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# $\mathcal{T}$-NORMAL DECOMPOSITIONS OF MODULES

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**Abstract.** Let $\tau$ be a hereditary torsion theory on the category of modules. A module $A$ is called finitely $\tau$-completely decomposable if it is a finite direct sum of $\tau$-uniform $\tau$-injective modules. For a submodule $B$ of a module $A$, we show that the $\tau$-injective envelope of $A/B$ is finitely $\tau$-completely decomposable if and only if $B$ has a so-called $\tau$-normal decomposition in $A$.

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**Key words.** Torsion theory, $\tau$-injective module, $\tau$-irreducible submodule, $\tau$-uniform module, $\tau$-completely decomposable module, $\tau$-normal decomposition.

## 1. INTRODUCTION

A classical result of E. Matlis states that every injective module over a left noetherian ring has a decomposition as a direct sum of indecomposable injective modules [9, Theorem 2.5]. Over an arbitrary ring, a module (not necessarily injective) having such a direct sum decomposition was called *completely decomposable* by C. Faith and E. Walker [6, p. 217], and played an important part in the theory of injective modules. For a hereditary torsion theory $\tau$ on the category of modules, a torsion-theoretic counterpart is the so-called $\tau$-*completely decomposable* module (i.e. direct sum of some minimal $\tau$-injective modules), which was studied by K. Masaike and T. Horigome [8], S. Mohamed, B.J. Müller and S. Singh [10, 11], J.L. Bueso, P. Jara and B. Torrecillas [1], S. Crivei [3, 4] or S. Charalambides and J. Clark [2]. For a submodule $B$ of a module $A$, our main result characterizes when the $\tau$-injective envelope of $A/B$ is finitely $\tau$-completely decomposable in terms of the existence of a so-called $\tau$-*normal decomposition* of $B$ in $A$. By specialising our results to the hereditary torsion theory having all modules torsion, one obtains properties from the classical theory of injective modules.

## 2. PRELIMINARIES

Throughout the paper $R$ is an associative ring with non-zero identity and all modules are unitary left $R$-modules. We recall some terminology on torsion theories following [5, 7]. By $\tau = (\mathcal{T}, \mathcal{F})$ we denote a torsion theory on the category of modules. The modules in the class $\mathcal{T}$ are called $\tau$-*torsion*, and the modules in the class $\mathcal{F}$ are called $\tau$-*torsionfree*. The class $\mathcal{T}$ is closed under direct sums, factor modules and extensions, while the class $\mathcal{F}$ is closed under direct products, submodules and extensions. If the class $\mathcal{T}$ is closed under submodules, then $\tau$ is called *hereditary*. Throughout the paper $\tau$ will be a hereditary torsion theory on the category of left $R$-modules.

The following definitions give the torsion-theoretic versions of the notions of essential submodule, irreducible submodule, uniform module and injective module.

DEFINITION 2.1. A submodule $B$ of a module $A$ is called:

(1) $\tau$-*dense* if $A/B$ is $\tau$-torsion.
(2) $\tau$-*essential* if $B$ is essential and $\tau$-dense in $A$.
(3) $\tau$-*irreducible* if $B \neq A$ and for every submodules $B_1$ and $B_2$ of $A$ with $B \subset B_1$ and $B \subset B_2$, we have $B \subset B_1 \cap B_2$ and $B_1 \cap B_2$ is a $\tau$-dense submodule of $A$.

DEFINITION 2.2. A module $A$ is called:

(1) $\tau$-*uniform* if $A \neq 0$ and every non-zero submodule $B$ of $A$ is $\tau$-essential in $A$.
(2) $\tau$-*injective* if it is injective with respect to every monomorphism with a $\tau$-torsion cokernel.

The following known proposition will be frequently used [7, p. 84].

PROPOSITION 2.3. *Every module $A$ has a $\tau$-injective envelope, which is unique up to isomorphism and denoted by $E_\tau(A)$. The $\tau$-injective envelope of $A$ is a $\tau$-essential extension of $A$.*

PROPOSITION 2.4. *Let $A$ be a $\tau$-injective module. Then the following are equivalent:*

(i) *$A$ is $\tau$-uniform.*
(ii) *$A \neq 0$ and $A = E_\tau(B)$ for every non-zero submodule $B$ of $A$.*
(iii) *The zero submodule of $A$ is $\tau$-irreducible.*

*Proof.* (i)⇒(ii) Assume that $A$ is $\tau$-uniform. Then $A \neq 0$. Let $B$ be a non-zero submodule of $A$. Then $B$ is $\tau$-essential in $A$. We may consider $E_\tau(B) \subseteq A$. Since $B$ is $\tau$-dense in $A$, $A/B$ is $\tau$-torsion. Then the factor module $A/E_\tau(B)$ is $\tau$-torsion, hence $E_\tau(B)$ is $\tau$-dense in $A$. Then $\tau$-injectivity implies that $E_\tau(B)$ is a direct summand of $A$. Since $B$ is essential in $A$, so is $E_\tau(B)$. It follows that $E_\tau(B) = A$.

(ii)⇒(iii) Assume that (ii) holds. Let $B_1$ and $B_2$ be non-zero submodules of $A$. Then $A = E_\tau(B_1)$ by hypothesis. Since $B_1$ is $\tau$-dense in $E_\tau(B_1) = A$, $B_1$ is essential in $A$, hence we have $B_1 \cap B_2 \neq 0$. Then $A = E_\tau(B_1 \cap B_2)$ by hypothesis, hence $B_1 \cap B_2$ is $\tau$-essential in $E_\tau(B_1 \cap B_2) = A$. It follows that the zero submodule of $A$ is $\tau$-irreducible.

(iii)⇒(i) Assume that the zero submodule of $A$ is $\tau$-irreducible. Then $A \neq 0$. Let $B$ be a non-zero submodule of $A$. By hypothesis, for every non-zero submodule $C$ of $A$, $B \cap C$ is a non-zero $\tau$-dense submodule of $A$. Then $B$ is essential in $A$. Since $B \cap C$ is $\tau$-dense in $A$, $A/(B \cap C)$ is $\tau$-torsion. It follows that the factor module $A/B$ is $\tau$-torsion, hence $B$ is $\tau$-dense in $A$. Therefore, $B$ is $\tau$-essential in $A$, and so $A$ is $\tau$-uniform. □

COROLLARY 2.5. *Let $A$ be a module. Then:*

(1) *The zero submodule of $A$ is $\tau$-irreducible if and only if $E_\tau(A)$ is $\tau$-uniform.*
(2) *A submodule $B$ of $A$ is $\tau$-irreducible if and only if $E_\tau(A/B)$ is $\tau$-uniform.*

*Proof.* (1) Assume that the zero submodule of $A$ is $\tau$-irreducible. Then $A \neq 0$. By Proposition 2.4, in order to show that $E_\tau(A)$ is $\tau$-uniform it is enough to prove that the zero submodule of $E_\tau(A)$ is $\tau$-irreducible. Let $B_1, B_2$ be non-zero submodules of $E_\tau(A)$. Since $A$ is essential in $E_\tau(A)$, we have $B_1 \cap A \neq 0$ and $B_2 \cap A \neq 0$. By hypothesis, $B_1 \cap B_2 \cap A$ is a non-zero $\tau$-dense submodule of $A$. Then we have $B_1 \cap B_2 \neq 0$. In the short exact sequence

$$0 \to A/(B_1 \cap B_2 \cap A) \to E_\tau(A)/(B_1 \cap B_2 \cap A) \to E_\tau(A)/A \to 0$$

the first and the last terms are $\tau$-torsion, hence the middle one must also be $\tau$-torsion. Then the factor module $E_\tau(A)/(B_1 \cap B_2)$ must be $\tau$-torsion, hence $B_1 \cap B_2$ is $\tau$-dense in $E_\tau(A)$. Therefore, the zero submodule of $E_\tau(A)$ is $\tau$-irreducible.

Conversely, assume that $E_\tau(A)$ is $\tau$-uniform. Then the zero submodule of $E_\tau(A)$ is $\tau$-irreducible by Proposition 2.4. Let $B_1, B_2$ be non-zero submodules of $A$. It follows that $B_1 \cap B_2$ is a non-zero $\tau$-dense submodule of $E_\tau(A)$. Since $E_\tau(A)/(B_1 \cap B_2)$ is $\tau$-torsion, so is its submodule $A/(B_1 \cap B_2)$. Hence $B_1 \cap B_2$ is a $\tau$-dense submodule of $A$. Therefore, the zero submodule of $A$ is $\tau$-irreducible.

(2) Let $B_1$ and $B_2$ be submodules of $A$ such that $B \subset B_1$ and $B \subset B_2$. Then we have $B \subset B_1 \cap B_2$ if and only if $B_1/B \cap B_2/B \neq 0$. Also, $B_1 \cap B_2$ is $\tau$-dense in $A$ if and only if $A/(B_1 \cap B_2)$ is $\tau$-torsion if and only if $(B_1/B) \cap (B_2/B)$ is $\tau$-dense in $A/B$. It follows that the submodule $B$ of $A$ is $\tau$-irreducible if and only if the zero submodule of $A/B$ is $\tau$-irreducible. But this is equivalent to $E_\tau(A/B)$ being $\tau$-uniform by Proposition 2.4. □

Another needed notion is that of irredundant intersection of submodules. If $B_1, \ldots, B_n$ are submodules of a module $A$, then the intersection $B_1 \cap \ldots \cap B_n$ is called *irredundant* if

$$B_i \not\supseteq B_1 \cap \ldots \cap B_{i-1} \cap B_{i+1} \cap \ldots \cap B_n$$

for every $i \in \{1, \ldots, n\}$ [12, p. 91]. Note that one can always refine a finite intersection of submodules to an irredundant intersection by omitting certain submodules.

## 3. $\tau$-NORMAL DECOMPOSITION

In this section we characterize when the $\tau$-injective envelope of a factor module is finitely $\tau$-completely decomposable in the sense of the following definition.

Definition 3.1. A module $A$ is called *finitely $\tau$-completely decomposable* if it is isomorphic to a finite direct sum of $\tau$-uniform $\tau$-injective modules, say $E_1, \ldots, E_n$. Then $\{E_1, \ldots, E_n\}$ is called a *complete set of associated $\tau$-uniform $\tau$-injective modules* of $A$.

If $B$ is a submodule of a module $A$ and $A \cong E_\tau(A/B) \cong E_1 \oplus \cdots \oplus E_n$ is a finite direct sum of $\tau$-uniform $\tau$-injective modules such that $E_1 \cong \ldots \cong E_n \cong E$, then $B$ is called *$E$-isotopic*.

The following lemma is immediate from the above definition.

Lemma 3.2. *Let $A$ be a module and let $B_1, \ldots B_n$ be $E$-isotopic submodules of $A$ for some $\tau$-uniform $\tau$-injective module $E$. Then $B = B_1 \cap \ldots \cap B_n$ is an $E$-isotopic submodule of $A$.*

Definition 3.3. Let $A$ be a module and let $B = B_1 \cap \ldots \cap B_n$ be an irredundant intersection of submodules of $A$. This intersection is called a *$\tau$-normal decomposition* of $B$ in $A$ if for every $i \in \{1, \ldots, n\}$, $B_i$ is $E_i$-isotopic for some non-isomorphic $\tau$-uniform $\tau$-injective modules $E_i$.

Now we may give our main result.

Theorem 3.4. *Let $B$ be a submodule of a module $A$. The following are equivalent:*

(i) *$E_\tau(A/B)$ is finitely $\tau$-completely decomposable.*
(ii) *$B$ is a finite intersection of $\tau$-irreducible submodules of $A$.*
(iii) *$B$ has a $\tau$-normal decomposition in $A$.*

*Proof.* (i)⇒(ii) Suppose that $E_\tau(A/B)$ is finitely $\tau$-completely decomposable, say $E_\tau(A/B) = E_1 \oplus \cdots \oplus E_n$ for some $\tau$-uniform $\tau$-injective modules $E_1, \ldots, E_n$. Let $p : A \to A/B$ be the natural homomorphism and $j : A/B \to E_\tau(A/B)$ the inclusion homomorphism. For every $i \in \{1, \ldots, n\}$, let $q_i : E_\tau(A/B) \to E_i$ be the canonical projection, $f_i = q_i jp : A \to E_i$ and $B_i = \mathrm{Ker}(f_i)$. Then we have $B = B_1 \cap \ldots \cap B_n$.

Let $i \in I$. Since $A/B$ is $\tau$-essential in $E_\tau(A/B)$ and $E_i \neq 0$, we have $(A/B) \cap E_i \neq 0$. Then we must have $B_i \neq A$. It follows that $0 \neq A/B_i \cong \mathrm{Im}(f_i) \subseteq E_i$. Since $E_i$ is $\tau$-uniform $\tau$-injective, we have $E_i \cong E_\tau(A/B_i)$ by Proposition 2.4. Then $B_i$ is $\tau$-irreducible by Corollary 2.5, and so $B$ is a finite intersection of $\tau$-irreducible submodules of $A$.

(ii)⇒(i) Suppose that $B = B_1 \cap \ldots \cap B_n$ is an intersection of $\tau$-irreducible submodules of $A$. We may assume that the intersection is already irredundant. By Proposition 2.4, a non-zero module is $\tau$-uniform $\tau$-injective if and only if it is the $\tau$-injective envelope of each of its non-zero submodules, that is, it is minimal $\tau$-injective in the sense of [4]. By Corollary 2.5 it follows that each $E_\tau(A/B_i)$ is minimal $\tau$-injective. By [4, Theorem 3.6] we have

$$E_\tau(A/B) \cong E_\tau(A/B_1) \oplus \cdots \oplus E_\tau(A/B_n),$$

hence $E_\tau(A/B)$ is finitely $\tau$-completely decomposable.

(ii)⇒(iii) Suppose that $B = B_1 \cap \ldots \cap B_n$ is an intersection of $\tau$-irreducible submodules of $A$. We may assume that the intersection is already irredundant. Then each $B_i$ is $E_\tau(A/B_i)$-isotopic. We may group together the submodules $B_i$ having isomorphic associated $\tau$-uniform $\tau$-injective modules and use Lemma 3.2 to produce a writing of $B$ as a finite intersection of isotopic submodules with non-isomorphic associated $\tau$-uniform $\tau$-injective modules. Finally, one obtains a $\tau$-normal decomposition of $B$ in $A$ by making the intersection irredundant.

(iii)⇒(ii) Suppose that $B = B_1 \cap \ldots \cap B_n$ is a $\tau$-normal decomposition of $B$ in $A$. Then for every $i \in \{1, \ldots, n\}$, $B_i$ is $E_i$-isotopic for some non-isomorphic $\tau$-uniform $\tau$-injective modules $E_i$. If $B_i = B_{i1} \cap \ldots \cap B_{im}$, then $E_\tau(A/B_{ij}) \cong E_i$ for every $j \in \{1, \ldots, m\}$ and each $B_{ij}$ is $\tau$-irreducible by Corollary 2.5. Hence $B$ may be written as a finite intersection of $\tau$-irreducible submodules of $A$. □

Corollary 3.5. *Let $B$ be a submodule of a module $A$ and let*

$$B = B_1 \cap \ldots \cap B_n = B'_1 \cap \ldots \cap B'_m$$

*be $\tau$-normal decompositions of $B$ in $A$ such that $B_i$ is $E_i$-isotopic and $B_j$ is $E'_j$-isotopic for some $\tau$-uniform $\tau$-injective modules $E_i$ and $E'_j$ for every $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Then $n = m$ and there is a one-to-one correspondence between the modules $E_i$ and $E'_j$ such that the corresponding modules are isomorphic.*

*Proof.* By Proposition 2.4 and [7, Proposition 8.16], the endomorphism ring of a $\tau$-uniform $\tau$-injective module is local. Then the corollary follows by the Krull-Remak-Schmidt-Azumaya Theorem. □

As already mentioned in the introduction, one may specialize our results to the hereditary torsion theory $\tau$ having all modules $\tau$-torsion in order to obtain similar properties from the classical theory of injective modules (e.g. see [12, Chapter 4]).

We end with an example for another hereditary torsion theory.

Example 3.6. Following [4, Example 3.7], consider the polynomial ring $R = K[X_1, \ldots, X_{n+2}]$, where $K$ is a field and $n$ is a positive integer. Consider the prime ideals $p = (X_1X_2, X_1X_3)$, $p_1 = (X_1)$ and $p_2 = (X_2, X_3)$ of $R$ generated by the specified elements. Then we have $p = p_1 \cap p_2$. Let $\tau_n$ be the hereditary torsion theory generated by all modules $M$ of Krull dimension $\dim M \leq n$. Then $E_{\tau_n}(R/p_1)$ and $E_{\tau_n}(R/p_2)$ are $\tau_n$-uniform $\tau_n$-injective and

$$E_{\tau_n}(R/p) \cong E_{\tau_n}(R/p_1) \oplus E_{\tau_n}(R/p_2)$$

by [4, Example 3.7]. Hence $p_1$ and $p_2$ are $\tau_n$-irreducible by Corollary 2.5, and so $p = p_1 \cap p_2$ is an irredundant intersection of $\tau_n$-irreducible ideals of $R$. Moreover, $p_1$ is $E_{\tau_n}(R/p_1)$-isotopic and $p_2$ is $E_{\tau_n}(R/p_2)$-isotopic. Hence $p = p_1 \cap p_2$ is a $\tau_n$-normal decomposition of $p$ in $R$ by Theorem 3.4.

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