MATHEMATICA, Tome 55 (78), N° 2, 2013, pp. 186–194

FIELD EXTENSIONS AND CLIFFORD THEORY

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Abstract. We study Clifford theory in connection with the action of the Galois group of a field extension in the context of group graded algebras.

MSC 2010. 16W50, 16D90, 16W22.

Key words. Group graded algebras, G-algebras.

1. INTRODUCTION

Let G be a finite group, let K/F be an algebraic field extension, and let $R = \bigoplus_{g\in G} R_g$ be a finite dimensional strongly G-graded F-algebra. A simple R_1-module, as well as a simple $K\otimes_F R_1$-module, define a "Clifford theory". The main idea is that the R-module induced from a simple R_1-module generates an abelian subcategory of the category of R-module which is equivalent to the category of modules over its endomorphism algebra. In this paper we investigate the relationships between these theories. One of the main results below says that a G-graded derived equivalence over F preserves the Clifford theory defined by corresponding simple modules, and also preserves Galois actions and Schur indices.

In Section 2 we present Dade's treatment [1], [2] of Clifford theory for strongy G-graded F-algebras, while in Section 3 we discuss the scalar extension from F to K and the action of the Galois group $\mathrm{Gal}(K/F)$ on $K\otimes_F R$-modules. Section 4 is devoted to the relationship between the Clifford theories over F and over K.

An important motivation for this paper is Turull's approach to Clifford theory and Schur indices via G-algebras. He considers the Clifford theory defined by an R-module lying over a simple (or semisimple) R_1-module and introduces the notion of endoisomorphism to formalize the idea of two modules determining the same Clifford theory. We show in Section 5 that a G-graded derived equivalence over F induces an endoisomorphism between two corresponding simple R-modules. This is related to the results of [7].

In what follows, groups are finite, and algebras and modules are finite dimensional. We consider only algebras over fields, but this is enough for our purposes, as we essetially deal with simple modules. Our notations are standard. If H is a subgroup of the group G, we denote by $[G/H]$ a set of representatives for the left cosets of H in G. The reader is referred to [4] for general

This work was supported by a grant of the Ministry of National Education, CNCS-UEFISCDI, project number PN-II-ID-PCE-2012-4-0100.

results on field extensions and Schur indices, and to [5] for results on G-graded algebras.

2. CLIFFORD THEORY FOR STRONGLY G-GRADED ALGEBRAS

The results presented in this section are Dade's version [1], [2] of the Clifford correspondence for group graded algebras.

2.1. As in the introduction, let G be a finite group, F a field, and let $R = \bigoplus_{g\in G} R_g$ be a finite dimensional strongly G-graded F-algebra.

The group G acts on the set of isomorphism classes of simple R_1-modules. If V be a simple R_1-module, we denote ${}^gV = R_g \otimes_{R_1} V$, and let

$$G_V := \{g \in G \mid R_g \otimes_{R_1} V \simeq V \text{ as } R_1\text{-modules}\}$$

be the stabilizer in G of V.

Theorem 2.2. *If M is a simple R-module, then there exists a simple R_1-module V such that V is a direct summand in M. More precisely, ${}_{R_1}M$ is a semisimple R_1-module and has the structure*

$${}_{R_1}M \simeq n \bigoplus_{g\in[G/G_V]} {}^gV,$$

for some positive integer n.

2.3. Let M and V be as above. Because we have a monomorphism $V \overset{\iota}{\hookrightarrow} {}_{R_1}M$, there exists the surjective R-homomorphism

$$R \otimes_{R_1} V \to M, \qquad r \otimes v \mapsto r\iota(v).$$

Definition 2.4. We denote by $(R|V)$-mod the full subcategory of R-mod consiting of R-modules M for which there exists an R-epimorphism

$$(R \otimes_{R_1} V)^{(I)} \to M \to 0$$

for some set I. Then $(R|V)$-mod is called the category of *R-modules above V*.

Theorem 2.5. *The category $(R|V)$-mod is abelian, and coincides with the full subcategory of R-mod consisting of R-modules M that viewed as R_1-modules have the structure as in Theorem (2.2).*

2.6. If we denote

$$E := \mathrm{End}_R(R \otimes_{R_1} V)^{\mathrm{op}},$$

then E is a G-graded algebra, and $R \otimes_{R_1} V$ is a G-graded (R, E)-bimodule. Moreover, we have that $E_g = 0$ for $g \in G\backslash G_V$ (because in this case $V \not\simeq {}^gV$), hence $E = E_{G_V}$ may be regarded as a strongly G_V-graded algebra.

Theorem 2.7. *We have the commutative diagram of equivalences of categories*

$$\begin{array}{ccc}
(R|V)\text{-mod} & \underset{(R\otimes_{R_1}V)\otimes_E -}{\overset{\mathrm{Hom}_R(R\otimes_{R_1}V,-)}{\rightleftarrows}} & E\text{-mod} \\
{\scriptstyle R\otimes_{R_{G_V}}-}\uparrow\ \downarrow{\scriptstyle (-)_{G_V}} & & \Vert \\
(R_{G_V}|V)\text{-mod} & \underset{(R_{G_V}\otimes_{R_1}V)\otimes_{E_{G_V}} -}{\overset{\mathrm{Hom}_{G_V}(R_{G_V}\otimes_{R_1}V,-)}{\rightleftarrows}} & E_{G_V}\text{-mod.}
\end{array}$$

Remark 2.8. Recall that the inverse of the equivalence $R\otimes_{R_{G_V}}-$ is defined as follows. For any object M in $(R|V)$-mod there exists a unique R_{G_V}-submodule U of M such that ${}_{R_1}U \simeq nV$, and then $M_{G_V} = U$ (see for instance [5, Theorem 2.3.10]. In other words, any object M of $(R|V)$-mod is naturally (G/G_V)-graded, so we may write

$$M = \bigoplus_{x\in G/G_V} M_x,$$

and then the inverse of $R\otimes_{R_{G_V}}-$ is $(-)_{G_V}$.

3. FIELD EXTENSIONS AND GALOIS ACTION

Let A be a finite dimensional F-algebra. We suppose that A is defined over a perfect subfield of F, that is, $A \simeq F\otimes_{F_0} A_0$, where F_0 is a perfect field and A_0 is an F_0-algebra. This is obviously the case when A is a group algebra.

3.1. Let $F \leq K$ be an algebraic normal field extension, and consider the Galois group $\hat{G} := \mathrm{Gal}(K/F)$. Then $\hat{G}$ acts on the set of isomorphism classes of simple $K\otimes_F A$-modules, and if W is a simple $K\otimes_F A$-module, denote

$$\hat{G}_W := \{\sigma\in\hat{G} \mid {}^{\sigma}W \simeq W \text{ as } K\otimes_F A\text{-modules}\}$$

the stabilizer of W, and let

$$\hat{W} := \bigoplus_{\sigma\in[\hat{G}/\hat{G}_W]} {}^{\sigma}W$$

be the sum of distinct $\hat{G}$-conjugates of W.

In this case we have results similar to Clifford theory, due to Schur and Noether (see [4, Theorem 8.1.11]).

Theorem 3.2. *With the above notations, the following statements hold.*

1) *If V is a simple A-module, then $K\otimes_F V$ is a semisimple $K\otimes_F A$-module.*
2) *Let W be a simple $K\otimes_F A$-module that is a direct summand of $K\otimes_F V$, where V is a simple A-module. Then*

$$K\otimes_F V \simeq m \bigoplus_{\sigma\in[\hat{G}/\hat{G}_W]} {}^{\sigma}W$$

for some positive integer m.

3) *For any simple $K \otimes_F A$-module W, there exists a simple A-module V, unique up to isomorphism, such that W is a summand of $K \otimes_F V$.*

4. GALOIS ACTION AND CLIFFORD CORRESPONDENCE

Let $R = \bigoplus_{g\in G} R_g$ be a finite dimensional strongly G-graded F-algebra, and let $F \leq K$ be an algebraic normal field extension. Denote $\hat{G} := \mathrm{Gal}(K/F)$ and $KR := K \otimes_F R$, which is a strongly G-graded K-algebra. We suppose that R_1 (and hence R) is defined over a perfect subfield of F.

4.1. Let W be a simple KR_1-module and V a simple R_1-module. Denote also

$$\hat{E} := \mathrm{End}_{KR}(KR \otimes_{KR_1} \hat{W})^{\mathrm{op}}.$$

One can see that $\hat{E} = \hat{E}_{G_{\hat{W}}}$, so $\hat{E}$ is strongly $G_{\hat{W}}$-graded.

Notation 4.2. We consider the following stabilizers, also called *inertia groups*:

- $I_G(V) := G_V = \{g \in G \mid R_g \otimes_{R_1} V \simeq V \text{ as } R_1\text{-modules}\}$,
- $I_G(W) := G_W = \{g \in G \mid KR_g \otimes_{KR_1} W \simeq W \text{ as } KR_1\text{-modules}\}$,
- $I_{G,F}(W) := \{g \in G \mid$ there exists $\sigma \in \hat{G}$ such that $KR_g \otimes_{KR_1} W \simeq {}^{\sigma}W$ as KR_1-modules$\}$,
- $I_G(K\otimes_F V) := \{g \in G \mid KR_g \otimes_{KR_1} K\otimes_F V \simeq K\otimes_F V \text{ as } KR_1\text{-modules}\}$.

We will also denote $T := I_{G,F}(W)$. We obviously have that

$$I_G(W) \leq I_{G,F}(W) = T \leq G.$$

Notation 4.3. Apart from the subcategory $(R|V)$-mod introduced in Section 2, we consider the following full subcategories:

- $(KR|W)$-mod, consisting of KR-modules M such that there exists an epimorphism
$$(KR \otimes_{KR_1} W)^{(I)} \to M \to 0$$
of KR-modules, for some set I,
- $(KR|W, F)$-mod consisting of KR-modules M such that ${}_{KR_1}M$ is isomorphic to a direct sum of $G \times \hat{G}$-conjugates of W,
- $(KR|K\otimes_F V)$-mod consisting of KR-modules M such that there exists an epimorphism
$$(KR \otimes_{KR_1} K \otimes_F V)^{(I)} \to M \to 0$$
of KR-modules, for some set I.

Theorem 4.4. *With the above notation, assume that W is a direct summand of $K \otimes_F V$. Then the following statements hold.*

1) $I_{G,F}(W) = I_G(K \otimes_F V)$.

2) *The categories* $(KR|W, F)$-mod *and* $(KR|K \otimes_F V)$-mod *coincide.*

3) *We have the following commutative diagram of equivalences of categories:*

$$\begin{array}{ccc}
(KR|W,F)\text{-mod} & \underset{(KR\otimes_{KR_1}\hat{W})\otimes_{\hat{E}}-}{\overset{\mathrm{Hom}_{KR}(KR\otimes_{KR_1}\hat{W},-)}{\rightleftarrows}} & \hat{E}\text{-mod} \\
{\scriptstyle KR\otimes_{KR_{G_{\hat{W}}}}-}\uparrow\downarrow\simeq & & \Vert \\
(KR_{G_{\hat{W}}}|W,F)\text{-mod} & \underset{(KR_{G_{\hat{W}}}\otimes_{KR_1}\hat{W})\otimes_{\hat{E}}-}{\overset{\mathrm{Hom}_{KR_{G_{\hat{W}}}}(KR_{G_{\hat{W}}}\otimes_{KR_1}\hat{W},-)}{\rightleftarrows}} & \hat{E}_{G_{\hat{W}}}\text{-mod.}
\end{array}$$

Proof. 1) The actions of G and $\hat{G}$ on KR_1-modules commute, so, for any $g \in G$, we have the isomorphisms

$$^g(K\otimes_F V) \simeq {}^g(m \bigoplus_{\sigma\in[\hat{G}/\hat{G}_W]} {}^\sigma W) \simeq m \bigoplus_{\sigma\in[\hat{G}/\hat{G}_W]} {}^\sigma({}^g W)$$

of KR_1-modules, and therefore the statement follows now immediately from the definitions. Note also that the group $I_G(K\otimes_F V)$ equals the stabilizer $G_{\hat{W}}$ of $\hat{W}$ in G.

2) We have that

$$(KR|\hat{W})\text{-mod} = (KR|K\otimes_F V)\text{-mod},$$

since, by Theorem 3.2, $K\otimes_F V \simeq m\hat{W}$ as KR_1-modules, and hence the G-graded KR-modules $KR\otimes_{KR_1}\hat{W}$ and $KR\otimes_{KR_1} V$ generate the same full subcategory of KR-mod. Now the equality

$$(KR|\hat{W})\text{-mod} = (KR|W,F)\text{-mod}$$

is a consequence of Theorem 2.5.

3) This follows by statements 1) and 2), and by Theorem 2.7, with KR instead of R and $\hat{W}$ instead of V. $\square$

Remark 4.5. It is easy to see that we have the isomorphism

$$\mathrm{End}_{KR_1}(K\otimes_F V) \simeq M_m(\mathrm{End}_{KR_1}(\hat{W}))$$

of K-algebras, and moreover,

$$\mathrm{End}_{KR}(K\otimes_F(R\otimes_{R_1} V)) \simeq M_m(\mathrm{End}_{KR}(KR\otimes_{KR_1}\hat{W}))$$

as $G_{\hat{W}}$-graded K-algebras.

4.6. We next discuss the relationship between the inertia groups $I_G(V)$ and $T = I_G(K\otimes_F V)$, and between the subcategories $(R|V)$-mod and $(KR|K\otimes_F V)$-mod. We denote

$$KE := \mathrm{End}_{KR}(KR\otimes_{KR_1}(K\otimes_F V))^{\mathrm{op}},$$

which is justified, since we have the isomorphisms

$$\begin{aligned} KE &\simeq \mathrm{End}_{KR}(K\otimes_F(R\otimes_{R_1} V))^{\mathrm{op}} \\ &\simeq K\otimes_F \mathrm{End}_R(R\otimes_{R_1} V)^{\mathrm{op}} = K\otimes_F E. \end{aligned}$$

As before, $KE = KE_T$ may be regarded as a strongly T-graded K-algebra.

Lemma 4.7. 1) $I_G(V) \leq I_G(K \otimes_F V) = T$.

2) *The extension of scalars* $K \otimes_F - : R$-mod $\to KR$-mod *induces by restriction a functor*

$$K \otimes_F - : (R|V)\text{-mod} \to (KR|K \otimes_F V)\text{-mod}.$$

Proof. 1) If $g \in I_G(V)$ then $R_g \otimes_{R_1} V \simeq V$ as R_1-modules, and

$$KR_g \otimes_{KR_1} (K \otimes_F V) \simeq K \otimes_F (R_g \otimes_{R_1} V) \simeq K \otimes_F V$$

as KR_1-modules, hence $g \in I_G(K \otimes_F V)$.

2) If there is an epimorphism

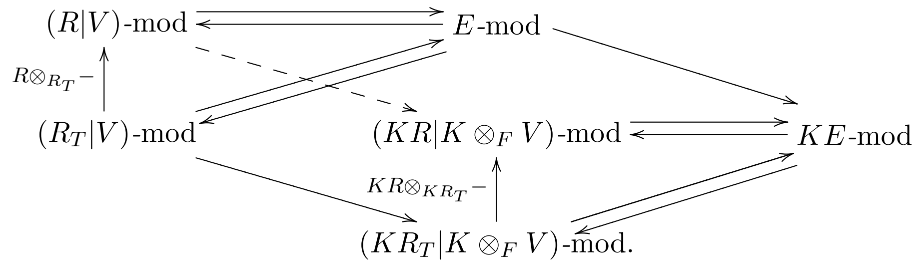
$$(R \otimes_{R_1} V)^{(I)} \to M \to 0$$

of R-modules, then we also have an epimorphism

$$(KR \otimes_{KR_1} K \otimes_F V)^{(I)} \to K \otimes_F M \to 0$$

of KR-modules. □

Corollary 4.8. *We have the following commutative diagram of categories and functors:*



Proof. The commutativity of the first triangle follows from Theorem 2.7, by noting that, since G_V is a subgroup of G, the functor

$$R \otimes_{R_T} - : (R_T|V)\text{-mod} \to (R|V)\text{-mod}.$$

The commutativity of the second triangle follows by Theorem 4.4 and Remark 4.5. For the remaining three diagrams, it is known that we have the following isomorphisms

$$K \otimes_F \mathrm{Hom}_R(R \otimes_{R_1} V, -) \simeq \mathrm{Hom}_{KR_1}(KR \otimes_{KR_1} V, K \otimes_F -)$$

and

$$K \otimes_F (R \otimes_{R_T} -) \simeq KR \otimes_{KR_T} -$$

of functors. □

Remark 4.9. Finally, we describe the relationship between the Clifford theory of $K \otimes_F V$ (that is, of $\hat{W}$) and the Clifford theory of W.

a) If $\sigma \in \hat{G}$, then $I_G(W) = I_G({}^\sigma W)$, because the actions of G and $\hat{G}$ on KR_1-modules commute. Indeed, if W is a KR_1-module, we have

$$KR_g \otimes_{KR_1} {}^\sigma W \simeq {}^\sigma(KR_g \otimes_{KR_1} W)$$

for all $g \in G$ and $\sigma \in \hat{G}$, since KR is defined over F.

b) The element σ induces an F-linear equivalence of categories

$$(KR|W)\text{-mod} \xrightarrow{\sigma(-)} (KR|{}^\sigma W)\text{-mod}.$$

5. QUASIHOMOGENEOUS R-MODULES

Turull [8], [9] considers the "Clifford theory determined by an R-module" instead of an R_1-module. We discuss here the connections with the point of view of the preceding sections. Let W be a simple KR_1-module as before.

Definition 5.1. Let M be an R-module. We say M is *W-quasihomogeneous* if $K \otimes_F M \in (KR|W)$-mod.

The first question is whether the Clifford theory depends on the choice of a W-quasihomogeneous module. The next result says that it does not depend.

Theorem 5.2. *Assume that M and M' are W-quasihomogeneous R-modules. Then there exists a G-equivariant Morita equivalence between the G-algebras* $\mathrm{End}_{R_1}(M)$ *and* $\mathrm{End}_{R_1}(M')$.

Proof. There is a unique simple R_1-module V such that W is a summand of $K \otimes_F V$. The restriction $\mathrm{Res}^R_{R_1} -$ and the scalar extension $K \otimes_F -$ commute, and, since W is also a summand of $K \otimes_F M$, it follows that V is an R_1-submodule of M. Moreover, any simple R_1-submodule of M must be a G-conjugate of V. This implies that there is an R-epimorphism

$$(R \otimes_{R_1} V)^{(I)} \to M \to 0$$

for some set I, that is, M belongs to the subcategory $(R|V)$-mod. But this subcategory coincides with $(R|_{R_1}M)$-mod, which, by Theorem 2.7, is equivalent to $\mathrm{End}_R(R \otimes_{R_1} M)^{\mathrm{op}}$. Clearly, this equivalence preserves grading of modules.

The same is true for M' instead of M, hence there is a G-graded Morita equivalence between $\mathrm{End}_R(R \otimes_{R_1} M)$ and $\mathrm{End}_R(R \otimes_{R_1} M')$.

Since M is an R-module, we have that $\mathrm{End}_{R_1}(M)$ is a G-algebra, and we have the isomorphism

$$\mathrm{End}_{R_1}(M) * G \simeq \mathrm{End}_R(R \otimes_{R_1} M)$$

of G-graded algebras. By [6, Theorem 2.13], a G-graded Morita equivalence between $\mathrm{End}_R(R \otimes_{R_1} M)$ and $\mathrm{End}_R(R \otimes_{R_1} M')$ is the same as a G-equivariant Morita equivalence between $\mathrm{End}_{R_1}(M)$ and $\mathrm{End}_{R_1}(M)$. $\square$

5.3. We next consider the following context. Let R and R' be G-graded F-algebras. Let M be a W-quasihomogeneous R-module and M' be a W'-quasihomogeneous R'-module, where W is a KR_1-module and W' a KR'_1-module. Then the question is: when is the Clifford theory determined by M equivalent to the Clifford theory determined by M'?

COROLLARY 5.4. *If there exists an isomorphism*

$$\varepsilon : \mathrm{End}_{R_1}(M) \to \mathrm{End}_{R'_1}(M'),$$

of G-algebras over F, then there exist an equivalence of categories

$$(KR|W,F)\text{-mod} \simeq (KR'|W',F)\text{-mod}.$$

that preserves the gradings of modules and commutes with the action of the Galois group $\hat{G}$.

Proof. The G-algebra isomorphism ε induces an insomorphism

$$\mathrm{End}_R(R \otimes_{R_1} M) \simeq \mathrm{End}_R(R \otimes_{R_1} M')$$

of G-graded algebras. We have that $\mathrm{End}_R(R \otimes_{R_1} M)^{\mathrm{op}}$-mod is equivalent to $(R|_{R_1}M)$-mod, while by the proof of Theorem 5.2, we have that $(KR|W,F)$-mod is equivalent to $(KR|_{KR_1}(K \otimes_F M))$-mod. The statement now follows by Corollary 4.8.

Alternatively, observe that scalar extension $K \otimes_F -$ induces an isomorphism

$$\mathrm{End}_{KR}(KR \otimes_{KR_1} (K \otimes_F M)) \simeq \mathrm{End}_{KR'}(KR' \otimes_{KR'_1} (K \otimes_F M'))$$

of G-graded K-algebras, and use that $K \otimes_F M$ (respectively $K \otimes_F M'$) is a direct sum of copies of $\hat{W}$ (respectively, $\hat{W}'$).

The compatibility with Galois action easily follows since the isomorphism ε is defined over F. □

An isomorphism

$$\mathrm{End}_R(R \otimes_{R_1} M) \simeq \mathrm{End}_R(R \otimes_{R_1} M')$$

of G-graded algebras is called an *endoisomorphism* in [9].

Finally, when does an endoisomorphism exist?

THEOREM 5.5. *Assume that there is a Rickard equivalence between the G-graded F-algebras R and R'. Let M be a simple W-quasihomogeneous R-module and let M' be the corresponding R'-module. Then the following assertions hold.*

1) *There exists an isomorphism*

$$\varepsilon : \mathrm{End}_{R_1}(M) \to \mathrm{End}_{R'_1}(M')$$

of G-algebras over F, induced by Rickard equivalences.

2) *The simple KR_1-module W also corresponds to a simple KR'_1-module W', and the derived equivalence induces the equivalence*

$$(KR|W,F)\text{-mod} \simeq (KR'|W',F)\text{-mod}.$$

of Corollary 5.4.

Proof. 1) Use the well-known fact that a derived equivalence sends a simple module to a simple module (as the category of R-modules naturally embeds into the derived category), and since the equivalence is G-graded, it commutes with the restriction from R to R_1 and preserves the action of G on R_1-modules (see also [5, Corollary 5.2.6]).

2) Clearly, by applying the scalar extension $K \otimes_F -$, we also obtain a G-graded derived equivalence between KR and KR', and these equivalences preserve the action of the Galois group $\hat{G}$. □

REMARK 5.6. We have that $\mathrm{End}_{R_1}(M)$ is a central simple G-algebra, and it can be regarded as representative for the Brauer-Clifford class of W (see Turull [8], and also Gliţia [3] for the version for strongly G-graded algebras). Then Theorem 5.2 says that the Brauer-Clifford class of W does not depend on the choice of a W-quasihomogeneous R-module, while Theorem 5.5 says that a G-graded derived equivalence over F "preserves Brauer-Clifford classes" (see also [7, Theorem 5.3]).

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Received August 1, 2013
Accepted November 3, 2013

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