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SOME FIXED POINT THEOREMS IN TERMS OF TWO MEASURES OF NONCOMPACTNESS

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Abstract. In this paper several fixed point theorems of Sadovskii type are obtained for operators on spaces endowed with two norms and two corresponding measures of noncompactness. An application to Hammerstein integral equations in a Banach space is included to illustrate the theory.

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1. INTRODUCTION

In recent years much work has been devoted to establish fixed point theorems in the terms of some abstract measure of noncompactness (see, e.g. [1, 4, 5, 8, 20, 21]). In this paper we introduce a less restrictive notion of abstract measure of noncompactness. We consider on a linear space endowed with two norms, two corresponding such abstract measures of noncompactness. In terms of these measures we give several fixed point theorems of Sadovskii type. Similar results are given in a set with two metrics. An application to Hammerstein integral equations in a Banach space illustrates our abstract results.

2. PRELIMINARIES

2.1. Notations. Let (X, d) be a metric space. We will use the following notations:

$\mathcal{P}(X) := \{Y \mid Y \subset X\}$,
$P(X) := \{Y \subset X \mid Y \text{ is nonempty }\}$,
$P_b(X) := \{Y \in P(X) \mid Y \text{ is bounded }\}$,
$P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed }\}$.

If X is a linear space, then $P_{cv}(X) := \{Y \in P(X) \mid Y \text{ is convex }\}$.
If $f : X \to X$ is an operator, then $F_f := \{x \in X \mid f(x) = x\}$.

2.2. Invariant subsets in terms of closure operators. Let X be a nonempty set. An operator $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ is called a *closure operator* if the following conditions are satisfied:

(i) $Y \subset \eta(Y)$ for every $Y \in \mathcal{P}(X)$;
(ii) $\eta(Y) \subset \eta(Z)$ for every $Y, Z \in \mathcal{P}(X)$ with $Y \subset Z$;
(iii) $\eta \circ \eta = \eta$.

Our results are based on the following lemma (see [21, p. 21]).

Lemma 2.1 (General Invariant Subset Lemma). *Let X be a nonempty set, $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ a closure operator, $Y \in F_\eta$ a set, $y \in X$ a point and $f : Y \to Y$ an operator. Then there exists a subset $Y_0 \subset Y$ such that:*

(1) $y \in Y_0$;
(2) $Y_0 \in F_\eta$;
(3) $Y_0 \in I(f)$;
(4) $\eta(f(Y_0) \cup \{y\}) = Y_0$.

2.3. Rectractible operators. Let X be a nonempty set and $Y \subset X$ a nonempty subset. An operator $\rho : X \to Y$ is said to be a *set retraction* if its restriction to Y is the identity map of Y, i.e. $\rho|_Y = 1_Y$. In case that X is a structured set (for instance, an ordered set, a topological space etc), we say that a set retraction ρ is a *retraction* with respect to that structure (an ordered set retraction, a topological retraction etc) if in addition ρ is a morphism with respect to that structure (increasing, continuous etc). By definition, an operator $f : Y \to X$ is *retractible* with respect to a retraction $\rho : X \to Y$, if $F_f = F_{\rho \circ f}$. For examples of retractible operators, see [6, 21, 22, 23].

For the radial retraction, we have the following result.

Lemma 2.2 (see [18]). *Let X be a linear normed space and $\alpha : P_b(X) \to \mathbf{R}_+$ be the Kuratowski measure of noncompactness on X, and $\rho : X \to B_R(0)$ the radial retraction. Then $\alpha(\rho(Y)) \le \alpha(Y)$ for every $Y \in P_b(X)$.*

2.4. Abstract measures of noncompactness. Let (X, d) be a metric space. There are known several notions of abstract measures of noncompactness on X (see, e.g. [1, 4, 5, 7, 20, 21]). In this paper we shall use a less restrictive one.

Definition 2.3. A functional $\theta : P_b(X) \to R_+$ is an *abstract measure of noncompactness* on (X, d) if the following conditions are satisfied:

(i) $\theta(Y) = 0$, $Y \in P_b(X)$ imply that Y is totally bounded;
(ii) $\theta(Y_1) \le \theta(Y_2)$ for every $Y_1, Y_2 \in P_b(X)$ with $Y_1 \subset Y_2$;
(iii) $\theta(Y \cup \{x\}) = \theta(Y)$ for every $Y \in P_b(X)$ and $x \in X$;
(iv) $\theta\left(\overline{Y} = \theta(Y)\right)$ for every $Y \in P_b(X)$.

If X is a normed linear space, then an additional axiom is added:

(v) $\theta(coY) = \theta(Y)$ for every $Y \in P_b(X)$.

Remark 2.4. If θ is an abstract measure of noncompactness on a normed linear space, then in Lemma 2.2 we can put θ instead of α.

3. MAIN RESULTS

3.1. Fixed point theorems in a linear space with two norms. Let X be a linear space and $\|.\|_1$, $\|.\|_2$ be two norms on X. Let θ_1 and θ_2 be two abstract measures of noncompactness on $(X, \|.\|_1)$ and $(X, \|.\|_2)$, respectively. Our first result is the following fixed point theorem for a self operator.

Theorem 3.1. *Let* $Y \subset X$ *and* $f : Y \to Y$. *Assume that the following conditions are satisfied:*

(i) $(X, \|.\|_1)$ *is a Banach space;*
(ii) *there exists* $c_1 > 0$ *such that* $\|.\|_2 \le c_1 \|.\|_1$;
(iii) $Y \in P_{b,cl,cv}(X, \|.\|_1)$;
(iv) f *is continuous with respect to* $\|.\|_1$;
(v) *there exists* $c_2 > 0$ *such that* $\theta_1(f(A)) \le c_2\theta_2(A)$ *for every* $A \in I(f)$;
(vi) *for each* $A \in I(f)$ *with* $\theta_2(A) \ne 0$, *one has* $\theta_2(f(A)) < \theta_2(A)$.

Then $F_f \ne \emptyset$ *and* $\theta_1(F_f) = 0$, *i.e.* F_f *is compact with respect to* $\|.\|_1$.

Proof. Denote by cl_i the topological closure operator on $(X, \|.\|_i)$, $i = 1, 2$. Let y_0 be any element of Y. By the General Invariant Subset Lemma for the closure operator $\mathrm{cl}_1\mathrm{co}$, there exists $Y_0 \subset Y$ such that

$$\mathrm{cl}_1\mathrm{co}(f(Y_0) \cup \{y_0\}) = Y_0.$$

From (ii) and the axioms in Definition 2.3, we have that

$$\begin{aligned} \theta_2(\mathrm{cl}_2\mathrm{cl}_1\mathrm{co}(f(Y_0) \cup \{y_0\})) &= \theta_2(\mathrm{cl}_2 Y_0) = \theta_2(Y_0) \\ &= \theta_2(\mathrm{cl}_2\mathrm{co}(f(Y_0) \cup \{y_0\})) \\ &= \theta_2(f(Y_0) \cup \{y_0\}) \\ &= \theta_2(f(Y_0)). \end{aligned}$$

Hence $\theta_2(f(Y_0)) = \theta_2(Y_0)$, and in view of (vi), $\theta_2(Y_0) = 0$. Then by (iv), $\theta_1(f(Y_0)) = 0$. Then $\theta_1(\mathrm{cl}_1\mathrm{co}(f(Y_0))) = 0$, that is $\mathrm{cl}_1\mathrm{co}(f(Y_0))$ is compact (also convex) in $(X, \|.\|_1)$. Being also an invariant set for f, we may apply Schauder's fixed point theorem and deduce that $F_f \ne \emptyset$. Since $F_f \in I(f)$ and $f(F_f) = F_f$, from (vi) we have $\theta_2(F_f) = 0$, and then from (v), $\theta_1(F_f) = 0$. □

The following particular case appears to be useful in applications. Let E be a Banach space and $X = C([a, b]; E)$. Consider on X the following two norms:

$$\|.\|_1 = \|.\|_\infty \quad \text{and} \quad \|.\|_2 = \|.\|_p$$

for some $p \in [1, \infty)$. In this case, Theorem 3.1 takes the following form:

Theorem 3.2. *Assume that:*

(i) $Y \in P_{b,cl,cv}(X, \|.\|_\infty)$;
(ii) $f : Y \to Y$ *is continuous with respect to* $\|.\|_\infty$;
(iii) *there exists* $c_2 > 0$ *such that* $\theta_1(f(A)) \le c_2\theta_2(A)$ *for every* $A \in I(f)$;
(iv) *for each* $A \in I(f)$ *with* $\theta_2(A) \ne 0$, *one has* $\theta_2(f(A)) < \theta_2(A)$.

Then $F_f \ne \emptyset$ *and* $\theta_1(F_f) = 0$, *i.e.* F_f *is compact with respect to* $\|.\|_\infty$.

Remark 3.3. In Theorems 3.1 and 3.2 it is sufficient that θ_1 satisfies the axioms (i), (ii) and (iv) from Definition 2.3.

3.2. The case of nonself operators. Let X be a linear space and $\|.\|_1$ and $\|.\|_2$ be two norms on X. Let $\rho : X \to B_R(0; \|.\|_1)$ be the radial retraction. Denote by α_i the Kuratowski measure of compactness on $(X, \|.\|_i)$, $i = 1, 2$. From Theorem 3.1 and Lemma 2.1 we have the following result:

THEOREM 3.4. *Let $f : B_R(0; \|.\|_1) \to X$ be an operator and assume that the following conditions are satisfied:*

(i) $(X, \|.\|_1)$ *is a Banach space;*
(ii) *there exists $c_1 > 0$ such that $\|.\|_2 \le c_1 \|.\|_1$;*
(iii) *f is continuous with respect to $\|.\|_1$;*
(iv) $f(B_R(0; \|.\|_1))$ *is bounded in* $(X, \|.\|_1)$;
(v) *there exists $c_2 > 0$ such that $\alpha_1(f(A)) \le c_2\alpha_2(A)$ for every $A \subset B_R(0; \|.\|_1)$;*
(vi) *for each $A \subset B_R(0; \|.\|_1)$ with $\alpha_2(A) \ne 0$, one has $\alpha_2(\rho \circ f(A)) < \alpha_2(A)$;*
(vii) *f is retractible with respect to ρ.*

Then $F_f \ne \emptyset$ and $\alpha_1(F_f) = 0$, i.e. F_f is compact with respect to $\|.\|_1$.

Proof. This follows by applying Theorem 3.1 to the self operator $\rho \circ f : B_R(0; \|.\|_1) \to B_R(0; \|.\|_1)$. □

REMARK 3.5. One can state a similar result on $C([a, b]; E)$, corresponding to Theorem 3.2.

3.3. The case of a set with two metrics. Let X be a nonempty set, d_1, d_2 two metrics on X and θ_1, θ_2 two measures of noncompactness on (X, d_1) and (X, d_2), respectively.

THEOREM 3.6. *Let $f : X \to X$ and assume that the following conditions are satisfied:*

(i) (X, d_1) *is a complete metric space;*
(ii) *there exists $c_1 > 0$ such that $d_2 \le c_1 d_1$;*
(iii) *$Y \in P_{cl_1}(X) \cap I(f)$ and $\theta_1(Y) = 0$ imply $F_f \cap Y \ne \emptyset$;*
(iv) *$f : (X, d_1) \to (X, d_1)$ is bounded and there exists $c_2 > 0$ such that $\theta_1(f(A)) \le c_2\theta_2(A)$ for every $A \in P_b(X, d_1) \cap I(f)$;*
(v) *$\theta_2(A) = 0$ implies $A \in P_b(X, d_1)$;*
(vi) *for each $A \in P_b(X, d_2) \cap I(f)$ with $\theta_2(A) \ne 0$, one has $\theta_2(f(A)) < \theta_2(A)$.*

Then $F_f \ne \emptyset$ and $\theta_1(F_f) = 0$, i.e. F_f is compact with respect to d_1.

Proof. The proof is similar to that of Theorem 3.1. □

REMARK 3.7. For the condition (iii) in Theorem 3.6, see [8, 9, 10, 14, 20].

4. APPLICATION TO INTEGRAL EQUATIONS IN BANACH SPACES

We present an application of Theorem 3.2 to the Hammerstein integral equation

$$u(t) = \int_0^T k(t,s)\, g(s,u(s))\, \mathrm{d}s, \quad t \in [0,T], \tag{4.1}$$

in a Banach space E with the norm $|.|$.

Theorem 4.1. *Let $k : [0,T]^2 \to \mathbf{R}$, $g_1, g_2 : [0,T] \times B \to E$, where $B = \{u \in E : |u| \le R\}$, and $g = g_1 + g_2$. Assume that the following conditions are satisfied:*

(a) *There exists $q \in (1,\infty)$ such that $k(t,.) \in L^q[0,T]$ for every $t \in [0,T]$, and the map $t \mapsto k(t,.)$ is continuous from $[0,T]$ to $L^q[0,T]$;*

(b) *g_1 is a Carathéodory function and there exists $\delta \in L^r[0,T]$ with $r \in (\frac{q}{q-1},\infty)$ such that*

$$|g_1(t,u) - g_1(t,v)| \le \delta(t)\, |u-v| \tag{4.2}$$

for all $u,v \in B$, a.a. $t \in [0,T]$, and

$$\lambda^p := \int_0^T \left(\int_0^T [|k(t,s)|\, \delta(s)]^{\frac{p}{p-1}}\, \mathrm{d}s \right)^{p-1} \mathrm{d}t < 1, \tag{4.3}$$

where $p = \frac{qr}{qr-q-r}$;

(c) *g_2 is a Carathéodory function and for each $A \subset B$,*

$$\alpha(g_2(t,A)) = 0$$

for a.a. $t \in [0,T]$, where α is the Kuratowski measure of noncompactness on E;

(d) *there exists $\delta_0 \in L^{\frac{q}{q-1}}[0,T]$ and $\psi : [0,R] \to \mathbf{R}_+$ continuous and nondecreasing with $\psi(\tau) > \tau$ for $\tau > 0$, such that*

$$|g(t,u)| \le \delta_0(t)\, \psi(|u|) \tag{4.4}$$

for all $u \in B$, a.a. $t \in [0,T]$, and

$$\max_{t\in[0,T]} \int_0^T |k(t,s)|\, \delta_0(s)\, \mathrm{d}s \le \frac{R}{\psi(R)}.$$

Then (4.1) has a solution in $C([0,T];B)$.

Proof. First note that from $q \in (1,\infty)$ and $r \in (q',\infty)$, where $q' = \frac{q}{q-1}$, one has $qr > q+r$, hence $p \in (1,\infty)$. Also note that Hölder's inequality guarantees that $k(t,.)\,\delta(.) \in L^{\frac{p}{p-1}}$ for each t.

We shall apply Theorem 3.2. Hence $X = C([0,T];E)$, $\|.\|_1$ is the sup-norm $\|.\|_\infty$, and $\|.\|_2$ is the L^p-norm

$$\|u\|_2 = \left(\int_0^T |u(t)|^p\, \mathrm{d}t \right)^{1/p},$$

with $p = \frac{qr}{qr-q-r}$. We take $Y := B_R(0; \|.\|_1)$, hence condition (i) of Theorem 3.2 holds. Let $f_i : B_R(0; \|.\|_1) \to C([0,T]; E)$, $i = 1, 2$, be defined by

$$f_i(u)(t) = \int_0^T k(t,s)\, g_i(s, u(s))\, \mathrm{d}s \quad (t \in [0,T]),$$

and let $f = f_1 + f_2$. From (4.4) we have that g_1, g_2 are $L^{q'}$-Carathéodory. Consequently, the operators f_1, f_2 are well defined and continuous with respect to $\|.\|_1$ (see [15]). Using (d) we find that for each $u \in B_R(0; \|.\|_1)$ and every $t \in [0,T]$,

$$\begin{aligned}
|f(u)(t)| &\leq \int_0^T |k(t,s)|\, |g(s, u(s))|\, \mathrm{d}s \\
&\leq \int_0^T |k(t,s)|\, \delta_0(s)\, \psi(|u(s)|)\, \mathrm{d}s \\
&\leq \psi(R) \max_{t\in[0,T]} \int_0^T |k(t,s)|\, \delta_0(s)\, \mathrm{d}s \leq R.
\end{aligned}$$

Thus $f(B_R(0; \|.\|_1)) \subset B_R(0; \|.\|_1)$ and so the condition (ii) is satisfied. Recall that, in Theorem 3.2, by θ_1, θ_2 we have understood the Kuratowski measures of noncompactness on $C([0,T]; E)$ with respect to the norms $\|.\|_1$, $\|.\|_2$. For any set $A \subset B_R(0; \|.\|_1)$, in view of (a), the set $f_2(A)$ is equicontinuous and thus, according to a result by Ambrosetti [2],

$$\theta_1(f_2(A)) = \max_{t\in[0,T]} \alpha(f_2(A)(t)).$$

On the other hand, for each countable set $C \subset A$, in view of a result by Heinz [12] (see also [15]), we have

$$\begin{aligned}
\alpha(f_2(C)(t)) &= \alpha\left(\int_0^T k(t,s)\, g_2(s, C(s))\, \mathrm{d}s\right) \\
&\leq 2\int_0^T |k(t,s)|\, \alpha(g_2(s, C(s)))\, \mathrm{d}s.
\end{aligned}$$

Hence from (c), we deduce that $\alpha(f_2(C)(t)) = 0$ for every t. Thus $\theta_1(f_2(C)) = 0$ for each countable set $C \subset A$. This shows that $f_2(A)$ is relatively compact with respect to $\|.\|_1$. Then, the comparison relation between the two norms implies that $f_2(A)$ is also relatively compact with respect to $\|.\|_2$, hence

$$\theta_1(f_2(A)) = \theta_2(f_2(A)) = 0. \tag{4.5}$$

Next, for any $u, v \in B_R(0; \|.\|_1)$, we have

$$
\begin{aligned}
&|f_1(u)(t) - f_1(v)(t)| \\
&\le \int_0^T |k(t,s)|\,|g_1(s,u(s)) - g_1(s,v(s))|\,\mathrm{d}s \\
&\le \int_0^T |k(t,s)|\,\delta(s)\,|u(s) - v(s)|\,\mathrm{d}s \qquad (4.6)\\
&\le \|u - v\|_2 \left(\int_0^T [|k(t,s)|\,\delta(s)]^{\frac{p}{p-1}}\,\mathrm{d}s\right)^{\frac{p-1}{p}}.
\end{aligned}
$$

Taking the supremum with respect to t, we deduce that

$$\|f_1(u) - f_1(v)\|_1 \le \|u - v\|_2 \max_{t\in[0,T]} \left(\int_0^T [|k(t,s)|\,\delta(s)]^{\frac{p}{p-1}}\,\mathrm{d}s\right)^{\frac{p-1}{p}}.$$

It follows that $\theta_1(f_1(A)) \le c_2\theta_2(A)$ for every $A \subset B_R(0; \|.\|_1)$, where

$$c_2 := \max_{t\in[0,T]} \left(\int_0^T [|k(t,s)|\,\delta(s)]^{\frac{p}{p-1}}\,\mathrm{d}s\right)^{\frac{p-1}{p}}.$$

Then, using (4.5), we obtain $\theta_1(f(A)) \le c_2\theta_2(A)$, that is condition (iii) holds. Finally, if we take the L^p-norm in (4.6) we obtain

$$\|f_1(u) - f_1(v)\|_2 \le \lambda\,\|u - v\|_2.$$

Then $\theta_2(f_1(A)) \le \lambda\theta_2(A)$ for every $A \subset B_R(0; \|.\|_1)$, and consequently

$$\theta_2(f(A)) \le \lambda\theta_2(A),$$

whence (iv) follows. Now the conclusion follows from Theorem 3.2. □

Remark 4.2. In fact, the operator f is the sum of the completely continuous operator f_2 and the operator f_1 which is condensing (even a set-contraction) with respect the L^p-norm. Note that the condensing condition (4.3) corresponding to the L^p-norm is in general better (less restrictive) than the similar condition

$$\max_{t\in[0,T]} \int_0^T [|k(t,s)|\,\delta(s)]^{\frac{p}{p-1}}\,\mathrm{d}s < 1$$

guaranteeing the condensing property with respect to the sup-norm.

For other applications of Darbo type and Sadovskii type fixed point theorems to integral equations see: [2, 3, 11, 12, 13], [16]-[19].

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