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EXACT SOLUTIONS OF SOME NONLINEAR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS BY USING THE FUNCTIONAL VARIABLE METHOD

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Abstract. In this paper, we will employ the functional variable method for solving some nonlinear systems of partial differential equations which are very important in applied sciences, namely, the generalized Drinfel'd-Sokolov-Wilson system, Bogoyavlenskii equations and Davey-Sterwatson equations. This approach provides a more powerful mathematical tool for solving nonlinear differential equations which can be converted to a second-order ordinary differential equation through the travelling wave transformation.

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1. INTRODUCTION

In the theoretical investigation of the dynamics of nonlinear waves, coupled nonlinear partial differential equations are of great importance, due to their very wide applications in many fields of physics. As a matter of fact, coupled nonlinear partial differential equations are used to model motions of waves in a great array of contexts, including plasma physics, fluid mechanics, optical fibers, hydrodynamics, quantum mechanics and many other nonlinear dispersive systems. These nonlinear partial differential equations play a key role in describing key scientific phenomena. For example, the dispersive long wave equation is very helpful for costal and civil engineers to apply the nonlinear water wave model in harbor and coastal design. Recently, many kinds of powerful methods have been proposed to find exact solutions of nonlinear partial differential equations, for example, Variational iteration method [15], Algebraic method [19], Jacobi elliptic function expansion method [11], F-expansion method [13], Auxiliary equation method [20], Tanh method [5], Generalized hyperbolic function [12] and Functional variable method [18]. Among these methods, the Functional variable method is a powerful mathematical tool to solve nonlinear partial differential equations. By using this method, many kinds of important nonlinear partial differential equations have been solved successfully [18, 17]. The aim of this paper is to construct exact solutions of the generalized Drinfel'd-Sokolov-Wilson system, the Bogoyavlenskii equations and the Davey-Sterwatson equations by using the functional variable method.

The rest of this paper is organized as follows. In Section 2, brief description of the functional variable method for finding traveling wave solutions of nonlinear system of partial differential equations is given. In Section 3, the method is employed for obtaining the exact solutions of the generalized Drinfel'd-Sokolov-Wilson system, the Bogoyavlenskii equations and the Davey-Sterwatson equations. Finally, some conclusions are given in Section 4.

2. THE FUNCTIONAL VARIABLE METHOD

The functional variable method were first proposed by Zerarka et al [18, 17] to find the exact solutions for a wide class of linear and nonlinear wave equations. This method was further developed by many authors [16, 1, 10]. The advantage of this method is that one treats nonlinear problems by essentially linear methods, based on which it is easy to construct in full the exact solutions such as soliton-like waves, compacton and noncompacton solutions, trigonometric function solutions, pattern soliton solutions, black solitons or kink solutions, and so on.

Now, we describe the main steps of the functional variable method for finding exact solutions of nonlinear system of partial differential equations.

Consider the following nonlinear system of partial differential equations with independent variables x and t and dependent variables u and v

$$\begin{aligned} &P_1(u, v, u_t, v_t, u_x, v_x, u_{tt}, v_{tt}, u_{xx}, v_{xx}, u_{xt}, \ldots) = 0, \\ &P_2(u, v, u_t, v_t, u_x, v_x, u_{tt}, v_{tt}, u_{xx}, v_{xx}, u_{xt}, \ldots) = 0. \end{aligned} \tag{1}$$

Applying the travelling wave transformations $u(x,t) = U(\xi)$ and $v(x,t) = V(\xi)$ where $\xi = x - wt$, converts Eq.(1) into a system of ordinary differential like

$$\begin{aligned} &G_1(U, V, U_\xi, V_\xi, U_{\xi\xi}, V_{\xi\xi}, \ldots) = 0, \\ &G_2(U, V, U_\xi, V_\xi, U_{\xi\xi}, V_{\xi\xi}, \ldots) = 0. \end{aligned} \tag{2}$$

Using some mathematical operations, the system (2) is converted into a second-order ordinary differential equation as

$$H(U, U_{\xi\xi}) = 0. \tag{3}$$

Then we make a transformation in which the unknown function U is considered as a functional variable in the form

$$U_\xi = F(U), \tag{4}$$

and some successive derivatives of U are

$$U_{\xi\xi} = \frac{1}{2}(F^2)', \tag{5}$$

$$U_{\xi\xi\xi} = \frac{1}{2}(F^2)''\sqrt{F^2}, \tag{6}$$

$$U_{\xi\xi\xi\xi} = \tfrac{1}{2}[(F^2)'''F^2 + (F^2)''(F^2)'], \tag{7}$$
$$\vdots$$

where " $'$ " stands for $\frac{\mathrm{d}}{\mathrm{d}U}$.

Substituting (5) into Eq.(3) and after the mathematical manipulations, we reduce the ordinary differential equation (3) in terms of U, F as

$$K(U, F) = 0. \tag{8}$$

The key idea of this particular form Eq.(8) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, Eq.(8) provides the expression of F, and this, together with Eq.(4), give appropriate solutions to the original problem.

Remark 1. The functional variable method definitely can be applied to nonlinear PDEs which can be converted to a second-order ordinary differential equation (ODE) through the travelling wave transformation.

Theorem 1. *Consider the following second-order ordinary differential equation*

$$U_{\xi\xi} = k_1 U - k_2 U^{n+1}, \quad n \neq 0, \tag{9}$$

where k_1 and k_2 are constants and U is a functional variable in the form (4). *Then using* (5)*, the exact solutions of the Eq.*(9) *are obtained as*

Type I. When $k_1 > 0$, the solutions of Eq.(9) *are*

$$U_1(\xi) = \left\{\frac{(n+2)k_1}{2k_2}\operatorname{sech}^2(\frac{n}{2}\sqrt{k_1}\xi)\right\}^{\frac{1}{n}}, \tag{10}$$

$$U_2(\xi) = \left\{-\frac{(n+2)k_1}{2k_2}\operatorname{sech}^2(\frac{n}{2}\sqrt{k_1}\xi)\right\}^{\frac{1}{n}}, \tag{11}$$

Type II. When $k_1 < 0$, the solutions of Eq.(9) *are*

$$U_3(\xi) = \left\{\frac{(n+2)k_1}{2k_2}\csc^2(\frac{n}{2}\sqrt{-k_1}\xi)\right\}^{\frac{1}{n}}, \tag{12}$$

$$U_4(\xi) = \left\{\frac{(n+2)k_1}{2k_2}\csc^2(\frac{n}{2}\sqrt{-k_1}\xi)\right\}^{\frac{1}{n}}. \tag{13}$$

Proof. According to Eq.(4), we get from (9) an expression for the function $F(U)$:

$$\frac{1}{2}\left(F^2(U)\right)' = k_1 U - k_2 U^{n+1}, \tag{14}$$

where the prime denotes differentiation with respect to ξ. Integrating Eq.(14) with respect to U and after the mathematical manipulations, we have

$$F(U) = \pm U\sqrt{k_1 - \frac{2k_2}{n+2}U^n}, \tag{15}$$

or

$$F(U) = \pm\sqrt{k_1}U\sqrt{1 - \frac{2k_2}{(n+2)k_1}U^n}, \tag{16}$$

after changing the variables

$$Z = \frac{2k_2}{(n+2)k_1}U^n, \tag{17}$$

or

$$\left(\frac{(n+2)k_1}{2k_2}Z\right)^{\frac{1}{n}} = U, \tag{18}$$

with differentiation from Eq.(18):

$$\frac{1}{n}\left(\frac{(n+2)k_1}{2k_2}\right)^{\frac{1}{n}} Z^{\frac{1-n}{n}}\mathrm{d}Z = \mathrm{d}U(\xi). \tag{19}$$

We use (19) transformation to the Eq.(16):

$$\frac{\mathrm{d}Z}{Z\sqrt{1-Z}} = \pm n\sqrt{k_1}\mathrm{d}\xi, \tag{20}$$

with integrating from Eq.(20) and with setting the constant of integration as zero:

$$\ln\left|\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}}\right| = \pm n\sqrt{k_1}\xi. \tag{21}$$

In this case we have:

$$\left|\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}}\right| = \mathrm{e}^{\pm n\sqrt{k_1}\xi}. \tag{22}$$

If $\theta = \pm n\sqrt{k_1}\xi$, two cases will be considered separately.

Case 1. Suppose that $k_1 > 0$. Then

$$\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}} = \mathrm{e}^{\theta}, \tag{23}$$

thus, according to (23), we have

$$Z = \frac{4}{\mathrm{e}^{-\theta}+\mathrm{e}^{\theta}+2} = \frac{2}{\cosh\theta+1} = \frac{1}{\cosh^2\left(\frac{\theta}{2}\right)+1} = \operatorname{sech}^2\left(\frac{\theta}{2}\right),$$

so

$$Z = \operatorname{sech}^2\left(\frac{n}{2}\sqrt{k_1}\xi\right). \tag{24}$$

Now, suppose that $k_1 < 0$. Then

$$\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}} = \mathrm{e}^{\mathrm{i}\theta}, \tag{25}$$

thus, according to (25), we have

$$Z = \frac{4}{\mathrm{e}^{-\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\theta}+2} = \frac{2}{\cos\theta+1} = \frac{1}{\cos^2\left(\frac{\theta}{2}\right)+1} = \sec^2\left(\frac{\theta}{2}\right),$$

hence

$$Z = \sec^2\left(\frac{n}{2}\sqrt{-k_1}\xi\right). \tag{26}$$

Case 2. Suppose that $k_1 > 0$. Then

$$\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}} = -\mathrm{e}^{\theta}, \tag{27}$$

therefore, according to (27), we have

$$Z = -\frac{4}{\mathrm{e}^{-\theta}+\mathrm{e}^{\theta}+2} = \frac{2}{\cosh\theta-1} = \frac{1}{\sinh^2\left(\frac{\theta}{2}\right)+1} = -\operatorname{csch}^2\left(\frac{\theta}{2}\right),$$

so

$$Z = -\operatorname{csch}^2\left(\frac{n}{2}\sqrt{k_1}\xi\right). \tag{28}$$

Now, assume that $k_1 < 0$. Then

$$\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}} = -\mathrm{e}^{-i\theta}, \tag{29}$$

thus, according to (29), we have

$$Z = -\frac{4}{\mathrm{e}^{-i\theta}+\mathrm{e}^{\mathrm{i}\theta}-2} = \frac{2}{1-\cos\theta} = \frac{1}{\sin^2\left(\frac{\theta}{2}\right)} = \csc^2\left(\frac{\theta}{2}\right),$$

so

$$Z = \csc^2\left(\frac{n}{2}\sqrt{-k_1}\xi\right). \tag{30}$$

Now, using the relations (18), (24), (26), (28) and (30), the solutions of Eq.(9) are in the following forms:

– when $k_1 > 0$, the solutions of Eq.(9) are

$$U_1(\xi) = \left\{\frac{(n+2)k_1}{2k_2}\operatorname{sech}^2\left(\frac{n}{2}\sqrt{k_1}\xi\right)\right\}^{\frac{1}{n}},$$

$$U_2(\xi) = \left\{-\frac{(n+2)k_1}{2k_2}\operatorname{sech}^2\left(\frac{n}{2}\sqrt{k_1}\xi\right)\right\}^{\frac{1}{n}}.$$

– when $k_1 < 0$, the solutions of Eq.(9) are

$$U_3(\xi) = \left\{\frac{(n+2)k_1}{2k_2}\csc^2\left(\frac{n}{2}\sqrt{-k_1}\xi\right)\right\}^{\frac{1}{n}},$$

$$U_4(\xi) = \left\{ \frac{(n+2)k_1}{2k_2} \csc^2 \left(\frac{n}{2} \sqrt{-k_1} \xi \right) \right\}^{\frac{1}{n}}.$$

□

3. EXAMPLES

In this Section, we demonstrate three nonlinear partial differential systems by using the functional variable method described in Section 2.

3.1. The generalized Drinfel'd-Sokolov-Wilson system. Consider the generalized Drinefel'd-Sokolov-Wilson system [14]:

$$\begin{cases} u_t + p(v^n)_x = 0, \\ v_t + qv_{xxx} + ruv_x + su_x v = 0, \end{cases} \tag{31}$$

where p, q, r and s are constants. Eq.(31) is a very important nonlinear evolution equation in mathematical physics and engineering. When $n = 2$, the Eq.(31) become the Drinfel'd-Sokolov-Wilson system [21, 6]. We use the wave transformations

$$u(x,t) = U(\xi), \quad v(x,t) = V(\xi), \quad \xi = x - wt. \tag{32}$$

Substituting (32) into (31), we obtain ordinary differential equations:

$$-wU_\xi + p(V^n)_\xi = 0, \tag{33}$$

$$-wV_\xi + qV_{\xi\xi\xi} + rUV_\xi + sU_\xi V = 0. \tag{34}$$

By integrating the Eq.(33) with respect to ξ, and neglecting the constant of integration, we have

$$U = \frac{p}{w} V^n. \tag{35}$$

Inserting Eq.(35) into Eq.(34) it yields

$$wqV_{\xi\xi\xi} - w^2 V_\xi + p(r+ns)V^n V_\xi = 0. \tag{36}$$

Integrating Eq.(36) with respect to ξ choosing constant of integration to zero, we obtain

$$wqV_{\xi\xi} - w^2 V + \frac{p(r+ns)}{(n+1)} V^{n+1} = 0, \tag{37}$$

or

$$V_{\xi\xi} = \frac{w}{q} V - \frac{p(r+ns)}{qw(n+1)} V^{n+1}. \tag{38}$$

Then we use the transformation

$$V_\xi = F(V), \tag{39}$$

and Eq.(5) to convert Eq.(38) to

$$\frac{1}{2} \left(F^2(V) \right)' = \frac{w}{q} V - \frac{p(r+ns)}{qw(n+1)} V^{n+1}, \tag{40}$$

where the prime denotes differentiation with respect to ξ. Integrating Eq.(40) with respect to $V$and after the mathematical manipulations, we have

$$F(V) = \pm\sqrt{\frac{w}{q}}\,V\sqrt{1-\frac{2p(r+ns)}{w^2(n+1)(n+2)}V^n}. \tag{41}$$

Using the relations (39), (10), (11), (12) and (13), when $\frac{w}{q} > 0$, the solution of Eq.(38) is in the following forms:

$$V_1(\xi) = \left\{\frac{w^2(n+1)(n+2)}{2p(r+ns)}\operatorname{sech}^2\left(\frac{n}{2}\sqrt{\frac{w}{q}}\xi\right)\right\}^{\frac{1}{n}}, \tag{42}$$

$$V_2(\xi) = \left\{-\frac{w^2(n+1)(n+2)}{2p(r+ns)}\operatorname{csch}^2\left(\frac{n}{2}\sqrt{\frac{w}{q}}\xi\right)\right\}^{\frac{1}{n}}, \tag{43}$$

and, when $\frac{w}{q} < 0$, the solution of Eq.(38) is in the following forms:

$$V_3(\xi) = \left\{\frac{w^2(n+1)(n+2)}{2p(r+ns)}\sec^2\left(\frac{n}{2}\sqrt{\frac{w}{q}}\xi\right)\right\}^{\frac{1}{n}}, \tag{44}$$

$$V_4(\xi) = \left\{\frac{w^2(n+1)(n+2)}{2p(r+ns)}\csc^2\left(\frac{n}{2}\sqrt{\frac{w}{q}}\xi\right)\right\}^{\frac{1}{n}}. \tag{45}$$

Also, by considering the solution U given by the relation (35), we have obtained

$$U_1(\xi) = \frac{w(n+1)(n+2)}{2(r+ns)}\operatorname{sech}^2\left(\frac{n}{2}\sqrt{\frac{w}{q}}\xi\right), \tag{46}$$

$$U_2(\xi) = -\frac{w(n+1)(n+2)}{2(r+ns)}\operatorname{csch}^2\left(\frac{n}{2}\sqrt{\frac{w}{q}}\xi\right), \tag{47}$$

$$U_3(\xi) = \frac{w(n+1)(n+2)}{2(r+ns)}\sec^2\left(\frac{n}{2}\sqrt{\frac{w}{q}}\xi\right), \tag{48}$$

$$U_4(\xi) = \frac{w(n+1)(n+2)}{2(r+ns)}\csc^2\left(\frac{n}{2}\sqrt{\frac{w}{q}}\xi\right). \tag{49}$$

For $\frac{w}{q} > 0$, using the travelling wave transformations (32), we obtain the following soliton solutions of the generalized Drinefel'd-Sokolov-Wilson system:

$$v_1(x,t) = \left\{\frac{w^2(n+1)(n+2)}{2p(r+ns)}\operatorname{sech}^2\left(\frac{n}{2}\sqrt{\frac{w}{q}}(x-wt)\right)\right\}^{\frac{1}{n}}, \tag{50}$$

$$u_1(x,t) = \frac{w(n+1)(n+2)}{2(r+ns)}\operatorname{sech}^2\left(\frac{n}{2}\sqrt{\frac{w}{q}}(x-wt)\right), \tag{51}$$

$$v_2(x,t) = \left\{ -\frac{w^2(n+1)(n+2)}{2p(r+ns)} \operatorname{csch}^2\left(\frac{n}{2}\sqrt{\frac{w}{q}}(x-wt)\right)\right\}^{\frac{1}{n}}, \tag{52}$$

$$u_2(x,t) = -\frac{w(n+1)(n+2)}{2(r+ns)} \operatorname{csch}^2\left(\frac{n}{2}\sqrt{\frac{w}{q}}(x-wt)\right). \tag{53}$$

For $\frac{w}{q} < 0$, we obtain the periodic wave solutions:

$$v_3(x,t) = \left\{ \frac{w^2(n+1)(n+2)}{2p(r+ns)} \sec^2\left(\frac{n}{2}\sqrt{-\frac{w}{q}}(x-wt)\right)\right\}^{\frac{1}{n}}, \tag{54}$$

$$u_3(x,t) = \frac{w(n+1)(n+2)}{2(r+ns)} \sec^2\left(\frac{n}{2}\sqrt{-\frac{w}{q}}(x-wt)\right), \tag{55}$$

$$v_4(x,t) = \left\{ \frac{w^2(n+1)(n+2)}{2p(r+ns)} \csc^2\left(\frac{n}{2}\sqrt{-\frac{w}{q}}(x-wt)\right)\right\}^{\frac{1}{n}}, \tag{56}$$

$$u_4(x,t) = \frac{w(n+1)(n+2)}{2(r+ns)} \csc^2\left(\frac{n}{2}\sqrt{-\frac{w}{q}}(x-wt)\right). \tag{57}$$

Note that the above two results in Eqs.(42) and (46) are the same as those obtained in [4].

3.2. The Bogoyavlenskii equations. Consider the Bogoyavlenskii equations [2]:

$$\begin{cases} 4u_t + u_{xxy} - 4u^2 u_y - 4u_x v = 0, \\ v_x = u u_y. \end{cases} \tag{58}$$

Eq. (58) were derived by Kudryashov and Pickering as a member of a (2+1) Schwarzian breaking soliton hierarchy [7]. Eq.(58) is the modified version of a breaking soliton equation, $4u_{xt} + 8u_x u_{xy} + 4u_y u_{xx} + u_{xxxy} = 0$, which describes the (2+1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis. To a certain extent, a similar interaction is observed in waves on the surface of the sea. It is well-known that the solution and its dynamics of the equation can make researchers deeply understand the described physical process [8]. Now, we apply the functional variable method to find the solitary wave solutions for Bogoyavlenskii equations. Firstly, we let

$$u(x,y,t) = U(\xi), \quad v(x,y,t) = V(\xi), \quad \xi = x + y - wt. \tag{59}$$

Substituting (59) into (58), we obtain ordinary differential equations:

$$-4wU_\xi + U_{\xi\xi\xi} - 4U^2 U_\xi - 4U_\xi V = 0, \tag{60}$$

$$UU_\xi = V_\xi. \tag{61}$$

By integrating the Eq.(61) with respect to ξ, and neglecting the constant of integration, we have

$$V = \frac{1}{2}U^2. \tag{62}$$

Substituting Eq.(62) into Eq.(60), after integrating with respect to ξ choosing constant of integration to zero, we obtain

$$U_{\xi\xi} - 2U^3 - 4\omega U = 0, \tag{63}$$

or

$$U_{\xi\xi} = 2U^3 + 4\omega U. \tag{64}$$

Then we use the transformation

$$U_\xi = F(U), \tag{65}$$

and Eq.(5) to convert Eq.(64) to

$$\frac{1}{2}\left(F^2(U)\right)' = 2U^3 + 4\omega U, \tag{66}$$

where the prime denotes differentiation with respect to ξ. Integrating Eq.(66) with respect to U and after the mathematical manipulations, we have

$$F(U) = U^4 + 4\omega U^2 = 4\omega U\sqrt{1 + \frac{1}{4\omega}U^2}. \tag{67}$$

Using the relations (57), (10), (11), (12) and (13), when $w > 0$, the solution of Eq.(62) is in the following forms:

$$U_1(\xi) = 2\sqrt{-w}\,\operatorname{sech}(2\sqrt{w}\xi), \tag{68}$$

$$U_2(\xi) = 2\sqrt{w}\,\operatorname{csch}(2\sqrt{w}\xi), \tag{69}$$

and, when $w < 0$, the solution of Eq.(62) is in the following forms:

$$U_3(\xi) = 2\sqrt{-w}\,\sec(2\sqrt{-w}\xi), \tag{70}$$

$$U_4(\xi) = 2\sqrt{-w}\,\sec(2\sqrt{-w}\xi). \tag{71}$$

Also, by considering the solution V given by the relation (62), we have obtained

$$V_1(\xi) = -2w\,\operatorname{sech}^2(2\sqrt{w}\xi), \tag{72}$$

$$V_2(\xi) = 2w\,\operatorname{csch}^2(2\sqrt{w}\xi), \tag{73}$$

$$V_3(\xi) = -2w\,\sec^2(2\sqrt{-w}\xi), \tag{74}$$

$$V_4(\xi) = -2w\,\csc^2(2\sqrt{-w}\xi). \tag{75}$$

For $w > 0$, using the travelling wave transformations (59), we obtain the following soliton solutions of the Bogoyavlenskii equations

$$u_1(x, y, t) = 2\sqrt{-w}\,\operatorname{sech}(2\sqrt{w}(x + y - wt)), \tag{76}$$

(77) $$v_1(x,y,t) = -2w\,\text{sech}^2(2\sqrt{w}(x+y-wt)),$$

(78) $$u_2(x,y,t) = 2\sqrt{w}\,\text{csch}(2\sqrt{w}(x+y-wt)),$$

(79) $$v_2(x,y,t) = 2w\,\text{csch}^2(2\sqrt{w}(x+y-wt)).$$

For $w < 0$, we obtain the periodic wave solutions:

(80) $$u_3(x,y,t) = 2\sqrt{-w}\sec(2\sqrt{-w}(x+y-wt)),$$

(81) $$v_3(x,y,t) = -2w\sec^2(2\sqrt{-w}(x+y-wt)),$$

(82) $$u_4(x,y,t) = 2\sqrt{-w}\csc(2\sqrt{-w}(x+y-wt)),$$

(83) $$v_2(x,y,t) = -2w\csc^2(2\sqrt{-w}(x+y-wt)).$$

These solutions are all new exact solutions.

3.3. Davey-Sterwatson equations. Consider the Davey-Sterwatson equations [9]:

(84) $$\begin{cases} iu_t + \frac{1}{2}k^2(u_{xx} + k^2 u_{yy}) + c\,|u|^2\,u - uv = 0, \\ v_{xx} - k^2 v_{yy} - 2c\left(|u|^2\right)_{xx} = 0, \end{cases}$$

where $c \neq 0$ is real constant and $k^2 = \pm 1$.

Davey and Stewartson first derived their model in the context of water wave, with purely physical considerations. In this context, u is the amplitude of a surface wave packet, while v is the velocity potential of the mean flow interacting with the surface wave [3]. At this time, by means of the functional variable method, we will find some solitary wave solutions of the Davey-Sterwatson system. In order to seek its travelling wave solution, we introduce a transformation

(85) $$u(x,y,t) = U(\xi)\mathrm{e}^{\mathrm{i}\beta}, \quad v(x,y,t) = V(\xi), \quad \xi = x+y-wt, \quad \beta = x+y-\alpha t.$$

Substituting (85) into Eq. (84), and cancelling $\mathrm{e}^{\mathrm{i}\beta}$ yields ordinary differential equations(ODES) for $U(\xi)$ and $V(\xi)$

(86) $$\frac{1}{2}k^2(1+k^2)U_{\xi\xi} + \left[\alpha - \frac{1}{2}k^2(1+k^2)\right]U + cU^3 + \\ i\left[-\omega + k^2(1+k^2)\right]U_\xi - UV = 0,$$

(87) $$(1-k^2)V_{\xi\xi} - 2c(U)^2_{\xi\xi} = 0.$$

Setting

(88) $$\omega = k^2(1+k^2),$$

then (86) and (87) reduce to

(89) $$\frac{1}{2}\omega U_{\xi\xi} + \left[\alpha - \frac{1}{2}\omega\right]U + cU^3 - UV = 0,$$

$$(1-k^2)V_{\xi\xi} - 2c(U^2)_{\xi\xi} = 0. \tag{90}$$

Integrating Eq.(90) with respect to ξ once and setting integration constant to zero, and integrating it again yields

$$(1-k^2)V - 2cU^2 = 0 \tag{91}$$

or

$$V = \frac{2cU^2}{(1-k^2)}, \tag{92}$$

substituting Eq.(92) into Eq.(89) yields

$$\frac{1}{2}\omega U_{\xi\xi} + \left[\alpha - \frac{1}{2}\omega\right] U + \left[c - \frac{2c}{(1-k^2)}\right] U^3 = 0. \tag{93}$$

Eq.(93) can be written as

$$U_{\xi\xi} = A_1 U - A_2 U^3, \tag{94}$$

where

$$A_1 = -\frac{2\alpha - \omega}{\omega}, \qquad A_2 = \frac{2c - \frac{4c}{(1-k^2)}}{\omega}. \tag{95}$$

Then we use the transformation

$$U_\xi = F(U), \tag{96}$$

and Eq.(5) to convert Eq.(94) to

$$\frac{1}{2}\left(F^2(U)\right)' = A_1 U - A_2 U^3, \tag{97}$$

where the prime denotes differentiation with respect to ξ. Integrating Eq.(97) with respect to U and after the mathematical manipulations, we have

$$F(U) = \pm U\sqrt{A_1 - \frac{A_2}{2}U^2}. \tag{98}$$

Using the relations (96), (10), (11), (12) and (13), when $A_1 > 0$, the solution of Eq.(94) is in the following forms:

$$U_1(\xi) = \sqrt{\frac{2A_1}{A_2}}\operatorname{sech}(\sqrt{A_1}\xi), \tag{99}$$

$$U_2(\xi) = \sqrt{-\frac{2A_1}{A_2}}\operatorname{csch}(\sqrt{A_1}\xi), \tag{100}$$

and, when $A_1 < 0$, the solution of Eq. (94) is in the following forms:

$$U_3(\xi) = \sqrt{\frac{2A_1}{A_2}}\sec(\sqrt{-A_1}\xi), \tag{101}$$

$$U_4(\xi) = \sqrt{\frac{2A_1}{A_2}}\sec(\sqrt{-A_1}\xi). \tag{102}$$

Also, by considering the solution V given by the relation (92), we have obtained

$$V_1(\xi) = \frac{4cA_1}{(1-k^2)A_2}\mathrm{sech}^2(\sqrt{A_1}\xi), \tag{103}$$

$$V_2(\xi) = -\frac{4cA_1}{(1-k^2)A_2}\,\mathrm{csch}^2(\sqrt{A_1}\xi), \tag{104}$$

$$V_3(\xi) = \frac{4cA_1}{(1-k^2)A_2}\,\sec^2(\sqrt{-A_1}\xi), \tag{105}$$

$$V_4(\xi) = \frac{4cA_1}{(1-k^2)A_2}\sec^2(\sqrt{-A_1}\xi). \tag{106}$$

For $A_1 > 0$, using the travelling wave transformations (69), we obtain the following soliton solutions of the Davey-Sterwatson equations

$$u_1(x,y,t) = \sqrt{\frac{2A_1}{A_2}}\,\mathrm{e}^{\mathrm{i}(x+y-\alpha t)}\mathrm{sech}(\sqrt{A_1}(x+y-k^2(1+k^2)t)), \tag{107}$$

$$v_1(x,y,t) = \frac{4cA_1}{(1-k^2)A_2}\mathrm{sech}^2(\sqrt{A_1}(x+y-k^2(1+k^2)t)), \tag{108}$$

$$u_2(x,y,t) = \sqrt{-\frac{2A_1}{A_2}}\,\mathrm{e}^{\mathrm{i}(x+y-\alpha t)}\mathrm{csch}(\sqrt{A_1}(x+y-k^2(1+k^2)t)), \tag{109}$$

$$v_2(x,y,t) = -\frac{4cA_1}{(1-k^2)A_2}\,\mathrm{csch}^2(\sqrt{A_1}(x+y-k^2(1+k^2)t)). \tag{110}$$

For $A_1 < 0$, we obtain the periodic wave solutions:

$$u_3(x,y,t) = \sqrt{\frac{2A_1}{A_2}}\mathrm{e}^{\mathrm{i}(x+y-\alpha t)}\,\sec(\sqrt{-A_1}(x+y-k^2(1+k^2)t)), \tag{111}$$

$$v_3(x,y,t) = \frac{4cA_1}{(1-k^2)A_2}\,\sec^2(\sqrt{-A_1}(x+y-k^2(1+k^2)t)), \tag{112}$$

$$u_4(x,y,t) = \sqrt{\frac{2A_1}{A_2}}\,\mathrm{e}^{\mathrm{i}(x+y-\alpha t)}\sec(\sqrt{-A_1}(x+y-k^2(1+k^2)t)), \tag{113}$$

$$v_4(x,y,t) = \frac{4cA_1}{(1-k^2)A_2}\sec^2(\sqrt{-A_1}(x+y-k^2(1+k^2)t)). \tag{114}$$

4. CONCLUSIONS

In this article, we applied the functional variable method to construct the exact solutions for three nonlinear partial differential systems, namely, the generalized Drinfel'd-Sokolov-Wilson system, the Bogoyavlenskii equations and the Davey-Sterwatson equations, which were not discussed elsewhere using that method. This method definitely can be applied to nonlinear partial differential systems which can be converted to a second-order ordinary differential equations through the travelling wave transformation. Also, we conclude that the proposed method used in this paper is very effective and can be extended to other kinds of nonlinear partial differential systems in mathematical physics. Finally, by using the Maple we have assured the correctness of the obtained solutions by putting them back into the original equations.

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