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HYERS-ULAM-RASSIAS STABILITY OF GENERALIZED CAUCHY FUNCTIONAL EQUATION

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Abstract. In this paper, the Hyers-Ulam-Rassias stability of generalized Cauchy functional equation $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$, $\alpha, \beta \in \mathbb{R} - \{0\}$, for A-linear mapping over C^*-algebras will be investigate.

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1. INTRODUCTION AND PRELIMINARIES

One of the interesting questions in the theory of functional equations concerning the problem of the stability of functional equations is as follows: when is it true that a mapping satisfying a functional equation approximately must be close to an exact solution of the given functional equation?

The first stability problem was raised by Ulam during his talk at the University of Wisconsin in 1940 [16].

Given a group G_1, a metric group (G_2, d), and a positive number ε, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \varepsilon$ for all $x, y \in G_1$?

Ulam's problem was partially solved by Hyers in 1941 in the context of Banach spaces with $\varepsilon = \delta$ as shown below [5].

THEOREM 1 ([5]). *Let E_1 be a normed vector space, E_2 a Banach space and suppose that the mapping $f : E_1 \to E_2$ satisfies the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$$

for all x, y in E_1, where $\varepsilon > 0$ is a constant. Then the limit

$$g(x) = \lim_n 2^{-n} f(2^n x)$$

exists for each $x \in E_1$, and g is the unique additive mapping satisfying

$$\|f(x) - g(x)\| \leq \varepsilon$$

for all $x \in E_1$. Also, if for each x the function $t \to f(tx)$ from $\mathbb{R}$ to E_2 is continuous for each fixed x, then g is linear. If f is continuous at a single point of E_1, then g is continuous in E_1.

Aoki [1] and Th.M. Rassias [14] provided a generalization of the Hyers' theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded.

Theorem 2 ([14]). *Let $f : E \to E_0$ be a mapping from a normed vector space E into a Banach space E_0 subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \tag{1}$$

for all $x, y \in E$, where ε and p are constants with $\varepsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E_0$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\varepsilon}{2-2^p}\|x\|^p \tag{2}$$

for all $x \in E$. If $p < 0$ then inequality (1) holds for $x, y \neq 0$ and (2) for $x \neq 0$. Also, if for each $x \in E$ the mapping $t \to f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

The above inequality has provided a lot of influence in the development of what is now known as a generalized Hyers-Ulam-Rassias stability of functional equations. P. Gavruta [4] provided a further generalization of the Th.M. Rassiass theorem. During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam-Rassias stability to a number of functional equations and mappings (see [9] $-$ [11],[16]). We also refer the readers to the books [2], [15] and [7].

Th. M. Rassias (1990) during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Z. Gajda (1991) gave an affirmative solution to this question for $p > 1$. It is shown that there is no analogue of Rassias result for $p = 1$ [3].

Definition 1. Let E be a set. A function $d : E \times E \to [0, \infty]$ is called a generalized metric on E if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in E$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in E$.

We recall the following theorem by Margolis and Diaz:

Theorem 3 ([8]). *Let (E, d) be a complete generalized metric space and let $J : E \to E$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in E$, either $d(J^n x, J^{n+1} x) = \infty$ for all non-negative integers n or there exists a non-negative integer n_0 such that*

(1) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J*;*
(3) y^* *is the unique fixed point of* J *in the set* $Y = y \in E : d(J^{n_0}x, y) < \infty$*;*
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$*.*

Throughout this paper, let A be a unital C^*-algebra with unitary group $U(A)$, unit e and norm $\|.\|$. Assume that X and Y are left Banach A-modules. An additive mapping $T : X \to Y$ is called A-linear if $T(ax) = aT(x)$ for all $a \in A$ and all $x \in X$.

In this paper, we investigate an A-linear mapping associated with the generalized Cauchy functional equation

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y), \tag{3}$$

where $\alpha, \beta \in \mathbb{R} - \{0\}$, and using the fixed point method, we prove the generalized Hyers-Ulam-Rassias stability of A-linear mappings in Banach A-modules associated with the functional equation (3). The first systematic study of fixed point theorems in nonlinear analysis is due to G. Isac and Th.M. Rassias [6].

Throughout this paper, α and β are fixed non-zero real numbers. For convenience, we use the following abbreviation for a given $a \in A$ and a mapping $f : X \to Y$,

$$\mathrm{D}_a f(x, y) := f(\alpha x + \beta a y) - \alpha f(x) - \beta a f(y) \tag{4}$$

for all $x, y \in X$.

2. MAIN RESULTS

At the first we need the following lemma:

LEMMA 1 ([12]). *Let* $f : X \to Y$ *be a mapping with* $f(0) = 0$ *such that* $\mathrm{D}_a f(x, y) = 0$ *for all* $x, y \in X$ *and all* $a \in U(A)$*. Then* f *is* A*-linear.*

Now we prove the generalized Hyers-Ulam-Rassias stability of A-linear mappings in Banach A-modules.

THEOREM 4. *Let* $f : X \to Y$ *be a mapping with* $f(0) = 0$ *for which there exists a function* $\varphi : X^2 \to [0, \infty)$ *such that*

$$\lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0,$$

$$\| \mathrm{D}_a f(x, y)\| \leq \varphi(x, y) \tag{5}$$

for all $x, y \in X$ *and all* $a \in U(A)$*. If there exists a constant* $L < 1$ *such that the function*

$$x \to \psi(x) := \varphi\left(\frac{x}{2\alpha}, \frac{x}{2\beta}\right) + \varphi\left(\frac{x}{2\alpha}, 0\right) + \varphi\left(0, \frac{x}{2\beta}\right)$$

has the property

$$2\psi(x) \leq L\psi(2x) \tag{6}$$

for all $x \in X$, then there exists a unique A-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{1}{1-L}\psi(x) \tag{7}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (6), we get

$$\|f(\alpha x) - \alpha f(x)\| \le \varphi(x, 0) \tag{8}$$

for all $x \in X$. Similarly, letting $x = 0$ and $a = e \in U(A)$ in (6), we get

$$\|f(\beta y) - \beta f(y)\| \ge \varphi(0, y) \tag{9}$$

for all $y \in$ X. So it follows from (6), (8) and (9) that

$$\|f(\alpha x + \beta y) - f(\alpha x) - f(\beta y)k\| \le \varphi(x, y) + \varphi(x, 0) + \varphi(0, y)$$

for all $x, y \in X$. Hence

$$\|f(x + y) - f(x) - f(y)\| \le \varphi\left(\frac{x}{\alpha}, \frac{y}{\beta}\right) + \varphi\left(\frac{x}{\alpha}, 0\right) + \varphi\left(0, \frac{y}{\beta}\right) \tag{10}$$

for all $x, y \in X$. Letting $y = x$ in (10), we get

$$\|f(2x) - 2f(x)\| \le \varphi\left(\frac{x}{\alpha}, \frac{x}{\beta}\right) + \varphi\left(\frac{x}{\alpha}, 0\right) + \varphi\left(0, \frac{x}{\beta}\right)$$

for all $x \in X$. Hence

$$\|f(x) - 2f\left(\frac{x}{2}\right)\| \le \psi(x) \tag{11}$$

for all $x \in X$. Let $E := \{g : X \to Y, g(0) = 0\}$. We introduce a generalized metric on E as follows:

$$d(g, h) := \inf\{C \in [0, \infty] : \|g(x) - h(x)\| \le C\psi(x) \quad \forall x \in X\}.$$

It is easy to show that (E, d) is a generalized complete metric space [3]. Now we consider the mapping $\Lambda : E \to E$ defined by

$$(\Lambda g)(x) = 2g\left(\frac{x}{2}\right), \quad \forall g \in E, x \in X.$$

Let $g, h \in E$ and let $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le C$. From the definition of d, we have $\|g(x) - h(x)\| \le C\psi(x)$ for all $x \in X$. By the assumption and last inequality, we have

$$\|(\Lambda g)(x) - (\Lambda h)(x)\| = 2\|g\left(\frac{x}{2}\right) - g\left(\frac{x}{2}\right)\| \le 2\psi\left(\frac{x}{2}\right) \le CL\psi\left(\frac{x}{2}\right)$$

for all $x \in X$. So $d(\Lambda g, \Lambda h) \le Ld(g, h)$ for any $g, h \in E$. It follows from (11) that $d(\Lambda f, f) \le 1$. Therefore according to Theorem 3, the sequence $\{\Lambda^n f\}$ converges to a fixed point T of Λ, i.e.,

$$T : X \to Y, \quad T(x) = \lim_n (\Lambda^n f)(x) = \lim_n 2^n f\left(\frac{x}{2^n}\right)$$

and $T(2x) = 2T(x)$ for all $x \in X$. Also T is the unique fixed point of Λ in the set $E^* = g \in E : d(f,g) < \infty$ and $d(T,f) \le \frac{1}{1-L} d(\Lambda f, f) \le \frac{1}{1-L}$, i.e., inequality (7) holds true for all $x \in X$. It follows from the definition of T, (5) and (6) that $\| \mathrm{D}_a T(x,y)\| = \lim 2^n \| \mathrm{D}_a f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\| \le \lim 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0$ for all $x, y \in X$ and all $a \in U(A)$. By Lemma 1, the mapping $T : X \to Y$ is A-linear. Finally it remains to prove the uniqueness of T. Let $P : X \to Y$ be another A-linear mapping satisfying (7). Since $d(f,P) \le \frac{1}{1-L}$ and P is additive, $P \in E^*$ and $(\Lambda P)(x) = 2P(x/2) = P(x)$ for all $x \in X$, i.e., P is a fixed point of Λ. Since T is the unique fixed point of Λ in E^*, $P = T$. □

Corollary 1. *Let $r > 1$ and θ be non-negative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and the inequality*

$$\| \mathrm{D}_a f(x,y)\| \le \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$ and all $a \in U(A)$. Then there exists a unique A-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{2(|\alpha|^r + |\beta|^r)\theta}{(2^r - 2)|\alpha\beta|^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows by letting $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ and $L = 2^{1-r}$ in the Theorem 4. □

By using the method of proof Theorem 4, we can prove:

Theorem 5. [12] *Let $f : X \to Y$ be a mapping with $f(0) = 0$ and there is a function $\Omega : X \times X \to [0,\infty)$ such that*

$$\lim_n \frac{1}{2^n}\Omega(2^n x, 2^n y) = 0, \ \| \mathrm{D}_a f(x,y)\| \le \Omega(x,y)$$

for all $x, y \in X$ and $a \in U(A)$. If there exists a constant $L < 1$ such that the function

$$x \mapsto \Psi(x) := \Omega\left(\frac{x}{\alpha}, \frac{x}{\beta}\right) + \Omega\left(\frac{x}{\alpha}, 0\right) + \Omega\left(0, \frac{x}{\beta}\right)$$

has the property

$$\Psi(2x) \le 2L\Psi(x)$$

for all $x \in X$, then there is a unique A-linear map $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{1}{2 - 2L}\Psi(x)$$

for all $x \in X$.

Corollary 2. *Let $0 < r < 1$ and θ, δ be non-negative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and the inequality*

$$\| \mathrm{D}_a f(x,y)\| \le \delta + \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$ *and all* $a \in U(A)$*. Then there exists a unique* A*-linear mapping* $T : X \to Y$ *such that*

$$\|f(x) - T(x)\| \leq \frac{3\delta}{2 - 2^r} + \frac{2(|\alpha|^r + |\beta|^r)\theta}{(2 - 2^r)|\alpha\beta|^r}\|x\|^r$$

for all $x \in X$*.*

Proof. The proof follows from Theorem 5 by taking $\Omega(x, y) := \delta + \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2r - 1$ and we get the desired result. □

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