2\pi}\ln\left|1-z^m\mathrm{e}^{-\mathrm{i}t}\right|^{\frac{\beta}{m}}\,\mathrm{d}\mu(t)\\ &\le |z|\int_0^{2\pi}|1-z^m\mathrm{e}^{-\mathrm{i}t}|^{\frac{\beta}{m}}\,\mathrm{d}\mu(t).\end{aligned}$$

Consequently

$$L_r(f)=\int_0^{2\pi}|zf'(z)|\,\mathrm{d}\theta\le|z|\int_0^{2\pi}\int_0^{2\pi}|1-z^m\mathrm{e}^{-\mathrm{i}t}|^{\frac{\beta}{m}}\,\mathrm{d}\mu(t)\mathrm{d}\theta,$$

where $z = re^{i\theta}$. By changing the order of integration and using the following identity

$$\int_0^{2\pi} |1 - z^m e^{-it}|^{\frac{\beta}{m}} \, d\theta = \int_0^{2\pi} |1 - z^m|^{\frac{\beta}{m}} \, d\theta,$$

we obtain

$$L_r(f) \le |z| \int_0^{2\pi} |1 - z^m|^{\frac{\beta}{m}} \, d\theta. \tag{10}$$

In order to show that the inequality (6) is sharp, we define

$$h_m(z) = z \; {}_2F_1\left(\frac{1}{m}, -\frac{\beta}{m}; 1 + \frac{1}{m}; z^m\right). \tag{11}$$

A simple computation gives

$$\mathrm{Re}\left(1 + \frac{z h_m''(z)}{h_m'(z)}\right) = 1 - \beta \; \mathrm{Re}\left(\frac{z^m}{1 - z^m}\right) < 1 + \frac{\beta}{2}.$$

From (11), it is not difficult to see that $h_m(e^{\frac{2\pi i}{m}} z) = e^{\frac{2\pi i}{m}} h_m(z)$. That is $h_m(z)$ is an m-fold symmetric function and hence $h_m \in \mathcal{F}_m(\beta)$. Finally, a simple computation gives

$$L_r(h_m) = \int_0^{2\pi} |z h'(z)| \, d\theta = |z| \int_0^{2\pi} |1 - z|^{\frac{\beta}{m}} \, d\theta.$$

Therefore $h_m(z)$ is a solution of the extremal problem

$$\max_{f \in \mathcal{F}_m(\beta)} L_r(f).$$

This shows that the inequality (6) is sharp. □

As a special case, for $m = 1$, Theorem 2 reduces to the following arclength problem for functions in $\mathcal{F}_1(\beta)$.

COROLLARY 1. *If* $f \in \mathcal{F}_1(\beta)$, *then for* $0 < r < 1$,

$$L_r(f) \le L_r(l_1), \tag{12}$$

where

$$l_1(z) = \frac{1}{\beta + 1}\left(1 - (1 - z)^{\beta + 1}\right).$$

The inequality in (12) *is sharp.*

Further, by taking $m = 1$ and $\beta = 1$ in Theorem 2, we obtain the following interesting arclength problem for functions in the class $\mathcal{F}_1$ which is a subclass of $\mathcal{S}^*$.

COROLLARY 2. *If* $f \in \mathcal{F}_1$, *then for* $0 < r < 1$,

$$L_r(f) \le L_r(l_2), \tag{13}$$

where

$$l_2(z) = \frac{1}{2}\left(1 - (1 - z)^2\right).$$

The inequality in (13) *is sharp.*

Next, our aim is to investigate the arclength problem for functions in the class $\mathcal{G}_m(\beta)$.

THEOREM 3. *If* $f \in \mathcal{G}_m(\beta)$, *then for* $0 < r < 1$,

$$L_r(f) \le L_r(\widetilde{J}_m), \tag{14}$$

where $\widetilde{J}_m(z) = z\ {}_2F_1\left(\frac{1}{m}, \frac{(4-\beta)}{m}; 1+\frac{1}{m}; z^m\right)$. *The inequality in* (14) *is sharp.*

Proof. If $f \in \mathcal{G}_m(\beta)$ then from (2) we have

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) = \operatorname{Re}\left(1+\frac{z^m f''(z^m)}{f'(z^m)}\right) > \frac{\beta}{2}-1, \quad \text{for } z \in \mathbb{D}. \tag{15}$$

Equivalently, (15) can be written as

$$2\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) - (\beta-2) = 2\operatorname{Re}\left(1+\frac{z^m f''(z^m)}{f'(z^m)}\right) - (\beta-2) > 0, \tag{16}$$

for $z \in \mathbb{D}$. Let $g_2(z) = 2\left(1+\frac{z^m f''(z^m)}{f'(z^m)}\right) - (\beta-2)$. Then clearly $g_2(0) = 4-\beta$ and $\operatorname{Re} g_2(z) > 0$ for $z \in \mathbb{D}$. Hence by well-known Herglotz representation, there exists a unique positive unit measure μ on $(0, 2\pi]$ such that

$$g_2(z) = (4-\beta)\int_0^{2\pi} \frac{1+z^m \mathrm{e}^{-\mathrm{i}t}}{1-z^m \mathrm{e}^{-\mathrm{i}t}}\,\mathrm{d}\mu(t).$$

Therefore in view of (16) we can easily see that

$$2\left(1+\frac{zf''(z)}{f'(z)}\right) - (\beta-2) = (4-\beta)\int_0^{2\pi}\left(\frac{1+z^m \mathrm{e}^{-\mathrm{i}t}}{1-z^m \mathrm{e}^{-\mathrm{i}t}}\right)\mathrm{d}\mu(t). \tag{17}$$

A simple computation of (17) gives

$$\frac{f''(z)}{f'(z)} = (4-\beta)\int_0^{2\pi} \frac{z^{m-1}\mathrm{e}^{-\mathrm{i}t}}{1-z^m \mathrm{e}^{-\mathrm{i}t}}\,\mathrm{d}\mu(t),$$

and by integrating both sides we obtain

$$f'(z) = \exp\int_0^{2\pi} \log(1-z^m \mathrm{e}^{-\mathrm{i}t})^{-\frac{(4-\beta)}{m}}\,\mathrm{d}\mu(t).$$

Thus by applying Lemma 1 for $f \in \mathcal{G}_m(\beta)$, we obtain

$$|zf'(z)| \quad\le\quad |z|\int_0^{2\pi} \left|1-z^m \mathrm{e}^{-\mathrm{i}t}\right|^{-\frac{(4-\beta)}{m}}\,\mathrm{d}\mu(t),$$

and consequently

$$L_r(f) = \int_0^{2\pi} |zf'(z)|\,\mathrm{d}\theta \le |z|\int_0^{2\pi}\int_0^{2\pi} \left|1-z^m \mathrm{e}^{-\mathrm{i}t}\right|^{-\frac{(4-\beta)}{m}}\,\mathrm{d}\mu(t)\,\mathrm{d}\theta,$$

where $z = re^{i\theta}$. By changing the order of integration and using the following identity

$$\int_0^{2\pi} |1 - z^m e^{-it}|^{-\frac{(4-\beta)}{m}} \, d\theta = \int_0^{2\pi} |1 - z^m|^{-\frac{(4-\beta)}{m}} \, d\theta,$$

we obtain

$$L_r(f) \le |z| \int_0^{2\pi} |1 - z^m|^{-\frac{(4-\beta)}{m}} \, d\theta. \tag{18}$$

To see the sharpness of the inequality (14), we let

$$\widetilde{J}_m(z) = z \; {}_2F_1\left(\frac{1}{m}, \frac{(4-\beta)}{m}; 1 + \frac{1}{m}; z^m\right).$$

Then a simple computation gives

$$\operatorname{Re}\left(1 + \frac{z\widetilde{J}_m''(z)}{\widetilde{J}_m'(z)}\right) = 1 + \beta \operatorname{Re}\left(\frac{z^m}{1 - z^m}\right) > \frac{\beta}{2} - 1.$$

Since $\widetilde{J}_m(e^{\frac{2\pi i}{m}} z) = e^{\frac{2\pi i}{m}} \widetilde{J}_m(z)$, we see that $\widetilde{J}_m(z)$ is an m-fold symmetric function. Thus

$$L_r(\widetilde{J}_m) = \int_0^{2\pi} |zh'(z)| \, d\theta = |z| \int_0^{2\pi} |1 - z^m|^{\frac{-(4-\beta)}{m}} \, d\theta,$$

which shows that (14) is sharp. □

In the case of $m = 1$, Theorem 3 reduces to the following interesting arclength problem for functions in the class $\mathcal{G}_1(\beta)$.

Corollary 3. *If $f \in \mathcal{G}_1(\beta)$, then for $0 < r < 1$,*

$$L_r(f) \le L_r(l_3), \tag{19}$$

where

$$l_3(z) = \frac{1}{3-\beta}\left(\frac{1}{(1-z)^{(3-\beta)}} - 1\right).$$

The inequality in (19) *is sharp.*

For $m = 1$ and $\beta = 1$, Theorem 3 reduces to the following interesting arclength problem for functions in the class $\mathcal{G}_1$ which is a subclass of $\mathcal{K}$.

Corollary 4. *If $f \in \mathcal{G}_1$, then for $0 < r < 1$,*

$$L_r(f) \le L_r(l_4), \tag{20}$$

where

$$l_4(z) = \frac{1}{2}\left(\frac{1}{(1-z)^2} - 1\right).$$

The inequality in (20) *is sharp.*

Since the proofs of the Corollaries 3 and 4 are on the same lines of that of Theorem 14, we omit the details.

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Received March 14, 2014
Accepted September 15, 2014

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