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ON $\mathfrak{m}$-IRREDUCIBILITY OF $\mathcal{M}$-SPACES

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Abstract. We introduce $\mathfrak{m}$-irreducible spaces and infratopological spaces. These notions generalize notions of topological space and irreducible space [T. Thompson, *Characterizations of irreducible spaces*, Kyungpook Math. J., **21**(2) (1981), 191–194]. Characterizations of $\mathfrak{m}$-T_2 spaces and characterizations of $\mathfrak{m}$-irreducible spaces are obtained. Our research leads to several generalizations of some well-known results.
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1. INTRODUCTION

In 1996 Haruo Maki [2] generalized the notion of topological space making use of four subfamilies of the powerset $\mathcal{P}(X)$ of a nonempty set X (see also [3]). In such spaces generalized closure and interior operators are defined and investigated. Maki generalized and studied the classical concepts of semi-open [1] and preopen set [4]. In 2000 Popa and Noiri introduced so-called minimal structure $\mathfrak{m}_X$ on $X \neq \varnothing$ as a subfamily of $\mathcal{P}(X)$ containing $\varnothing$ and X [10]. A pair $(X, \mathfrak{m}_X)$ we call here an $\mathcal{M}$-space; in [9] Popa and Noiri used the term $\mathfrak{m}$-*space* instead. For sets equipped with minimal structures, these authors introduced a continuity-like property, so-called $\mathcal{M}$-*continuity*, and provided several characterizations of functions with this property [10]. In the latter paper, Popa and Noiri also defined $\mathfrak{m}$-compactness, $\mathfrak{m}$-connectedness, $\mathfrak{m}$-T_2 spaces and investigated some of their properties. In 1983 Masshour et al. [6] studied properties of a particular case of $\mathcal{M}$-spaces, so-called supratopological spaces (briefly: supraspaces); see also [5]. In this paper we offer a new type of $\mathcal{M}$-spaces called *infratopological space* (briefly: infraspace). Generalized α-open [7] and semi-open sets are studied. We obtain some characterizations of $\mathfrak{m}$-T_2 spaces (analogous to the classical ones). We introduce $\mathfrak{m}$-irreducible spaces and give several characterizations of them. Some theorems in the paper are generalizations of already known results.

2. PRELIMINARIES

Let X be a nonempty set. A subfamily $\mathfrak{m}_X$ of the powerset $\mathcal{P}(X)$ is called a *minimal structure on* X if $\varnothing \in \mathfrak{m}_X$ and $X \in \mathfrak{m}_X$. The pair $(X, \mathfrak{m}_X)$ is called then an $\mathcal{M}$-*space*. $\mathfrak{m}_X$-closure and $\mathfrak{m}_X$-interior of any subset S of X are defined as follows:

$$\mathfrak{m}_X\text{-cl}(S) = \bigcap\{F : S \subset F \text{ and } X \setminus F \in \mathfrak{m}_X\};$$

$$\mathfrak{m}_X\text{-int}(S) = \bigcup\{U : S \supset U \text{ and } U \in \mathfrak{m}_X\}.$$

The following lemma lists all useful fundamental properties of $\mathfrak{m}_X$-cl$(\cdot)$ and $\mathfrak{m}_X$-int$(\cdot)$.

LEMMA 2.1. [2, 10] *Let* $(X, \mathfrak{m}_X)$ *be any* $\mathcal{M}$*-space. Then, for subsets* $A, B \subset X$ *the following hold:*

(1a) $\mathfrak{m}_X\text{-cl}(X \setminus A) = X \setminus \mathfrak{m}_X\text{-int}(A)$,
(1b) $\mathfrak{m}_X\text{-int}(X \setminus A) = X \setminus \mathfrak{m}_X\text{-cl}(A)$;
(2a) *if* $X \setminus A \in \mathfrak{m}_X$, *then* $\mathfrak{m}_X\text{-cl}(A) = A$,
(2b) *if* $A \in \mathfrak{m}_X$, *then* $\mathfrak{m}_X\text{-int}(A) = A$;
(3) $\mathfrak{m}_X\text{-cl}(\varnothing) = \varnothing$, $\mathfrak{m}_X\text{-cl}(X) = X$, $\mathfrak{m}_X\text{-int}(\varnothing) = \varnothing$, $\mathfrak{m}_X\text{-int}(X) = X$;
(4a) *if* $A \subset B$, *then* $\mathfrak{m}_X\text{-cl}(A) \subset \mathfrak{m}_X\text{-cl}(B)$,
(4b) *if* $A \subset B$, *then* $\mathfrak{m}_X\text{-int}(A) \subset \mathfrak{m}_X\text{-int}(B)$;
(5a) $A \subset \mathfrak{m}_X\text{-cl}(A)$,
(5b) $A \supset \mathfrak{m}_X\text{-int}(A)$;
(6a) $\mathfrak{m}_X\text{-cl}\big(\mathfrak{m}_X\text{-cl}(A)\big) = \mathfrak{m}_X\text{-cl}(A)$,
(6b) $\mathfrak{m}_X\text{-int}\,\big(\mathfrak{m}_X\text{-int}(A)\big) = \mathfrak{m}_X\text{-int}(A)$.

A function $f\colon (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$, where $(X, \mathfrak{m}_X)$, $(Y, \mathfrak{m}_Y)$ are two $\mathcal{M}$-spaces, is said to be *$\mathcal{M}$-continuous on* $(X, \mathfrak{m}_X)$ [10] if for each $x \in X$ and each $V \in \mathfrak{m}_Y$ containing $f(x)$, there exists $U \in \mathfrak{m}_X$ containing x such that $f(U) \subset V$.

An $\mathcal{M}$-space $(X, \mathfrak{m}_X)$ is said to be *supratopological space* [6] (briefly: supraspace) if for any family $\{U_i\}_{i \in I} \subset \mathfrak{m}_X$, $\bigcup_{i \in I} U_i \in \mathfrak{m}_X$.

It is necessary to recall the classical notions of semi-open and α-open subsets of a topological space (X, τ). A set $S \subset X$ is said to be *α-open* [7] (resp. *semi-open* [1]) in (X, τ) if $S \subset \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(S)))$ (resp. $S \subset \text{cl}_\tau(\text{int}_\tau(S))$). The folowing characterizations hold: (1) S is α-open in (X, τ) if and only if there exist $U \in \tau$ with $U \subset S \subset \text{int}_\tau(\text{cl}_\tau(U))$ [8, Lemma 4.12]; (2) S is semi-open in (X, τ) if and only if there is $U \in \tau$ such that $U \subset S \subset \text{cl}_\tau(U)$ [1, Theorem 1].

LEMMA 2.2. *Let* $(X, \mathfrak{m}_X)$ *be any* $\mathcal{M}$*-space and* $U, V \in \mathfrak{m}_X$. *If* $U \cap V = \varnothing$, *then* $U \cap \mathfrak{m}_X\text{-cl}(V) = \emptyset$ *(and* $V \cap \mathfrak{m}_X\text{-cl}(U) = \varnothing$*).*

Proof. Since $U \cap V = \varnothing$, then $U \subset X \setminus V$. By Lemma 2.1, (4b) and (1b), we obtain $\mathfrak{m}_X\text{-int}(U) \subset X \setminus \mathfrak{m}_X\text{-cl}(V)$. But $\mathfrak{m}_X\text{-int}(U) = U$ (Lemma 2.1 (2b)), thus $U \cap \mathfrak{m}_X\text{-cl}(V) = \varnothing$. □

3. INFRATOPOLOGICAL SPACES AND SEMI-OPEN SETS

DEFINITION 3.1. For any set X, a collection $\mathfrak{m}_X \subset \mathcal{P}(X)$ is said to be an *infratopology in* X if

(1) $\varnothing, X \in \mathfrak{m}_X$;
(2) for any $A_1, A_2 \in \mathfrak{m}_X$ the intersection $A_1 \cap A_2 \in \mathfrak{m}_X$.

A pair $(X, \mathfrak{m}_X)$ where $\mathfrak{m}_X$ is an infratopology in X, will be called an *infraspace*. Obviously, each topological space is an infraspace and each infraspace is an $\mathcal{M}$-space. The reverse implications are not true, in general.

DEFINITION 3.2. [10] An $\mathcal{M}$-space $(X, \mathfrak{m}_X)$ is said to be $\mathfrak{m}$-*Hausdorff* (equiv. $\mathfrak{m}$-T_2), if for each distinct $x, y \in X$ there exist $U, V \in \mathfrak{m}_X$, $x \in U$, $y \in V$, such that $U \cap V = \varnothing$.

EXAMPLE 3.3. Let $X = \{a, b, c, d\}$ and

$$\mathfrak{m}_X = \big\{\varnothing, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}\big\}.$$

The infraspace $(X, \mathfrak{m}_X)$ (that is not a supraspace) is $\mathfrak{m}$-T_2.

The infraspace given later on in Example 3.7 is not $\mathfrak{m}$-Hausdorff.

EXAMPLE 3.4. Let $X = \mathbb{R}$ and $\mathfrak{m}_X = \{[a, +\infty) : a \in \mathbb{R}\} \cup \{(-\infty, b] : b \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$. The $\mathcal{M}$-space $(X, \mathfrak{m}_X)$ is $\mathfrak{m}$-T_2, but it is not an infraspace (and not a supraspace).

The following lemma will be of use.

LEMMA 3.5. [10, Lemma 3.2] *Let $(X, \mathfrak{m}_X)$ be an $\mathcal{M}$-space and $S \subset X$. Then $x \in \mathfrak{m}_X\text{-cl}(S)$ if and only if $U \cap S \neq \varnothing$ for each $U \in \mathfrak{m}_X$ containing x.*

For infraspaces the following property holds.

PROPOSITION. *Let $(X, \mathfrak{m}_X)$ be an infraspace and $S \subset X$. Then*

$$U \cap \mathfrak{m}_X\text{-cl}(S) \subset \mathfrak{m}_X\text{-cl}(U \cap S)$$

for every $U \in \mathfrak{m}_X$.

Proof. Let $x \in U \cap \mathfrak{m}_X\text{-cl}(S)$ be such a point that $x \notin \mathfrak{m}_X\text{-cl}(U \cap S)$. By Lemma 3.5, for a certain $W \in \mathfrak{m}_X$ with $x \in W$ we have $W \cap (U \cap S) = \varnothing$. But, since $\mathfrak{m}_X$ is an infratopology on X, $x \in W \cap U \in \mathfrak{m}_X$. Thus we get a contradiction with the fact that $x \in \mathfrak{m}_X\text{-cl}(S)$. $\square$

DEFINITION 3.6. Let $(X, \mathfrak{m}_X)$ be an $\mathcal{M}$-space. A subset $S \subset X$ is said to be $\mathfrak{m}_X$-*open in* $(X, \mathfrak{m}_X)$ if for each point $x \in S$ there exists a set $U_x \in \mathfrak{m}_X$ with $x \in U_x \subset S$.

The family of all $\mathfrak{m}_X$-open subsets of an $\mathcal{M}$-space $(X, \mathfrak{m}_X)$ we denote as $\mathrm{O}(X, \mathfrak{m}_X)$. The following statements (we omit the proofs) hold for every $\mathcal{M}$-space $(X, \mathfrak{m}_X)$:
(1) $\mathfrak{m}_X \subset \mathrm{O}(X, \mathfrak{m}_X)$;
(2) $\mathfrak{m}_X\text{-int}(S) \in \mathrm{O}(X, \mathfrak{m}_X)$ for every $S \subset X$;
(3) for every $S \subset X$, $S \in \mathrm{O}(X, \mathfrak{m}_X)$ if and only if $\mathfrak{m}_X\text{-int}(S) = S$.

The inclusion $\mathfrak{m}_X \subset \mathrm{O}(X, \mathfrak{m}_X)$ is proper, in general, as the following example shows.

Example 3.7. Let $X = \mathbb{R}$ and $\mathfrak{m}_X = \{[a, +\infty) : a \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$. For any $a \in \mathbb{R}$ one has $(a, +\infty) \in \mathrm{O}(X, \mathfrak{m}_X) \setminus \mathfrak{m}_X$, since $(X, \mathfrak{m}_X)$ is not a topological space (it is an infraspace only).

It is worth to observe that every $\mathcal{M}$-space $(X, \mathfrak{m}_X)$ is a supraspace if and only if $\mathrm{O}(X, \mathfrak{m}_X) = \mathfrak{m}_X$.

In [2] the author has introduced a certain generalization of semi-open set, which – in particular case – can be applied in $\mathcal{M}$-spaces. In this section we define also a generalization of α-openness of subsets for $\mathcal{M}$-spaces and we study interrelationships between these notions.

Definition 3.8. Let $(X, \mathfrak{m}_X)$ be an $\mathcal{M}$-space. A subset $S \subset X$ is said to be:

(1) $\mathfrak{m}$-***semi-open in*** $(X, \mathfrak{m}_X)$ if there exists a set $U \in \mathfrak{m}_X$ such that

$$U \subset S \subset \mathfrak{m}_X\text{-cl}(U).$$

(2) $\mathfrak{m}$-α-***open in*** $(X, \mathfrak{m}_X)$ if there exists a set $U \in \mathfrak{m}_X$ such that

$$U \subset S \subset \mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-cl}(U)).$$

(3) ***weakly*** $\mathfrak{m}$-***semi-open in*** $(X, \mathfrak{m}_X)$ if

$$S \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S)).$$

(4) ***weakly*** $\mathfrak{m}$-α-***open in*** $(X, \mathfrak{m}_X)$ if

$$S \subset \mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S))).$$

The families of all subsets defined by (1)–(4) above are denoted respectively by $\mathrm{SO}(X, \mathfrak{m}_X)$, $\alpha\mathrm{O}(X, \mathfrak{m}_X)$, $\mathrm{wSO}(X, \mathfrak{m}_X)$, and $\mathrm{w}\alpha\mathrm{O}(X, \mathfrak{m}_X)$.

Theorem 3.9. *Let $(X, \mathfrak{m}_X)$ be an $\mathcal{M}$-space. Then the following hold:*

(a) $\mathfrak{m}_X \subset \mathrm{O}(X, \mathfrak{m}_X) \subset \mathrm{w}\alpha\mathrm{O}(X, \mathfrak{m}_X) \subset \mathrm{wSO}(X, \mathfrak{m}_X)$*;*

(b) $\mathfrak{m}_X \subset \alpha\mathrm{O}(X, \mathfrak{m}_X) \subset \mathrm{SO}(X, \mathfrak{m}_X) \subset \mathrm{wSO}(X, \mathfrak{m}_X)$*;*

(c) (1) $\mathrm{O}(X, \mathfrak{m}_X) \subset \mathrm{SO}(X, \mathfrak{m}_X)$ *and* (2) $\alpha\mathrm{O}(X, \mathfrak{m}_X) \subset \mathrm{w}\alpha\mathrm{O}(X, \mathfrak{m}_X)$*;*

(d) *if $(X, \mathfrak{m}_X)$ is a supraspace, then* (1) $\alpha\mathrm{O}(X, \mathfrak{m}_X) = \mathrm{w}\alpha\mathrm{O}(X, \mathfrak{m}_X)$ *and* (2) $\mathrm{SO}(X, \mathfrak{m}_X) = \mathrm{wSO}(X, \mathfrak{m}_X)$.

Proof. We use respective properties from Lemma 2.1 and the observation (3) before Example 3.7.

(a) Only the inclusion $\mathrm{O}(X, \mathfrak{m}_X) \subset \mathrm{w}\alpha\mathrm{O}(X, \mathfrak{m}_X)$ requires a proof. Let $S \in \mathrm{O}(X, \mathfrak{m}_X)$. Then $S = \mathfrak{m}_X\text{-int}(S)$. By Lemma 2.1 (5a), $S \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S))$ and by (4b) of that lemma we get $S = \mathfrak{m}_X\text{-int}(S) \subset \mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S)))$.

(b) We show the inclusion $\mathrm{SO}(X, \mathfrak{m}_X) \subset \mathrm{wSO}(X, \mathfrak{m}_X)$. Let $S \in \mathrm{SO}(X, \mathfrak{m}_X)$. Then for a certain $U \in \mathfrak{m}_X$ with $U \subset S$, $S \subset \mathfrak{m}_X\text{-cl}(U)$. Thus $U \subset \mathfrak{m}_X\text{-int}(S)$ and consequently $S \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S))$. To show $\mathfrak{m}_X \subset \alpha\mathrm{O}(X, \mathfrak{m}_X)$, let $S \in \mathfrak{m}_X$. Obviously $S \subset \mathfrak{m}_X\text{-cl}(S)$ and by Lemma 2.1 (4b), $\mathfrak{m}_X\text{-int}(S) \subset \mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-cl}(S))$. Since $S = \mathfrak{m}_X\text{-int}(S)$ (Lemma 2.1 (3)) finally we obtain $S \subset S \subset \mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-cl}(S))$ and so $S \in \alpha\mathrm{O}(X, \mathfrak{m}_X)$.

(c) The case (1) is clear and the proof of (2) is similar to the proof of $\mathrm{SO}(X,\mathfrak{m}_X) \subset \mathrm{wSO}(X,\mathfrak{m}_X)$.

(d) We shall show only that $\mathrm{w}\alpha\mathrm{O}(X,\mathfrak{m}_X) \subset \alpha\mathrm{O}(X,\mathfrak{m}_X)$. Namely, for $S \in \mathrm{w}\alpha\mathrm{O}(X,\mathfrak{m}_X)$ we have what follows (use Lemma 2.1 (5b)): $\mathfrak{m}_X\text{-int}(S) \subset S \subset \mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S)))$, where $\mathfrak{m}_X\text{-int}(S) \in \mathfrak{m}_X$ since $(X,\mathfrak{m}_X)$ is a supraspace. □

We shall show now that there is, in general, no relationship between families $\mathrm{O}(X,\mathfrak{m}_X)$ and $\alpha\mathrm{O}(X,\mathfrak{m}_X)$.

Example 3.10. Let $X = \{a,b,c,d\}$ and $\mathfrak{m}_X = \big\{\varnothing, X, \{b\}, \{c\}\big\}$. $(X,\mathfrak{m}_X)$ is an infraspace which is not a supraspace, and for $S = \{b,c\}$ we have $S \in \mathrm{O}(X,\mathfrak{m}_X) \setminus \alpha\mathrm{O}(X,\mathfrak{m}_X)$ since $S \not\subset \{b\} = \mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-cl}(\{b\}))$ and $S \not\subset \{c\} = \mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-cl}(\{c\}))$. Observe also that by inclusion $\mathrm{O}(X,\mathfrak{m}_X) \subset \mathrm{SO}(X,\mathfrak{m}_X)$ we get $S \in \mathrm{SO}(X,\mathfrak{m}_X) \setminus \alpha\mathrm{O}(X,\mathfrak{m}_X)$, and by $\mathrm{O}(X,\mathfrak{m}_X) \subset \mathrm{w}\alpha\mathrm{O}(X,\mathfrak{m}_X)$, $S \in \mathrm{w}\alpha\mathrm{O}(X,\mathfrak{m}_X) \setminus \alpha\mathrm{O}(X,\mathfrak{m}_X)$.

Example 3.11. Consider the infraspace $(X,\mathfrak{m}_X)$ from Example 3.7. One may easily check that for each of the kind $S = \{a\} \cup (b,+\infty)$, where $a < b$, we have $S \in \alpha\mathrm{O}(X,\mathfrak{m}_X) \setminus \mathrm{O}(X,\mathfrak{m}_X)$.

Observe that by the inclusion $\alpha\mathrm{O}(X,\mathfrak{m}_X) \subset \mathrm{SO}(X,\mathfrak{m}_X)$ we have $S \in \mathrm{SO}(X,\mathfrak{m}_X) \setminus \mathrm{O}(X,\mathfrak{m}_X)$, and by inclusion $\alpha\mathrm{O}(X,\mathfrak{m}_X) \subset \mathrm{w}\alpha\mathrm{O}(X,\mathfrak{m}_X)$, $S \in \mathrm{w}\alpha\mathrm{O}(X,\mathfrak{m}_X) \setminus \mathrm{O}(X,\mathfrak{m}_X)$.

Example 3.12. Let $X = \{a,b,c,d\}$ and $\mathfrak{m}_X = \big\{\varnothing, X, \{c\}, \{d\}, \{b,c\}\big\}$. $(X,\mathfrak{m}_X)$ is an infraspace (not a supraspace) and for $S = \{a,b,c\}$ we have

$$S = \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(\{a,b,c\})) = \{a,b,c\},$$
$$S \not\subset \{b,c\} = \mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S))).$$

So, $S \in \mathrm{wSO}(X,\mathfrak{m}_X) \setminus \mathrm{w}\alpha\mathrm{O}(X,\mathfrak{m}_X)$.

Example 3.13. Let $X = \{a,b,c\}$, $\mathfrak{m}_X = \big\{\varnothing, X, \{a\}, \{c\}\big\}$. For the subset $S = \{a,c\}$ one gets $S \in \mathrm{wSO}(X,\mathfrak{m}_X)$ because $S \subset X = \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S))$. At the same time $S \notin \mathrm{SO}(X,\mathfrak{m}_X)$, since $S \not\subset \{a,b\} = \mathfrak{m}_X\text{-cl}(\{a\})$, $S \not\subset \mathfrak{m}_X\text{-cl}(\{c\}) = \{b,c\}$.

Some results of the next lemma will be useful in the sequel.

Lemma 3.14. *Let $(X,\mathfrak{m}_X)$ be an $\mathcal{M}$-space. Then the following hold:*

(a) $\mathfrak{m}_X\text{-int}(S) \in \mathrm{wSO}(X,\mathfrak{m}_X)$ *for each* $S \subset X$*;*

(b) $\mathfrak{m}_X\text{-int}(S) \in \mathrm{SO}(X,\mathfrak{m}_X)$ *for each* $S \in \mathfrak{m}_X$*;*

(c) $\mathfrak{m}_X\text{-cl}(S) \in \mathrm{SO}(X,\mathfrak{m}_X)$ *for each* $S \in \mathfrak{m}_X$*;*

(d) $\mathfrak{m}_X\text{-cl}(S) \in \mathrm{wSO}(X,\mathfrak{m}_X)$ *for each* $S \in \mathrm{wSO}(X,\mathfrak{m}_X)$.

Proof. (a) Let $S \subset X$. Clearly, we have $\mathfrak{m}_X\text{-int}(S) \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S)) = \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-int}(S)))$ (by Lemma 2.1 (6b)).

(b) Let $S \in \mathfrak{m}_X$. Then by Lemma 2.1 (2b), (5a) we get $S = \mathfrak{m}_X\text{-int}(S) \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S)) = \mathfrak{m}_X\text{-cl}(S)$.

(c) Directly follows by Lemma 2.1 (5a): $S \subset \mathfrak{m}_X\text{-cl}(S) \subset \mathfrak{m}_X\text{-cl}(S)$.

(d) Let $S \in \text{wSO}(X, \mathfrak{m}_X)$. Then $S \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S))$ and by Lemma 2.1, (4a) and (6a), $\mathfrak{m}_X\text{-cl}(S) \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S))) = \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S))$. Using again Lemma 2.1 (5a), (4), $\mathfrak{m}_X\text{-cl}(S) \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-cl}(S)))$. □

4. $\mathfrak{M}$-IRREDUCIBLE SPACES

Definition 4.1. An $\mathcal{M}$-space $(X, \mathfrak{m}_X)$ is said to be ***$\mathfrak{m}$-irreducible*** if for every two nonempty sets $S_1, S_2 \in \mathfrak{m}_X$, $S_1 \cap S_2 \neq \varnothing$.

In general, infraspaces need not be $\mathfrak{m}$-irreducible and conversely, as it is shown below.

Example 4.2. Let $X = \mathbb{R}$ and $\mathfrak{m}_X = \big\{\{0\} \cup [a, +\infty) : a \geq 2\big\} \cup \{\varnothing, X\} \cup \big\{\{0\} \cup (-\infty, -a] : a \geq 2\big\}$. One checks that $(X, \mathfrak{m}_X)$ is $\mathfrak{m}$-irreducible and neither an infraspace nor a supraspace.

Example 4.3. Let $X = \mathbb{R}$ and $\mathfrak{m}_X = \big\{\{-1\} \cup [a, +\infty) : a \geq 2\big\} \cup \big\{(-\infty, -a] \cup \{1\} : a \geq 2\big\} \cup \{\varnothing, X\}$. $(X, \mathfrak{m}_X)$ is an infraspace and not a supraspace, and it is not $\mathfrak{m}$-irreducible.

There exists an infraspace which is $\mathfrak{m}$-irreducible.

Example 4.4. Let $X = \mathbb{R}$ and $\mathfrak{m}'_X = \mathfrak{m}_X \cup \{\{0\}\}$, where $\mathfrak{m}_X$ is the minimal structure from Example 4.2. $(X, \mathfrak{m}'_X)$ is an $\mathfrak{m}$-irreducible infraspace.

Remark 4.5. Each $\mathfrak{m}$-irreducible $\mathcal{M}$-space is not $\mathfrak{m}$-T_2, but the converse is not true, in general. It is enough to consider Example 4.2.

Definition 4.6. An $\mathcal{M}$-space $(X, \mathfrak{m}_X)$ is said to be ***weakly $S\mathfrak{m}$-connected*** (briefly: $wS\mathfrak{m}$-connected) if there are no two nonempty $S_1, S_2 \in \text{wSO}(X, \mathfrak{m}_X)$ such that $X = S_1 \cup S_2$ and $S_1 \cap S_2 = \varnothing$.

Theorem 4.7. *Let $(X, \mathfrak{m}_X)$ be an $\mathcal{M}$-space. The following statements are equivalent:*

1. *$(X, \mathfrak{m}_X)$ is $\mathfrak{m}$-irreducible,*
2. *$S_1 \cap S_2 \neq \varnothing$ for any nonempty sets $S_1, S_2 \in \text{wSO}(X, \mathfrak{m}_X)$,*
3. *$(X, \mathfrak{m}_X)$ is $wS\mathfrak{m}$-connected,*
4. *there is no surjection $f\colon X \to \{a, b\}$ such that $f^{-1}(\{a\}), f^{-1}(\{b\}) \in \text{wSO}(X, \mathfrak{m}_X)$.*

Proof. (1)⇒(2) Let $S_1, S_2 \in \text{wSO}(X, \mathfrak{m}_X)$ be nonempty sets. So, $S_1 \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S_1))$, $S_2 \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S_2))$, where

$$\mathfrak{m}_X\text{-int}(S_1) \neq \varnothing \neq \mathfrak{m}_X\text{-int}(S_2)$$

(see Lemma 2.1 (3)). Hence there exist nonempty sets $U_1, U_2 \in \mathfrak{m}_X$ with $U_1 \subset \mathfrak{m}_X\text{-int}(S_1) \subset S_1$ and $U_2 \subset \mathfrak{m}_X\text{-int}(S_2) \subset S_2$. Thus by $\mathfrak{m}$-irreducibility of $(X, \mathfrak{m}_X)$ we infer that $S_1 \cap S_2 \neq \varnothing$.

(2)⇒(3) Clear.

(3)⇒(4) If there exists a surjection f fulfilling the condition in (4), then it can be easily seen that $(X, \mathfrak{m}_X)$ is not $wS\mathfrak{m}$-connected.

(4)⇒(1) Suppose $(X, \mathfrak{m}_X)$ is not $\mathfrak{m}$-irreducible. Then there are two nonempty sets $S_1, S_2 \in \mathfrak{m}_X$ such that $S_1 \cap S_2 = \varnothing$. By Lemma 2.2 we get $S_1 \cap \mathfrak{m}_X\text{-cl}(S_2) = \varnothing$. By Lemma 3.14 (c), $\varnothing \neq \mathfrak{m}_X\text{-cl}(S_2) \in \mathrm{SO}(X, \mathfrak{m}_X) \subset \mathrm{wSO}(X, \mathfrak{m}_X)$. On the other hand, by Lemma 2.1 (1b), $X \setminus \mathfrak{m}_X\text{-cl}(S_2) = \mathfrak{m}_X\text{-int}(X \setminus S_2) \supset S_1 \neq \varnothing$. Using Lemma 3.14 (a) we have that $\varnothing \neq \mathfrak{m}_X\text{-int}(X \setminus S_2) \in \mathrm{wSO}(X, \mathfrak{m}_X)$. To obtain a contradiction it is enough to define a surjection $f\colon X \to \{a, b\}$ as follows: $f = a$ on $\mathfrak{m}_X\text{-cl}(S_2)$, $f = b$ on $X \setminus \mathfrak{m}_X\text{-cl}(S_2)$. □

Remark 4.8. Theorem 4.7 generalizes [12, Theorem 17].

Theorem 4.9. *Let $(X, \mathfrak{m}_X)$ be any $\mathcal{M}$-space. The following statements are equivalent:*

(1) *$(X, \mathfrak{m}_X)$ is $\mathfrak{m}$-irreducible,*
(2) *$S_1 \cap S_2 \neq \varnothing$ for every nonempty sets $S_1, S_2 \in \mathrm{w}\alpha\mathrm{O}(X, \mathfrak{m}_X)$,*
(3) *$S_1 \cap S_2 \neq \varnothing$ for every nonempty sets $S_1, S_2 \in \mathrm{O}(X, \mathfrak{m}_X)$.*

Proof. (1)⇒(2) The implication (1)⇒(2) of Theorem 4.7 is true for arbitrary $\mathcal{M}$-space (Remark 4.8 (1)). Therefore the result follows from $\mathrm{w}\alpha\mathrm{O}(X, \mathfrak{m}_X) \subset \mathrm{wSO}(X, \mathfrak{m}_X)$ (Theorem 3.9 (a)).

(2)⇒(3) Use the inclusion $\mathrm{O}(X, \mathfrak{m}_X) \subset \mathrm{w}\alpha\mathrm{O}(X, \mathfrak{m}_X)$ (Theorem 3.9).

(3)⇒(1) By the inclusion $\mathrm{O}(X, \mathfrak{m}_X) \supset \mathfrak{m}_X$. □

The proof of the next theorem is similar to the proof of Theorem 4.9 – we use inclusions (b) from Theorem 3.9.

Theorem 4.10. *Let $(X, \mathfrak{m}_X)$ be any $\mathcal{M}$-space. The following are equivalent:*

(1) *$(X, \mathfrak{m}_X)$ is $\mathfrak{m}$-irreducible,*
(2) *$S_1 \cap S_2 \neq \varnothing$ for every nonempty sets $S_1, S_2 \in \mathrm{SO}(X, \mathfrak{m}_X)$,*
(3) *$S_1 \cap S_2 \neq \varnothing$ for every nonempty sets $S_1, S_2 \in \alpha\mathrm{O}(X, \mathfrak{m}_X)$.*

In the class of non-infraspaces we can indicate both $wS\mathfrak{m}$-connected and non-$wS\mathfrak{m}$-connected $\mathcal{M}$-spaces.

Example 4.11. Let $X = \{a, b, c, d\}$ and $\mathfrak{m}_X = \big\{\varnothing, X, \{b, c\}, \{c, d\}\big\}$. It can be easily checked that this non-infraspace $(X, \mathfrak{m}_X)$ (which is also not a supraspace) is $wS\mathfrak{m}$-connected.

Example 4.12. Let $X = \{a, b, c, d, e\}$, $\mathfrak{m}_X = \big\{\varnothing, X, \{d, e\}, \{a, b, c\}, \{c, d, e\}\big\}$. The non-infraspace $(X, \mathfrak{m}_X)$ (which is a supraspace) is not $wS\mathfrak{m}$-connected.

Definition 4.13. A subset S of an $\mathcal{M}$-space $(X, \mathfrak{m}_X)$ is said to be ***$\mathfrak{m}$-dense in*** $(X, \mathfrak{m}_X)$ if $\mathfrak{m}_X\text{-cl}(S) = X$.

Theorem 4.14. *Let $(X, \mathfrak{m}_X)$ be an $\mathcal{M}$-space. The following are equivalent:*

(1) *$(X, \mathfrak{m}_X)$ is $\mathfrak{m}$-irreducible.*
(2) *Every nonempty set $S \in \mathrm{wSO}(X, \mathfrak{m}_X)$ is $\mathfrak{m}$-dense in $(X, \mathfrak{m}_X)$.*

(3) *Every nonempty set* $S \in \mathrm{SO}(X, \mathfrak{m}_X)$ *is* $\mathfrak{m}$*-dense in* $(X, \mathfrak{m}_X)$.
(4) *Every nonempty set* $S \in \alpha\mathrm{O}(X, \mathfrak{m}_X)$ *is* $\mathfrak{m}$*-dense in* $(X, \mathfrak{m}_X)$.
(5) *Every nonempty set* $S \in \mathrm{w}\alpha\mathrm{O}(X, \mathfrak{m}_X)$ *is* $\mathfrak{m}$*-dense in* $(X, \mathfrak{m}_X)$.
(6) *Every nonempty set* $S \in \mathrm{O}(X, \mathfrak{m}_X)$ *is* $\mathfrak{m}$*-dense in* $(X, \mathfrak{m}_X)$.
(7) *Every nonempty set* $S \in \mathfrak{m}_X$ *is* $\mathfrak{m}$*-dense in* $(X, \mathfrak{m}_X)$.

Proof. (1)⇒(2) Suppose there exits a nonempty set $S \in \mathrm{wSO}(X, \mathfrak{m}_X)$ such that $\mathfrak{m}_X\text{-cl}(S) \neq X$. Hence $\mathfrak{m}_X\text{-int}(X \setminus S) = X \setminus \mathfrak{m}_X\text{-cl}(S) \neq \varnothing$. It implies the existence of a nonempty set $U \in \mathfrak{m}_X$ with $U \subset \mathfrak{m}_X\text{-int}(X \setminus S)$. On the other hand, $\mathfrak{m}_X\text{-cl}(S) \in \mathrm{wSO}(X, \mathfrak{m}_X)$ by Lemma 3.14 (d). Thus there are two nonempty sets $S_1 = U$, $S_2 = \mathfrak{m}_X\text{-cl}(S) \in \mathrm{wSO}(X, \mathfrak{m}_X)$ such that $S_1 \cap S_2 = \varnothing$. Consequently by Theorem 4.7 (2), $(X, \mathfrak{m}_X)$ is not $\mathfrak{m}$-irreducible. Implications (2)⇒(5)⇒(6)⇒(7) are obvious by Theorem 3.9 (a). Implications (2)⇒(3)⇒(4)⇒(7) hold by Theorem 3.9 (b). Thus it is enough to show (7)⇒(1). Suppose $(X, \mathfrak{m}_X)$ is not $\mathfrak{m}$-irreducible. Then for some two nonempty sets $S_1, S_2 \in \mathfrak{m}_X$, $S_1 \cap S_2 = \varnothing$. Using now Lemma 2.2 we get $S_1 \cap \mathfrak{m}_X\text{-cl}(S_2) = \varnothing$ which shows that $\mathfrak{m}_X\text{-cl}(S_2) \neq X$. □

Problem. It is an open problem to find a non-infraspace being not $\mathfrak{m}$-irreducible, for which nevertheless the condition (7) holds.

Any filterbase given on an arbitrary $\mathcal{M}$-space we will call $\mathfrak{m}$***-filterbase***. Let $(X, \mathfrak{m}_X)$ be an infraspace and $\mathcal{F}_{\mathfrak{m}_X} = \{A_i \in \mathfrak{m}_X : i \in I \text{ and } A_{i_1} \cap A_{i_2} \neq \varnothing \text{ for } i_1, i_2 \in I\}$. Obviously $\mathcal{F}_{\mathfrak{m}_X}$ is an $\mathfrak{m}$-filterbase on $(X, \mathfrak{m}_X)$. Define also the folowing two families of subsets of any $\mathcal{M}$-space $(X, \mathfrak{m}_X)$: $\mathfrak{m}\mathcal{N}(y) = \{S \in \mathfrak{m}_X : y \in S\}$ and $\mathfrak{m}\mathcal{U}(y) = \{U \in \mathrm{O}(X, \mathfrak{m}_X) : y \in U\}$.

Definition 4.15. Let $\mathcal{F}$ be an $\mathfrak{m}$-filterbase on $(X, \mathfrak{m}_X)$ and $y_0 \in X$. We say that $\mathcal{F}$ ***accumulates at*** y_0 if for each $U \in \mathfrak{m}\mathcal{U}(y_0)$ and each $A \in \mathcal{F}$, $A \cap U \neq \varnothing$.

Theorem 4.16. *An infraspace* $(X, \mathfrak{m}_X)$ *is* $\mathfrak{m}$*-irreducible if and only if every* $\mathfrak{m}$*-filterbase* $\mathcal{F} \subset \mathfrak{m}_X$ *accumulates at every point of* X.

Proof. (⇒) Let $(X, \mathfrak{m}_X)$ be $\mathfrak{m}$-irreducible and let $\mathcal{F} \subset \mathfrak{m}_X$ be arbitrary $\mathfrak{m}$-filterbase in it. For any $x \in X$ and every $U = \bigcup\{U_i \in \mathfrak{m}_X : U_i \subset U, i \in I\} \in \mathcal{U}(x)$ and $A \in \mathcal{F}$, one has $U_i \cap A \neq \varnothing$, $i \in I$, because of $\mathfrak{m}$-irreducibility of $(X, \mathfrak{m}_X)$. So, $A \cap U \neq \varnothing$.

(⇐) Suppose every $\mathfrak{m}$-filterbase $\mathcal{F} \subset \mathfrak{m}_X$ accumulates at every point $x \in X$. Let $S_1, S_2 \in \mathfrak{m}_X$ be arbitrary two nonempty sets and let $x \in S_1$, $y \in S_2$. Consider the family $\mathfrak{m}\mathcal{N}(x)$. Obviously, it is an $\mathfrak{m}$-filterbase in $(X, \mathfrak{m}_X)$, since $(X, \mathfrak{m}_X)$ is an infraspace. By assumption, $\mathfrak{m}\mathcal{N}(x)$ accumulates at y, that is for each $U \in \mathfrak{m}\mathcal{U}(y)$ and each $A \in \mathfrak{m}\mathcal{N}(x)$, $A \cap U \neq \varnothing$. In particular, for $U = S_2$ and $A = S_1$ one obtains $S_1 \cap S_2 \neq \varnothing$. This shows that $(X, \mathfrak{m}_X)$ is $\mathfrak{m}$-irreducible. □

Remark 4.17. Theorem 4.16 generalizes [12, Theorem 16].

5. $\mathfrak{M}$-IRREDUCIBILITY AND FUNCTIONS

Recall the following useful characterizations of $\mathcal{M}$-continuity.

LEMMA 5.1. [10, Theorem 3.1]. *Let $(X, \mathfrak{m}_X)$ and $(Y, \mathfrak{m}_Y)$ be two arbitrary $\mathcal{M}$-spaces. For a function $f\colon (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$ the following properties are equivalent:*

(a) *f is $\mathcal{M}$-continuous,*
(b) *$f^{-1}(V) = \mathfrak{m}_X\text{-int}\big(f^{-1}(V)\big)$ for every $V \in \mathfrak{m}_Y$,*
(c) *$f\big(\mathfrak{m}_X\text{-cl}(S)\big) \subset \mathfrak{m}_X\text{-cl}(f(S))$ for every $S \subset X$,*
(d) *$f^{-1}(V) \in \mathrm{O}(X, \mathfrak{m}_X)$ for every $V \in \mathfrak{m}_Y$.*

Equivalence (b)$\Leftrightarrow$(d) follows directly by the observation (3) in Section 3 (page 135).

COROLLARY 5.2. *Let $(X, \mathfrak{m}_X)$ and $(Y, \mathfrak{m}_Y)$ be two arbitrary $\mathcal{M}$-spaces. Then for each $\mathcal{M}$-continuous function $f\colon (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$ and any subset T of Y one has*

$$f\big(\mathfrak{m}_X\text{-cl}\big(f^{-1}(T)\big)\big) \subset \mathfrak{m}_Y\text{-cl}(T).$$

Proof. Omitted. □

THEOREM 5.3. *Let $(X, \mathfrak{m}_X)$ and $(Y, \mathfrak{m}_Y)$ be $\mathcal{M}$-spaces. If $(X, \mathfrak{m}_X)$ is $\mathfrak{m}$-irreducible and a surjection $f\colon (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$ is $\mathcal{M}$-continuous, then $(Y, \mathfrak{m}_Y)$ is $\mathfrak{m}$-irreducible.*

Proof. Let $V \in \mathfrak{m}_Y$ be arbitrarily chosen. Since f is $\mathcal{M}$-continuous, by Lemma 5.1, $f^{-1}(V) \in \mathrm{O}(X, \mathfrak{m}_X)$. Then by $\mathfrak{m}$-irreducibility of $(X, \mathfrak{m}_X)$, using Theorem 4.14 we get that $X = \mathfrak{m}_X\text{-cl}\big(f^{-1}(V)\big)$. By Corollary 5.2 one obtains that $Y \subset \mathfrak{m}_Y\text{-cl}(V)$. So, again by Theorem 4.14, $(Y, \mathfrak{m}_Y)$ is $\mathfrak{m}$-irreducible. □

The result of Theorem 5.3 may be extended for a class of functions defined as follows:

DEFINITION 5.4. Let $(X, \mathfrak{m}_X)$ and $(Y, \mathfrak{m}_Y)$ be two arbitrary $\mathcal{M}$-spaces. A function $f\colon (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$ is said to be ***$wS\mathfrak{m}$-continuous*** if $f^{-1}(V) \in \mathrm{wSO}(X, \mathfrak{m}_X)$ for every $V \in \mathfrak{m}_Y$.

$\mathcal{M}$-continuity implies $wS\mathfrak{m}$-continuity, but the converse is not true, in general.

EXAMPLE 5.5. Let $X = \{a, b, c, d\}$, $\mathfrak{m}_X = \big\{\varnothing, X, \{a\}, \{b\}\big\}$, $Y = \{a, b, c\}$ and $\mathfrak{m}_Y = \big\{\varnothing, Y, \{b\}, \{c\}\big\}$. Define a function $f\colon (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$ as follows: $f(a) = f(d) = b$, $f(b) = f(c) = c$. One checks that $f^{-1}(\{b\}), f^{-1}(\{c\}) \in \mathrm{wSO}(X, \mathfrak{m}_X) \setminus \mathrm{O}(X, \mathfrak{m}_X)$.

THEOREM 5.6. *Let $(X, \mathfrak{m}_X)$ and $(Y, \mathfrak{m}_Y)$ be $\mathcal{M}$-spaces. If $(X, \mathfrak{m}_X)$ is $\mathfrak{m}$-irreducible and a surjection $f\colon (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$ is $wS\mathfrak{m}$-continuous, then $(Y, \mathfrak{m}_Y)$ is $\mathfrak{m}$-irreducible.*

Proof. Suppose $(Y, \mathfrak{m}_Y)$ is not $\mathfrak{m}$-irreducible. Then for some nonempty sets $V_1, V_2 \in \mathfrak{m}_Y$, $V_1 \cap V_2 = \varnothing$. It is clear that by $wS\mathfrak{m}$-continuity of f,

$$\mathfrak{m}_X\text{-int}\left(f^{-1}(V_1)\right) \neq \varnothing \neq \mathfrak{m}_X\text{-int}\left(f^{-1}(V_2)\right).$$

Thus for some nonempty $U_1, U_2 \in \mathfrak{m}_X$ we have $U_1 \cap U_2 \subset \mathfrak{m}_X\text{-int}\left(f^{-1}(V_1)\right) \cap \mathfrak{m}_X\text{-int}\left(f^{-1}(V_2)\right) \subset f^{-1}(V_1 \cap V_2) = \varnothing$. A contradiction. □

Remark 5.7. Theorem 5.6 generalizes [12, Theorem 13].

Let us generalize the well-known notion of cluster set [13] for $\mathcal{M}$-spaces in the following fashion.

Definition 5.8. Let $(X, \mathfrak{m}_X)$ and $(Y, \mathfrak{m}_Y)$ be $\mathcal{M}$-spaces and $f\colon (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$. For $x_0 \in X$, the ***$\mathfrak{m}$-cluster set of f at*** x_0 is the set $\mathfrak{m}\text{-C}(f, x_0) = \bigcap \{\mathfrak{m}_Y\text{-cl}(f(U)) : U \in \mathfrak{m}\mathcal{N}(x_0)\}$.

The next theorem is a generalization of [11, Theorem 6]. First, we give the following definition:

Definition 5.9. Let $(X, \mathfrak{m}_X)$ be an $\mathcal{M}$-space. The ***$\mathfrak{m}$-spiral of a point*** $x_0 \in X$ is the set $\mathfrak{m}_X\text{-Sp}(x_0) = \bigcap \{\mathfrak{m}_X\text{-cl}(U) : U \in \mathfrak{m}\mathcal{N}(x_0)\}$.

Theorem 5.10. *Let a function $f\colon (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$ be $\mathcal{M}$-continuous, where $(X, \mathfrak{m}_X)$ and $(Y, \mathfrak{m}_Y)$ are $\mathcal{M}$-spaces. Then for an arbitrarily chosen $x_0 \in X$ we have*

$$f\left(\mathfrak{m}_X\text{-Sp}(x_0)\right) \subset \mathfrak{m}\text{-C}(f, x_0) \subset \mathfrak{m}_Y\text{-Sp}(f(x_0)).$$

Proof. Let us prove the left-hand inclusion. Using Lemma 5.1 we calculate as follows:

$$\begin{aligned} f\left(\mathfrak{m}_X\text{-Sp}(x_0)\right) &= f\Big(\bigcap \{\mathfrak{m}_X\text{-cl}(U) : U \in \mathfrak{m}\mathcal{N}(x_0)\}\Big) \subset \\ &\subset \bigcap f\left(\{\mathfrak{m}_X\text{-cl}(U) : U \in \mathfrak{m}\mathcal{N}(x_0)\}\right) \subset \\ &\subset \bigcap \{\mathfrak{m}_X\text{-cl}(f(U)) : U \in \mathfrak{m}\mathcal{N}(x_0)\} = \mathfrak{m}\text{-C}(f, x_0). \end{aligned}$$

To prove the second inclusion observe that for each $V \in \mathfrak{m}\mathcal{N}(f(x_0))$, $x_0 \in f^{-1}(V) = \mathfrak{m}_X\text{-int}\left(f^{-1}(V)\right)$ by Lemma 5.1. Then there exists a set $U_V \in \mathfrak{m}_X$ with $x_0 \in U_V$, that is $U_V \in \mathfrak{m}\mathcal{N}(x_0)$. Therefore we have:

$$\begin{aligned} \mathfrak{m}\text{-C}(f, x_0) &\subset \bigcap \{\mathfrak{m}_Y\text{-cl}(f(U_V)) : V \in \mathfrak{m}\mathcal{N}(f(x_0))\} \subset \\ &\subset \bigcap \left\{\mathfrak{m}_Y\text{-cl}\big(f\big(\mathfrak{m}_X\text{-int}\left(f^{-1}(V)\right)\big)\big) : V \in \mathfrak{m}\mathcal{N}(f(x_0))\right\} \subset \\ &\subset \bigcap \left\{\mathfrak{m}_Y\text{-cl}\left(f\left(f^{-1}(V)\right)\right) : V \in \mathfrak{m}\mathcal{N}(f(x_0))\right\} \subset \\ &\subset \bigcap \left\{\mathfrak{m}_Y\text{-cl}(V) : V \in \mathfrak{m}\mathcal{N}(f(x_0))\right\} = \mathfrak{m}_Y\text{-Sp}(f(x_0)). \end{aligned}$$ □

Recall the following property of the diagonal set $\Delta = \{(x, x) : x \in X\}$, where X is an arbitrary nonempty set: $(U \times V) \cap \Delta = \varnothing$ if and only if $U \cap V = \varnothing$ for arbitrarily chosen subsets $U, V \subset X$.

Let $(X, \mathfrak{m}_X)$ and $(Y, \mathfrak{m}_Y)$ be infraspaces. By $(X \times Y, \mathfrak{m}_{X\times Y})$ we mean the infraspace with $\mathfrak{m}_{X\times Y} = \{U \times V : U \in \mathfrak{m}_X,\ V \in \mathfrak{m}_Y\}$.

THEOREM 5.11. *Let $(X, \mathfrak{m}_X)$ be an $\mathcal{M}$-space. The following properties are equivalent:*

(a) $(X, \mathfrak{m}_X)$ *is $\mathfrak{m}$-Hausdorff;*

(b) *for any distinct points $x_1, x_2 \in X$ there exist sets $U_{x_1}, U_{x_2} \in \mathfrak{m}_X$ with $x_1 \in U_{x_1}$, $x_2 \in U_{x_2}$, such that $x_1 \notin \mathfrak{m}_X\text{-cl}(U_{x_2})$ and $x_2 \notin \mathfrak{m}_X\text{-cl}(U_{x_1})$;*

(c) *for each point $x \in X$, $\bigcap\{\mathfrak{m}_X\text{-cl}(U) : x \in U \in \mathfrak{m}_X\} = \{x\}$;*

(d) *the set $(X \times X) \setminus \Delta \in \mathrm{O}(X \times X, \mathfrak{m}_{X\times X})$.*

Proof. (a)⇒(b). Let $x_1 \neq x_2$. By assumption there exist disjoint $U_{x_1}, U_{x_2} \in \mathfrak{m}_X$ with $x_1 \in U_{x_1}$, $x_2 \in U_{x_2}$. Using Lemma 2.2 we get $x_1 \notin \mathfrak{m}_X\text{-cl}(U_{x_2})$ and $x_2 \notin \mathfrak{m}_X\text{-cl}(U_{x_1})$.

(b)⇒(c). If $y \neq x$, then there exists a set $U_x \in \mathfrak{m}_X$ with $x \in U_x$ such that $y \notin \mathfrak{m}_X\text{-cl}(U_x)$. So, $y \notin \bigcap\{\mathfrak{m}_X\text{-cl}(U) : x \in U \in \mathfrak{m}_X\}$.

(c)⇒(d). Let $(x, y) \notin \Delta$. Then $x \neq y$ and by assumption there is a certain set $U \in \mathfrak{m}_X$ with $x \in U$ such that $y \notin \mathfrak{m}_X\text{-cl}(U)$. Clearly, we obtain $U \cap \mathfrak{m}_X\text{-int}(X \setminus U) = \varnothing$, where $y \in \mathfrak{m}_X\text{-int}(X \setminus U) = X \setminus \mathfrak{m}_X\text{-cl}(U)$ (use Lemma 2.1 (1b)). Thus for some $W \subset \mathfrak{m}_X\text{-int}(X \setminus U)$, where $W \in \mathfrak{m}_X$, we get $(x, y) \in U \times W$. Since $U \cap W = \varnothing$, one has $U \times W \subset (X \times X) \setminus \Delta$ and so the desired result follows.

(d)⇒(a). Let $x \neq y$. Hence $(x, y) \in (X \times X) \setminus \Delta$ and there exist sets $U, V \in \mathfrak{m}_X$ with $x \in U$, $y \in V$, such that $(U \times V) \cap \Delta = \varnothing$. Therefore $U \cap V = \varnothing$. This completes the proof. □

THEOREM 5.12. *Let $(X, \mathfrak{m}_X)$ be an $\mathfrak{m}$-irreducible $\mathcal{M}$-space and $(Y, \mathfrak{m}_Y)$ be an $\mathfrak{m}$-T_2 $\mathcal{M}$-space. Then each $\mathcal{M}$-continuous function $f\colon (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$ is constant.*

Proof. Let a point $x_0 \in X$ be arbitrary. By Theorem 5.11, $\mathfrak{m}_Y\text{-Sp}(f(x_0)) = \{f(x_0)\}$. But by using Theorem 4.14, $\mathfrak{m}_X\text{-Sp}(x_0) = X$. Therefore, by Theorem 5.10 one gets $f(X) = \{f(x_0)\}$. □

COROLLARY 5.13. *Let X, Y be topological spaces. If X is irreducible and Y is Hausdorff, then every continuous function from X to Y is constant.*

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