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COMMON FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS IN MENGER SPACES VIA COMMON LIMIT RANGE PROPERTY

BADRIDATT D. PANT, SUNNY CHAUHAN, and HUMA SAHPER

Abstract. Inspired by the well-known concept of common limit range property due to Sintunavarat and Kumam, we prove common fixed point theorems for two pairs of weakly compatible mappings in Menger spaces. Some illustrative examples are given in order to demonstrate the validity of our main result. We extend our results to four finite families of self-mappings.

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Key words. t-norm, Menger space, weakly compatible mappings, common limit range property, common fixed point.

1. INTRODUCTION

Sehgal [32] initiated the probabilistic version of the celebrated Banach Contraction Principle in his doctoral dissertation. Since then the subject has been further investigated by various authors (see [31, 33]). The study of common fixed points of compatible mappings satisfying contractive conditions emerged as an area of vigorous research activity after Jungck [12] introduced the notion of compatibility in metric spaces. In the study of common fixed points of compatible mappings we often require assumptions on completeness of the whole space and continuity of the mappings, besides some contractive conditions. Later, Jungck and Rhoades [13] weakened the notion of compatibility, by introducing the notion of weakly compatible mappings. They proved common fixed point theorems without assuming continuity of the mappings or completeness of the whole space.

However, the study of common fixed points of non-compatible mappings is also interesting due to Pant [23]. In 2002, Aamri and Moutawakil [1] introduced the property (E.A) for self-maps which applies to the class of non-compatible mappings. In [19], Liu et al. introduced the notion of the common property (E.A) which implies the property (E.A), and proved several fixed point theorems for single-valued and multi-valued mappings under hybrid contractive conditions. Many mathematicians exploited these concepts (see, for example, [2, 14, 15, 21, 35]) in the framework of probabilistic metric spaces in order to obtain a number of common fixed point results. Recent literature on fixed point in probabilistic metric spaces, using different approaches, can be found in [3, 6, 7, 8, 9, 10, 16, 17, 18, 22, 24, 25, 26, 28, 29, 30]. Both property (E.A) and the common property (E.A) require the closeness of the underlying subspaces for the existence of coincidence points. Sintunavarat and Kumam introduced in [36] the notion of "common limit range property" which doesn't require the closeness of the subspaces for the existence of fixed points in fuzzy metric spaces (also see [37]). Recently, Imdad et al. [11] utilized the notion of common limit in the range property for two pairs of self-mappings and proved some interesting theorems in Menger spaces (also see [4, 5, 34]).

The aim of this paper is to prove some fixed point theorems for weakly compatible mappings in Menger spaces satisfying a common limit range property. We also give some examples which demonstrate the validity of the hypotheses and the generality of our results. As an application, we present a fixed point theorem for four finite families of self-mappings.

2. PRELIMINARIES

DEFINITION 1 ([31]). A mapping $* : [0,1] \times [0,1] \to [0,1]$ is called a *triangular norm* (briefly, t-norm) if it satisfies the following conditions: for all $a, b, c, d \in [0,1]$

(1) $a * 1 = a$,
(2) $a * b = b * a$,
(3) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$,
(4) $a * (b * c) = (a * b) * c$.

Some examples of t-norms are $a * b = \min\{a, b\}$, $a * b = ab$ and $a * b = \max\{a + b - 1, 0\}$.

DEFINITION 2 ([31]). A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a *distribution function* if it is non-decreasing, left continuous, and such that $\inf\{F(t) : t \in \mathbb{R}\} = 0$ and $\sup\{F(t) : t \in \mathbb{R}\} = 1$. We denote the set of all distribution functions on $(-\infty, \infty)$ by $\Im$, while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1, & \text{if } t > 0. \end{cases}$$

If X is a non-empty set, $\mathcal{F} : X \times X \to \Im$ is called a *probabilistic distance on* X, and $F(x, y)$ is usually denoted by $F_{x,y}$.

DEFINITION 3 ([20]). The ordered pair $(X, \mathcal{F})$ is called a *probabilistic metric space* (briefly, PM-space) if X is a non-empty set and F is a probabilistic distance satisfying, for all $x, y, z \in X$ and all $t, s > 0$, the following conditions:

(1) $F_{x,y}(t) = H(t)$ if and only if $x = y$,
(2) $F_{x,y}(t) = F_{y,x}(t)$,
(3) $F_{x,z}(t) = 1, F_{z,y}(s) = 1 \Rightarrow F_{x,y}(t + s) = 1$.

The ordered triplet $(X, \mathcal{F}, *)$ is called a *Menger space* if $(X, \mathcal{F})$ is a PM-space, $*$ is a t-norm and the following inequality holds for all $x, y, z \in X$ and all $t, s > 0$

$$F_{x,y}(t + s) \geq F_{x,z}(t) * F_{z,y}(s).$$

Every metric space (X, d) can be realized as a PM-space, by defining the map $\mathcal{F} : X \times X \to \Im$ by $F_{x,y}(t) = H\,(t - d(x, y))$ for all $x, y \in X$.

Definition 4 ([31]). Let $(X, \mathcal{F}, *)$ be a Menger space and $*$ be a continuous t-norm. A sequence $\{x_n\}$ in X is said to *converge to a point* x in X if, for every $\varepsilon > 0$ and every $\lambda \in (0, 1)$, there exists an integer $\mathbb{N}$ such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ for all $n \geq \mathbb{N}$.

Definition 5 ([21]). A pair (A, S) of self-mappings of a Menger space $(X, \mathcal{F}, *)$ is said to be *compatible* if $F_{ASx_n,SAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to z$, for some $z \in X$, as $n \to \infty$.

Definition 6 ([15]). A pair (A, S) of self-mappings of a Menger space $(X, \mathcal{F}, *)$ is said to satisfy property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$.

Definition 7 ([3]). A pair (A, S) of self-mappings of a Menger space $(X, \mathcal{F}, *)$ is said to be *non-compatible* if there exists at least one sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} Ax_n = z = \lim\limits_{n\to\infty} Sx_n$, for some $z \in X$, but, for some $t > 0$, either $\lim\limits_{n\to\infty} F_{ASx_n,SAx_n}(t) \neq 1$ or the limit does not exist.

In view of Definition 6, it is easy to see that any two non-compatible self-mappings of $(X, \mathcal{F}, *)$ satisfy the property (E.A). But two mappings satisfying the property (E.A) need not be non-compatible (see [9, Example 1]).

Definition 8 ([3]). Two pairs (A, S) and (B, T) of self-mappings of a Menger space $(X, \mathcal{F}, *)$ are said to satisfy the common property (E.A) if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

for some $z \in X$.

Definition 9 ([13]). A pair (A, S) of self-mappings of a non-empty set X is said to be *weakly compatible* (or coincidentally commuting) if they commute at their coincidence points, that is, if $Az = Sz$, for some $z \in X$, then $ASz = SAz$.

If self-mappings A and S of a Menger space $(X, \mathcal{F}, *)$ are compatible, then they are weakly compatible, but the reverse need not be true (see [35, Example 1]). Moreover, the weak compatibility and the property (E.A) are independent of each other (see [27, Example 2.2]).

Definition 10 ([11]). A pair (A, S) of self-mappings of a Menger space $(X, \mathcal{F}, *)$ is said to satisfy the *common limit range property with respect to the mapping* S, i.e., the (CLR_S) property, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

where $z \in S(X)$.

It is evident that a pair (A, S) of self-mappings satisfying the property (E.A), along with the closeness of the subspace $S(X)$, implies the (CLR_S) property.

Definition 11 ([11]). Two pairs (A, S) and (B, T) of self-mappings of a Menger space $(X, \mathcal{F}, *)$ are said to satisfy the *common limit range property with respect to the mappings S and T*, i.e., the (CLR_{ST}) property, if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$.

Lemma 1 ([21]). *Let $(X, \mathcal{F}, *)$ be a Menger space. If there exists a constant $k \in (0, 1)$ such that, for fixed $x, y \in X$,*

$$F_{x,y}(kt) \geq F_{x,y}(t)$$

for all $t > 0$, then $x = y$.

Definition 12 ([10]). Two families of self-mappings $\{A_i\}$ and $\{B_j\}$ are said to be *pairwise commuting* if:

(1) $A_iA_j = A_jA_i$, $i, j \in \{1, 2, \ldots, m\}$,
(2) $B_iB_j = B_jB_i$, $i, j \in \{1, 2, \ldots, n\}$,
(3) $A_iB_j = B_jA_i$, $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$.

3. RESULTS

We first prove the following lemma.

Lemma 2. *Let A, B, S, and T be self-mappings of a Menger space $(X, \mathcal{F}, *)$ with a continuous t-norm $* = \min$, satisfying the following conditions:*

(1) $A(X) \subset T(X)$ $\Big($*or* $B(X) \subset S(X)\Big)$,
(2) $T(X)$ $\Big($*or* $S(X)\Big)$ *is a closed subset of* X,
(3) $B(y_n)$ *converges for every sequence* $\{y_n\}$ *in* X, *whenever* $T(y_n)$ *converges (or* $A(x_n)$ *converges for every sequence* $\{x_n\}$ *in* X *whenever* $S(x_n)$ *converges),*
(4) *the pair* (A, S) *satisfies the* (CLR_S) *property (or the pair* (B, T) *satisfies the* (CLR_T) *property),*
(5) *there exists a constant* $k \in (0, 1)$ *such that*

$$[1 + aF_{Sx,Ty}(kt)] * F_{Ax,By}(kt) \geq a \left\{ \begin{array}{c} F_{Ax,Sx}(kt) * F_{By,Ty}(kt) * \\ F_{Ax,Ty}(2kt) * F_{By,Sx}(2kt) \end{array} \right\} + \left\{ \begin{array}{c} F_{Sx,Ty}(t) * F_{Ax,Sx}(t) * F_{By,Ty}(t) * \\ F_{Ax,Ty}(2t) * F_{By,Sx}(2t) \end{array} \right\}$$

for all $t > 0$, $x, y \in X$ *and* $a \geq 0$.

Then (A, S) *and* (B, T) *satisfy the* (CLR_{ST}) *property.*

Proof. Since the pair (A,S) satisfies the (CLR_S) property with respect to S, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

where $z \in S(X)$. As $A(X) \subset T(X)$ (where $T(X)$ is a closed subset of X), we can find a sequence $\{y_n\} \subset X$ such that $Ax_n = Ty_n$. Therefore,

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = z,$$

for some $z \in S(X)\cap T(X)$. Thus we have $Ax_n \to z$, $Sx_n \to z$ and $Ty_n \to z$. Now we show that $By_n \to z$. Using inequality (5) with $x = x_n$, $y = y_n$, we get

$$[1 + aF_{Sx_n,Ty_n}(kt)] * F_{Ax_n,By_n}(kt) \geq a\left\{\begin{array}{c} F_{Ax_n,Sx_n}(kt) * F_{By_n,Ty_n}(kt) * \\ F_{Ax_n,Ty_n}(2kt) * F_{By_n,Sx_n}(2kt) \end{array}\right\} + \left\{\begin{array}{c} F_{Sx_n,Ty_n}(t) * F_{Ax_n,Sx_n}(t) * F_{By_n,Ty_n}(t) * \\ F_{Ax_n,Ty_n}(2t) * F_{By_n,Sx_n}(2t) \end{array}\right\}.$$

For $t > 0$, let $By_n \to l$ $(\neq z)$, as $n \to \infty$. Then, passing to limit with $n \to \infty$, we get

$$[1 + aF_{z,z}(kt)] * F_{z,l}(kt) \geq a\{F_{z,z}(kt) * F_{l,z}(kt) * F_{z,z}(2kt) * F_{l,z}(2kt)\} + \left\{\begin{array}{c} F_{z,z}(t) * F_{z,z}(t) * F_{l,z}(t) * \\ F_{z,z}(2t) * F_{l,z}(2t) \end{array}\right\}.$$

Since $F_{z,l}(kt) \leq F_{z,l}(2kt)$ and $F_{z,l}(t) \leq F_{z,l}(2t)$, we obtain

$$(1 + a) * F_{z,l}(kt) \geq aF_{l,z}(kt) + F_{l,z}(t).$$

Taking into account that $a \geq 0$, it follows that

$$(1 + a) * F_{z,l}(kt) \leq F_{z,l}(kt) + aF_{z,l}(kt).$$

The above inequalities imply then

$$F_{z,l}(kt) + aF_{z,l}(kt) \geq aF_{l,z}(kt) + F_{l,z}(t), \quad F_{z,l}(kt) \geq F_{z,l}(t).$$

In view of Lemma 1, we obtain $l = z$. Thus we conclude that the pairs (A,S) and (B,T) satisfy the (CLR_{ST}) property. □

THEOREM 1. *Let $A, B, S,$ and T be self mappings of a Menger space $(X, \mathcal{F}, *)$ with a continuous t-norm $* = \min$, satisfying the inequality* (5) *of Lemma* 2. *If the pairs (A,S) and (B,T) share the (CLR_{ST}) property, then each of the pairs (A,S) and (B,T) has a coincidence point. Moreover, $A, B, S,$ and T have a unique common fixed point, provided both pairs (A,S) and (B,T) are weakly compatible.*

Proof. Since the pairs (A,S) and (B,T) satisfy the (CLR_{ST}) property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where, $z \in S(X)\cap T(X)$. Since $z \in S(X)$, there exists a point $u \in X$ such that $Su = z$. First we show that $Au = Su$. Putting $x = u$ and $y = y_n$ in inequality (5), we get

$$[1 + aF_{Su,Ty_n}(kt)] * F_{Au,By_n}(kt) \geq a \left\{ \begin{array}{c} F_{Au,Su}(kt) * F_{By_n,Ty_n}(kt) * \\ F_{Au,Ty_n}(2kt) * F_{By_n,Su}(2kt) \end{array} \right\} + \left\{ \begin{array}{c} F_{Su,Ty_n}(t) * F_{Au,Su}(t) * F_{By_n,Ty_n}(t) * \\ F_{Au,Ty_n}(2t) * F_{By_n,Su}(2t) \end{array} \right\}.$$

Passing now to limit with $n \to \infty$, we obtain

$$[1 + aF_{z,z}(kt)] * F_{Au,z}(kt) \geq a \left\{ \begin{array}{c} F_{Au,z}(kt) * F_{z,z}(kt) * \\ F_{Au,z}(2kt) * F_{z,z}(2kt) \end{array} \right\} + \left\{ \begin{array}{c} F_{z,z}(t) * F_{Au,z}(t) * F_{z,z}(t) * \\ F_{Au,z}(2t) * F_{z,z}(2t) \end{array} \right\}.$$

The inequalities $F_{Au,z}(kt) \leq F_{Au,z}(2kt)$ and $F_{Au,z}(t) \leq F_{Au,z}(2t)$ yield

$$(1 + a) * F_{Au,z}(kt) \geq aF_{Au,z}(kt) + F_{Au,z}(t).$$

Since $a \geq 0$, it follows that $(1 + a) * F_{Au,z}(kt) \leq F_{Au,z}(kt) + aF_{Au,z}(kt)$. The above inequality implies

$$F_{Au,z}(kt) + aF_{Au,z}(kt) \geq aF_{Au,z}(kt) + F_{Au,z}(t), \quad F_{Au,z}(kt) \geq F_{Au,z}(t).$$

In view of Lemma 1, we get $Au = z$. Therefore $Au = Su = z$, i.e., u is a coincidence point of the pair (A, S).

Since $z \in T(X)$, there exists a point $v \in X$ such that $Tv = z$. We assert that $Bv = Tv$. Putting $x = u$ and $y = v$ in inequality (5), we have

$$[1 + aF_{Su,Tv}(kt)] * F_{Au,Bv}(kt) \geq a \left\{ \begin{array}{c} F_{Au,Su}(kt) * F_{Bv,Tv}(kt) * \\ F_{Au,Tv}(2kt) * F_{Bv,Su}(2kt) \end{array} \right\} + \left\{ \begin{array}{c} F_{Su,Tv}(t) * F_{Au,Su}(t) * F_{Bv,Tv}(t) * \\ F_{Au,Tv}(2t) * F_{Bv,Su}(2t) \end{array} \right\},$$

so

$$[1 + aF_{z,z}(kt)] * F_{z,Bv}(kt) \geq a \left\{ \begin{array}{c} F_{z,z}(kt) * F_{Bv,z}(kt) * \\ F_{z,z}(2kt) * F_{Bv,z}(2kt) \end{array} \right\} + \left\{ \begin{array}{c} F_{z,z}(t) * F_{z,z}(t) * F_{Bv,z}(t) * \\ F_{z,z}(2t) * F_{Bv,z}(2t) \end{array} \right\}.$$

The inequalities $F_{z,Bv}(kt) \leq F_{z,Bv}(2kt)$ and $F_{z,Bv}(t) \leq F_{z,Bv}(2t)$ yield

$$(1 + a) * F_{z,Bv}(kt) \geq aF_{Bv,z}(kt) + F_{Bv,z}(t).$$

Since $a \geq 0$, it follows that $(1 + a) * F_{z,Bv}(kt) \leq F_{z,Bv}(kt) + aF_{z,Bv}(kt)$. The above inequality implies

$$F_{z,Bv}(kt) + aF_{z,Bv}(kt) \geq aF_{Bv,z}(kt) + F_{Bv,z}(t), \quad F_{z,Bv}(kt) \geq F_{z,Bv}(t).$$

Thus, by Lemma 1, we conclude that $z = Bv$. Therefore $Bv = Tv = z$, i.e., v is a coincidence point of the pair (B, T).

Since the pair (A, S) is weakly compatible, the equalities $Az = ASu = SAu = Sz$ hold. Putting $x = z$ and $y = v$ in inequality (5), we have

$$[1+aF_{Sz,Tv}(kt)]*F_{Az,Bv}(kt)\ge a\left\{\begin{array}{c}F_{Az,Sz}(kt)*F_{Bv,Tv}(kt)*\\F_{Az,Tv}(2kt)*F_{Bv,Sz}(2kt)\end{array}\right\}+\left\{\begin{array}{c}F_{Sz,Tv}(t)*F_{Az,Sz}(t)*F_{Bv,Tv}(t)*\\F_{Az,Tv}(2t)*F_{Bv,Sz}(2t)\end{array}\right\},$$

so

$$[1+aF_{Az,z}(kt)]*F_{Az,z}(kt)\ge a\left\{\begin{array}{c}F_{Az,Az}(kt)*F_{z,z}(kt)*\\F_{Az,z}(2kt)*F_{z,Az}(2kt)\end{array}\right\}+\left\{\begin{array}{c}F_{Az,z}(t)*F_{Az,Az}(t)*F_{z,z}(t)*\\F_{Az,z}(2t)*F_{z,Az}(2t)\end{array}\right\}.$$

The inequalities $F_{Az,z}(kt)\le F_{Az,z}(2kt)$ and $F_{Az,z}(t)\le F_{Az,z}(2t)$ imply

$$[1+aF_{Az,z}(kt)]*F_{Az,z}(kt)\ge aF_{Az,z}(kt)+F_{Az,z}(t).$$

Since $a\ge 0$, we get $[1+aF_{Az,z}(kt)]*F_{Az,z}(kt)\le F_{Az,z}(kt)+aF_{Az,z}(kt)$. This implies

$$F_{Az,z}(kt)+aF_{Az,z}(kt)\ge aF_{Az,z}(kt)+F_{Az,z}(t),\quad F_{Az,z}(kt)\ge F_{Az,z}(t).$$

Applying again Lemma 1, we get $z=Az$. Therefore $Az=Sz=z$, thus z is a common fixed point of the pair (A,S).

The weak compatibility of the pair (B,T) implies $Bz=BTv=TBv=Tz$. Using inequality (5) with $x=u$, $y=z$, we obtain

$$[1+aF_{Su,Tz}(kt)]*F_{Au,Bz}(kt)\ge a\left\{\begin{array}{c}F_{Au,Su}(kt)*F_{Bz,Tz}(kt)*\\F_{Au,Tz}(2kt)*F_{Bz,Su}(2kt)\end{array}\right\}+\left\{\begin{array}{c}F_{Su,Tz}(t)*F_{Au,Su}(t)*F_{Bz,Tz}(t)*\\F_{Au,Tz}(2t)*F_{Bz,Su}(2t)\end{array}\right\},$$

so

$$[1+aF_{z,Bz}(kt)]*F_{z,Bz}(kt)\ge a\left\{\begin{array}{c}F_{z,z}(kt)*F_{Bz,Bz}(kt)*\\F_{z,Bz}(2kt)*F_{Bz,z}(2kt)\end{array}\right\}+\left\{\begin{array}{c}F_{z,Bz}(t)*F_{z,z}(t)*F_{Bz,Bz}(t)*\\F_{z,Bz}(2t)*F_{Bz,z}(2t)\end{array}\right\}.$$

From $F_{z,Bz}(kt)\le F_{z,Bz}(2kt)$ and $F_{z,Bz}(t)\le F_{z,Bz}(2t)$ we conclude

$$[1+aF_{z,Bz}(kt)]*F_{z,Bz}(kt)\ge aF_{Bz,z}(kt)+F_{Bz,z}(t).$$

Since $a\ge 0$, we get $[1+aF_{z,Bz}(kt)]*F_{z,Bz}(kt)\le F_{z,Bz}(kt)+aF_{z,Bz}(kt)$. This implies

$$F_{z,Bz}(kt)+aF_{z,Bz}(kt)\ge aF_{Bz,z}(kt)+F_{Bz,z}(t),\quad F_{z,Sz}(kt)\ge F_{z,Sz}(t).$$

Thus we get $z=Bz$, by Lemma 1. Therefore $Bz=Tz=z$, so z is a common fixed point of the pair (B,T). We conclude that z is a common fixed point of the pairs (A,S) and (B,T). The uniqueness of this common fixed point is an easy consequence of inequality (4). □

Remark 1. The above result shows that the (CLR_{ST}) property does not necessarily require continuity conditions or certain containments of the ranges of the involved mappings, and completeness (or closeness) of the underlying space (or subspaces). Theorem 1 improves the results of Pant and Chauhan [25, Theorem 3.1] and Pathak and Verma [28, Theorem 3.2].

The following example illustrates Theorem 1.

EXAMPLE 1. Let $X = [5, 21)$ and define $F_{x,y}(t) := H(t - d(x, y))$, where d is the usual metric, that is $d(x, y) =| x - y |$ for all $x, y \in X$. Then $(X, \mathcal{F}, *)$ is a Menger space, where $* = \min$ is a continuous t-norm. Let A, B, S, and T be self-mappings on X defined by

$$A(x) = \begin{cases} 5, & \text{if } x \in \{5\} \cup (9, 21) \\ 20, & \text{if } x \in (5, 9], \end{cases}$$

$$B(x) = \begin{cases} 5, & \text{if } x \in \{5\} \cup (9, 21) \\ 13, & \text{if } x \in (5, 9], \end{cases}$$

$$S(x) = \begin{cases} 5, & \text{if } x = 5 \\ 10, & \text{if } x \in (5, 9] \\ \frac{x+1}{2}, & \text{if } x \in (9, 21), \end{cases}$$

$$T(x) = \begin{cases} 5, & \text{if } x = 5 \\ 18, & \text{if } x \in (5, 9] \\ x - 4, & \text{if } x \in (9, 21). \end{cases}$$

Consider the sequences $\{x_n\} = \left\{9 + \frac{1}{n}\right\}_{n \in \mathbb{N}}$, $\{y_n\} = \{5\}$ or $\{x_n\} = \{5\}$, $\{y_n\} = \left\{9 + \frac{1}{n}\right\}_{n \in \mathbb{N}}$. It is obvious that both pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property:

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 5 \in S(X) \cap T(X).$$

Note that $A(X) = \{5, 20\} \nsubseteq [5, 17) \cup \{18\} = T(X)$ and $B(X) = \{5, 13\} \nsubseteq [5, 11) = S(X)$. Also observe that $S(X)$ and $T(X)$ are not closed subsets of X. All requirements of Theorem 1 are satisfied for some fixed $0 < k < 1$ and $a \geq 0$. Moreover, 5 is the unique common fixed point of the pairs (A, S) and (B, T), as well a point of coincidence. We point out that the involved mappings are discontinuous at their unique common fixed point 5.

THEOREM 2. *Let A, B, S, and T be self-mappings of a Menger space $(X, \mathcal{F}, *)$ with a continuous t-norm $* = \min$, satisfying the conditions* (1)-(5) *of Lemma* 2. *Then each of the pairs (A, S) and (B, T) has a coincidence point. Moreover, A, B, S, and T have a unique common fixed point, provided both pairs (A, S) and (B, T) are weakly compatible.*

Proof. In view of Lemma 2, both pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property. Therefore there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

for some $z \in S(X) \cap T(X)$. The rest of the proof is similar to the proof of Theorem 1, hence we omit it. □

EXAMPLE 2. In the setting of Example 1, replace the self-mappings A, B, S, and T by the following ones

$$A(x) = \begin{cases} 5, & \text{if } x \in \{5\} \cup (9, 21) \\ 15, & \text{if } x \in (5, 9], \end{cases}$$

$$B(x) = \begin{cases} 5, & \text{if } x \in \{5\} \cup (9, 21) \\ 10, & \text{if } x \in (5, 9], \end{cases}$$

$$S(x) = \begin{cases} 5, & \text{if } x = 5 \\ 11, & \text{if } x \in (5, 9] \\ \frac{x+1}{2}, & \text{if } x \in (9, 21), \end{cases}$$

$$T(x) = \begin{cases} 5, & \text{if } x = 5 \\ 17, & \text{if } x \in (5, 9] \\ x - 4, & \text{if } x \in (9, 21). \end{cases}$$

The pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property. Note that $A(X) = \{5, 15\} \subset [5, 17] = T(X)$ and $B(X) = \{5, 10\} \subset [5, 11] = S(X)$. The pairs (A, S) and (B, T) commute at 5 which is also their common coincidence point. Thus all hypotheses of Theorem 2 are satisfied for some fixed $0 < k < 1$ and $a \geq 0$. Moreover, 5 is a unique common fixed point of the mappings A, B, S, and T. We point out that Theorem 1 is not applicable in this case, since $S(X)$ and $T(X)$ are closed subsets of X.

By taking $A = B$ and $S = T$ in Theorem 1, we get the following immediate consequence of it.

COROLLARY 1. *Let A and S be self-mappings of a Menger space $(X, \mathcal{F}, *)$ with a continuous t-norm $* = \min$. Suppose that*

(1) *the pair (A, S) satisfies the (CLR_S) property,*
(2) *there exists a constant $k \in (0, 1)$ such that the inequality*

$$[1 + aF_{Sx,Sy}(kt)] * F_{Ax,Ay}(kt) \geq a \begin{Bmatrix} F_{Ax,Sx}(kt) * F_{Ay,Sy}(kt) * \\ F_{Ax,Sy}(2kt) * F_{Ay,Sx}(2kt) \end{Bmatrix} + \begin{Bmatrix} F_{Sx,Sy}(t) * F_{Ax,Sx}(t) * F_{Ay,Sy}(t) * \\ F_{Ax,Sy}(2t) * F_{Ay,Sx}(2t) \end{Bmatrix}$$

holds for all $t > 0$, $x, y \in X$ and $a \geq 0$.

Then the pair (A, S) has a coincidence point. Moreover, A and S have a unique common fixed point, provided the pair (A, S) is weakly compatible.

Now we utilize the notion of pairwise commuting pairs due to Imdad et al. [10], and extend Theorem 1 to six self-mappings in Menger spaces.

THEOREM 3. *Let A, B, R, S, H, and T be self-mappings of a Menger space $(X, \mathcal{F}, *)$ with a continuous t-norm $* = \min$. Suppose that*

(1) *the pairs (A, SR) and (B, TH) satisfy the $(CLR_{(SR)(TH)})$ property,*

(2) *there exists a constant* $k \in (0,1)$ *such that*

$$[1+aF_{SRx,THy}(kt)]*F_{Ax,By}(kt) \geq a\left\{\begin{array}{c} F_{Ax,SRx}(kt)*F_{By,THy}(kt)* \\ F_{Ax,THy}(2kt)*F_{By,SRx}(2kt) \end{array}\right\} + \left\{\begin{array}{c} F_{SRx,THy}(t)*F_{Ax,SRx}(t)*F_{By,THy}(t)* \\ F_{Ax,THy}(2t)*F_{By,SRx}(2t) \end{array}\right\}$$

holds for all $t > 0$, $x, y \in X$ *and* $a \geq 0$.

Then each of the pairs (A, SR) *and* (B, TH) *has a point of coincidence. Moreover,* $A, B, R, S, H,$ *and* T *have a unique common fixed point, provided the pairs* (A, S) *and* (B, T) *commute pairwise (that is,* $AS = SA$, $AR = RA$, $SR = RS$, $BT = TB$, $BH = HB$, *and* $TH = HT$*).*

Proof. Since the pairs (A, SR) and (B, TH) commute pairwise, both pairs are weakly compatible. By Theorem 1, the maps A, B, SR, and TH have a unique common fixed point z in X. We are going to prove that z is a unique common fixed point of the self-mappings A, R and S. Putting $x = Rz$ and $y = z$ in inequality (2), we have

$$\begin{aligned}[1+aF_{SR(Rz),THz}(kt)]*F_{A(Rz),Bz}(kt) &\geq a\left\{\begin{array}{c} F_{A(Rz),SR(Rz)}(kt)*F_{Bz,THz}(kt)* \\ F_{A(Rz),THz}(2kt)*F_{Bz,SR(Rz)}(2kt) \end{array}\right\} \\ &+ \left\{\begin{array}{c} F_{SR(Rz),THz}(t)*F_{A(Rz),SR(Rz)}(t)*F_{Bz,THz}(t)* \\ F_{A(Rz),THz}(2t)*F_{Bz,SR(Rz)}(2t) \end{array}\right\},\end{aligned}$$

so

$$[1+aF_{Rz,z}(kt)]*F_{Rz,z}(kt) \geq a\left\{\begin{array}{c} F_{Rz,Rz}(kt)*F_{z,z}(kt)* \\ F_{Rz,z}(2kt)*F_{z,Rz}(2kt) \end{array}\right\} + \left\{\begin{array}{c} F_{Rz,z}(t)*F_{Rz,Rz}(t)*F_{z,z}(t)* \\ F_{Rz,z}(2t)*F_{z,Rz}(2t) \end{array}\right\}.$$

The inequalities $F_{Rz,z}(kt) \leq F_{Rz,z}(2kt)$ and $F_{Rz,z}(t) \leq F_{Rz,z}(2t)$ imply

$$[1+aF_{Rz,z}(kt)]*F_{Rz,z}(kt) \geq aF_{Rz,z}(kt) + F_{Rz,z}(t).$$

Since $a \geq 0$, we get $[1+aF_{Rz,z}(kt)]*F_{Rz,z}(kt) \leq F_{Rz,z}(kt) + aF_{Rz,z}(kt)$. This implies

$$F_{Rz,z}(kt) + aF_{Rz,z}(kt) \geq aF_{Rz,z}(kt) + F_{Rz,z}(t), \quad F_{Rz,z}(kt) \geq F_{Rz,z}(t).$$

Thus $z = Rz$, by Lemma 2. Hence $S(z) = S(Rz) = z$. Therefore we have $z = Az = Sz = Rz$. Finally we show that z is a fixed point of B, T and H. For this we use inequality (3) with $x = z$, $y = Hz$. Hence we get

$$\begin{aligned}[1+aF_{SRz,TH(Hz)}(kt)]*F_{Az,B(Hz)}(kt) &\geq a\left\{\begin{array}{c} F_{Az,SRz}(kt)*F_{B(Hz),TH(Hz)}(kt)* \\ F_{Az,TH(Hz)}(2kt)*F_{B(Hz),SRz}(2kt) \end{array}\right\} \\ &+ \left\{\begin{array}{c} F_{SRz,TH(Hz)}(t)*F_{Az,SRz}(t)*F_{B(Hz),TH(Hz)}(t)* \\ F_{Az,TH(Hz)}(2t)*F_{B(Hz),SRz}(2t) \end{array}\right\},\end{aligned}$$

so $$[1+aF_{z,Hz}(kt)]*F_{z,Hz}(kt)\geq a\left\{\begin{array}{c}F_{z,z}(kt)*F_{Hz,Hz}(kt)*\\ F_{z,Hz}(2kt)*F_{Hz,z}(2kt)\end{array}\right\} +\left\{\begin{array}{c}F_{z,Hz}(t)*F_{z,z}(t)*F_{Hz,Hz}(t)*\\ F_{z,Hz}(2t)*F_{Hz,z}(2t)\end{array}\right\}.$$

The inequalities $F_{z,Hz}(kt)\leq F_{z,Hz}(2kt)$ and $F_{z,Hz}(t)\leq F_{z,Hz}(2t)$ yield

$$[1+aF_{z,Hz}(kt)]*F_{z,Hz}(kt)\geq aF_{z,Hz}(kt)+F_{z,Hz}(t).$$

Since $a\geq 0$, we get $[1+aF_{z,Hz}(kt)]*F_{z,Hz}(kt)\leq F_{z,Hz}(kt)+aF_{z,Hz}(kt)$. This implies

$$F_{z,Hz}(kt)+aF_{z,Hz}(kt)\geq aF_{Hz,z}(kt)+F_{Hz,z}(t),\quad F_{z,Hz}(kt)\geq F_{z,Hz}(t).$$

Thus $z=Hz$, by Lemma 2. Hence $T(z)=T(Hz)=z$. Therefore z is the unique common fixed point of the self-mappings A,B,R,S,H, and T. □

We can extend now Theorem 1 to four finite families of self-mappings.

Corollary 2. *Let $\{A_1,A_2,\ldots A_m\}$, $\{B_1,B_2,\ldots B_p\}$, $\{S_1,S_2,\ldots S_n\}$, and $\{T_1,T_2,\ldots T_q\}$ be four finite families of self-mappings of a Menger space $(X,\mathcal{F},*)$ with a continuous t-norm $*=\min$ such that the maps $A:=A_1A_2\ldots A_m$, $B:=B_1B_2\ldots B_p$, $S:=S_1S_2\ldots S_n$, and $T:=T_1T_2\ldots T_q$ satisfy inequality (5) of Lemma 2. If the pairs (A,S) and (B,T) share the (CLR_{ST}) property, then (A,S) and (B,T) have a coincidence point each.*

Moreover, if the family $\{A_i\}_{i=1}^m$ commutes pairwise with the family $\{S_i\}_{j=1}^n$, whereas the family $\{B_r\}_{r=1}^p$ commutes pairwise with the family $\{T_w\}_{w=1}^q$, then A_i, B_j, S_r, and T_w have a common fixed point in X, for all $i\in\{1,2,\ldots,m\}$, $j\in\{1,2,\ldots,n\}$, $r\in\{1,2,\ldots,p\}$, and $w\in\{1,2,\ldots,q\}$.

By setting $A_1=A_2=\ldots=A_m=A$, $B_1=B_2=\ldots=B_p=B$, $S_1=S_2=\ldots=S_n=S$, and $T_1=T_2=\ldots=T_q=T$ in Corollary 2, we deduce the following result.

Corollary 3. *Let A,B,S, and T be self-mappings of a Menger space $(X,\mathcal{F},*)$ with a continuous t-norm $*=\min$. Suppose that the pairs (A^m,S^p) and (B^n,T^q) satisfy the $(CLR_{S^pT^q})$ property, where m,n,p,q are fixed positive integers. Then there exists a constant $k\in(0,1)$ such that*

$$[1+aF_{S^nx,T^qy}(kt)]*F_{A^mx,B^py}(kt)\geq a\left\{\begin{array}{c}F_{A^mx,S^nx}(kt)*F_{B^py,T^qy}(kt)*\\ F_{A^mx,T^qy}(2kt)*F_{B^py,S^nx}(2kt)\end{array}\right\} +\left\{\begin{array}{c}F_{S^nx,T^qy}(t)*F_{A^mx,S^nx}(t)*F_{B^py,T^qy}(t)*\\ F_{A^mx,T^qy}(2t)*F_{B^py,S^nx}(2t)\end{array}\right\},$$

for all $t>0$, $x,y\in X$ and $a\geq 0$. Then A,B,S, and T have a unique common fixed point, provided both pairs (A^m,S^p) and (B^n,T^q) commute pairwise.

Remark 2. By taking the constant $a=0$ in our results, we obtain several corollaries which improve the result of Ali et al. [2, Theorem 2.1].

REFERENCES

[1] Aamri, M. and Moutawakil, D.El, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl., **270** (2002), 181–188.

[2] Ali, J., Imdad, M. and Bahuguna, D., *Common fixed point theorems in Menger spaces with common property (E.A)*, Comput. Math. Appl., **60** (2010), 3152–3159.

[3] Ali, J., Imdad, M., Miheţ, D. and Tanveer, M., *Common fixed points of strict contractions in Menger spaces*, Acta Math. Hungar., **132** (2011), 367–386.

[4] Chauhan, S., Beg, I. and Pant, B.D., *Impact of* (CLR_{ST}) *property and existence of fixed points using implicit relations*, J. Indian Math. Soc., **81** (2014), 45–59.

[5] Chauhan, S., Dalal, S., Sintunavarat, W. and Vujaković, J., *Common property (E.A) and existence of fixed points in Menger spaces*, J. Inequal. Appl., 2014, Article No. 56.

[6] Chauhan, S. and Pant, B.D., *Common fixed point theorem for weakly compatible mappings in Menger space*, J. Adv. Res. Pure Math., **3** (2011), 107–119.

[7] Chauhan, S. and Pant, B.D., *Existence of fixed points in Menger spaces using common property (E.A)*, Indian J. Math., **54** (2012), 321–342.

[8] Dalal, S., Chauhan, S. and Vujaković, J., *Employing common property (E.A) on new contraction condition in Menger spaces*, Far East J. Math. Sci. (FJMS), **83** (2013), 145–165.

[9] Fang, J.-x and Gao, Y., *Common fixed point theorems under strict contractive conditions in Menger spaces*, Nonlinear Anal., **70** (2009), 184–193.

[10] Imdad, M., Ali, J. and Tanveer, M., *Coincidence and common fixed point theorems for nonlinear contractions in Menger PM spaces*, Chaos Solitons Fractals, **42** (2009), 3121–3129.

[11] Imdad, M., Pant, B.D. and Chauhan, S., *Fixed point theorems in Menger spaces using the* (CLR_{ST}) *property and applications*, Journal of Nonlinear Analysis and Optimization, **3** (2012), 225–237.

[12] Jungck, G., *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci., **9** (1986), 771–779.

[13] Jungck, G. and Rhoades, B.E., *Fixed points for set valued functions without continuity*, Indian J. Pure Appl. Math., **29** (1998), 227–238.

[14] Kohli, J.K. and Vashistha, S., *Common fixed point theorems in probabilistic metric spaces*, Acta Math. Hungar., **115** (2007), 37–47.

[15] Kubiaczyk, I. and Sharma, S., *Some common fixed point theorems in Menger space under strict contractive conditions*, Southeast Asian Bull. Math., **32** (2008), 117–124.

[16] Kumar, S., Chauhan, S. and Pant, B.D., *Common fixed point theorem for noncompatible maps in probabilistic metric space*, Surv. Math. Appl., **8** (2013), 51–57.

[17] Kumar, S. and Pant, B.D., *Common fixed point theorems in probabilistic metric spaces using implicit relation and property (E.A)*, Bull. Allahabad Math. Soc., **25** (2010), 223–235.

[18] Kutukcu, S. and Sharma, S., *Compatible maps and common fixed points in Menger probabilistic metric spaces*, Commun. Korean Math. Soc., **24** (2009), 17–27.

[19] Liu, Y., Wu, J. and Li, Z., *Common fixed points of single-valued and multi-valued maps*, Int. J. Math. Math. Sci., **19** (2005), 3045–3055.

[20] Menger, K., *Statistical metrics*, Proc. Natl. Acad. Sci. USA, **28** (1942), 535–537.

[21] Mishra, S.N., *Common fixed points of compatible mappings in PM-spaces*, Sci. Math. Jpn., **36** (1991), 283–289.

[22] O'Regan, D. and Saadati, R., *Nonlinear contraction theorems in probabilistic spaces*, Appl. Math. Comput., **195** (2008), 86-93.

[23] Pant, R.P., *Common fixed point theorems for contractive maps*, J. Math. Anal. Appl., **226** (1998), 251–258.

[24] PANT, B.D. and CHAUHAN, S., *A contraction theorem in Menger space*, Tamkang J. Math., **42** (2011), 59–68.

[25] PANT, B.D. and CHAUHAN, S., *Common fixed point theorems for two pairs of weakly compatible mappings in Menger spaces and fuzzy metric spaces*, Sci. Stud. Res. Ser. Math. Inform., **21** (2011), 81–96.

[26] PANT, B.D. and CHAUHAN, S., *A common fixed point theorem in Menger space using implicit relation*, Surv. Math. Appl., **8** (2013), 1–10.

[27] PATHAK, H.K., LÓPEZ, R.R. and VERMA, R.K., *A common fixed point theorem using implicit relation and property (E.A) in metric spaces*, Filomat, **21** (2007), 211–234.

[28] PATHAK, H.K. and VERMA, R.K., *Common fixed point theorems for weakly compatible mappings on Menger spaces and application*, Int. J. Math. Anal. (Ruse), **3** (2009), 1199–1206.

[29] RAZANI, A. and SHIRDARYAZDI, M., *A common fixed point theorem of compatible maps in Menger space*, Chaos Solitons Fractals, **32** (2007), 26–34.

[30] SAADATI, R., O'REGAN, D., VAEZPOUR, S.M. and KIM, J.K., *Generalized distance and common fixed point theorems in Menger probabilistic metric spaces*, Bull. Iranian Math. Soc., **35** (2009), 97–117.

[31] SCHWEIZER, B. and SKLAR, A., *Statistical metric spaces*, Pacific J. Math., **10** (1960), 313–334.

[32] SEHGAL, V.M., *Some fixed point theorems in functional analysis and probability*, Ph.D. Thesis, Wayne State Univ., 1966.

[33] SEHGAL, V.M. and BHARUCHA-REID, A.T., *Fixed points of contraction mappings on probabilistic metric spaces*, Mathematical Systems Theory, **6** (1972), 97–102.

[34] SINGH, S.L., PANT, B.D. and CHAUHAN, S., *Fixed point theorems in Non-Archimedean Menger PM-spaces*, Journal of Nonlinear Analysis and Optimization, **3** (2012), 153–160.

[35] SINGH, B. and JAIN, S., *A fixed point theorem in Menger space through weak compatibility*, J. Math. Anal. Appl., **301** (2005), 439–448.

[36] SINTUNAVARAT, W. and KUMAM, P., *Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces*, J. Appl. Math., 2011, Article ID 637958, 14 pages.

[37] SINTUNAVARAT, W. and KUMAM, P., *Common fixed points for R-weakly commuting in fuzzy metric spaces*, Ann. Univ. Ferrara Sez. VII Sci. Mat., **58** (2012), 389–406.

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Government Degree College
Champawat-262523, Uttarakhand, India
E-mail: badridatt.pant@gmail.com

Near Nehru Training Centre
H. No. 274, Nai Basti B-14
Bijnor-246701, Uttar Pradesh, India
E-mail: sun.gkv@gmail.com

Aligarh Muslim University, Aligarh
Z. H. College of Engineering and Technology
Department of Applied Mathematics
Uttar Pradesh 202002, India