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NEW SUBCLASS OF UNIVALENT HOLOMORPHIC FUNCTIONS BASED ON SALAGEAN OPERATOR

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Abstract. By using generalized Salagean differential operator a new subclass of Univalent holomorphic functions with negative coefficients is defined. Coefficient estimates, weighted mean and arithmetic mean properties are proved. Finally, effect of two integral operators on functions of this subclass are investigated.

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1. INTRODUCTION AND MOTIVATION

Let A denote the class of functions $f(z)$ of the from

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^k \tag{1}$$

which are holomorphic in $\Delta = \{z \in C :| z |< 1\}$. We denote by T the subclass of A consisting of function $f(z) \in A$ which are holomorphic univalent in Δ and are of the from

$$f(z) = z - \sum_{n=2}^{\infty} a_k z^k, \quad a_k \geq 0. \tag{2}$$

The generalized Salagean differential operator is defined in [1] by

$$\mathrm{D}_\lambda^0 f(z) = f(z)$$

$$\mathrm{D}_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z)$$

$$\mathrm{D}_\lambda^n f(z) = \mathrm{D}\ \lambda^1(\mathrm{D}_\lambda^{n-1} f(z)), \quad \lambda \geq 0.$$

See also [3].

If $f(z)$ is given by (2), we see that

$$\mathrm{D}_\lambda^n f(z) = z - \sum_{n=2}^{\infty} [1 + (k-1)\lambda]^n a_k z^k. \tag{3}$$

When $\lambda = 1$, we get the classic "Salagean" differential operator [4].

A function $f(z) \in T$ is said to be in $S_n^\lambda(\alpha, \beta, \gamma)$ if and only if

$$\left| \frac{[\mathrm{D}_\lambda^{n+2} f(z)]' - \frac{1}{z}\mathrm{D}_\lambda^{n+1} f(z)}{2\alpha \frac{[\mathrm{D}_\lambda^{n+1} f(z)]}{z} - \beta(1+\theta)\alpha} \right| < \gamma \tag{4}$$

for $\alpha, \beta, \gamma, \theta$ belong to $[0, 1)$.

2. MAIN RESULT

First we obtain necessary and sufficient codition for the function $f(z)$ to be in class $S_n^\lambda(\alpha, \beta, \gamma, \theta)$.

Theorem 1. *A function $f(z)$ given by* (2) *is in the class $S_n^\lambda(\alpha, \beta, \gamma, \theta)$ if and only if*

$$(5)\quad \sum_{n=2}^{\infty}[(1+(k-1)\lambda)^{n+1}(k^2\lambda + k(1-\lambda) - 1 + 2\alpha\gamma)a_k] \leq \alpha\gamma(2-\beta(1+\theta)).$$

The result is best possible for the function

$$(6)\qquad G(z) = z - \frac{\alpha\gamma(2-\beta(1+\theta))}{(1+(k-1)\lambda)^{n+1}(k^2\lambda + k(1-\lambda) - 1 + 2\alpha\gamma)} z^k.$$

Proof. Let the inequality (5) holds true and suppose $|z| = 1$, then we obtain

$$\mid [\mathrm{D}_\lambda^{n+2} f(z)]' - \frac{\mathrm{D}_\lambda^{n+1} f(z)}{z}] \mid - \gamma \mid 2\alpha - 2\alpha \sum_{n=2}^{\infty}[1+(k-1)\lambda]^{n+1} a_k z^{k-1} - \beta(1+\theta)\alpha \mid$$

$$= \sum_{n=2}^{\infty}[1+(k-1)\lambda]^{n+1}(k^2\lambda + k(1-\lambda) - 1 + 2\alpha\gamma)a_k - \alpha\gamma(2-\beta(1+\theta))] \leq 0.$$

Hence, by maximum modulus theorem, we conclude that $f(z) \in S_n^\lambda(\alpha, \beta, \gamma, \theta)$.

Conversely, let $f(z)$ defined by (2) be in the class $S_n^\lambda(\alpha, \beta, \gamma, \theta)$, so the condition (4) yields

$$\left| \frac{[\mathrm{D}_\lambda^{n+2} f(z)]' - \frac{1}{z}\mathrm{D}_\lambda^{n+1} f(z)]}{2\alpha[\frac{\mathrm{D}_\lambda^{n+1} f(z)}{z}] - \beta(1+\theta)\alpha} \right| =$$

$$\left| \frac{\sum_{n=2}^{\infty}[(1+(k-1)\lambda)^{n+1}(k^2\lambda + k(1-\lambda) - 1)]a_k z^{k-1}}{2\alpha - \sum_{n=2}^{\infty} 2\alpha(1+(k-1)\lambda)^{n+1} - \beta(1+\theta)\alpha} \right| < \gamma, z \in \Delta.$$

Since for any z, we have $|\mathrm{Re}(z)| < |z|$, then

$$\mathrm{Re}\frac{\sum_{n=2}^{\infty}[(1+(k-1)\lambda)^{n+1}(k^2\lambda + k(1-\lambda) - 1]a_k z^{k-1}}{\alpha(2-\beta(1+\theta)) - \sum_{n=2}^{\infty} 2\alpha(1+(k-1)\lambda)^{n+1} a_k z^{k-1}} < \gamma.$$

By letting $z \to 1$ through real values, we have

$$\sum_{n=2}^{\infty}[(1+(k-1)\lambda)^{n+1}(k^2\lambda + k(1-\lambda) - 1]a_k$$

$$\leq \alpha\gamma(2-\beta(1+\theta)) - 2\alpha\gamma\sum_{n=2}^{\infty}(1+(k-1)\lambda)^{n+1}a_k,$$

and this completes the proof. □

COROLLARY 1. *Suppose that the function $f(z)$ belongs to T and is in the class $S_n^\lambda(\alpha,\beta,\gamma,\theta)$. Then*

$$a_k \le \frac{\alpha\gamma(2-\beta(1+\theta))}{(1+(k-1)\lambda)^{n+1}(k^2\lambda+k(1-\lambda)-1+2\alpha\gamma)}, \quad k\ge 2. \tag{7}$$

3. WEIGHTED MEAN, ARITHEOREMETIC MEAN AND OPERATOR ON $S_N^\lambda(\alpha,\beta,\gamma,\theta)$

In the last section we prove that the class $S_n^\lambda(\alpha,\beta,\gamma,\theta)$ is closed under weighted mean and aritheoremetic mean. Also we verify the effect of two operators on functions in the same class.

THEOREM 2. *If $f(z)$ and $g(z)$ be in the class $S_n^\lambda(\alpha,\beta,\gamma,\theta)$, then the weighted mean of $f(z)$ and $g(z)$ defined by $h(z)=\frac{1}{2}((1-j)f(z)+(1+j)g(z))$ is also in $S_n^\lambda(\alpha,\beta,\gamma,\theta)$.*

Proof. By a direct calculation we obtain $h(z)=z-\sum_{n=2}^{\infty}\frac{1}{2}[(1-j)a_k+(1+j)b_k]z^k$. Since $f(z)$ and $h(z)$ are in the class $S_n^\lambda(\alpha,\beta,\gamma,\theta)$, so by Theorem 1 we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}[(1+(k-1)\lambda)^{n+1}(k^2\lambda+k(1-\lambda)-1+2\alpha\gamma)(\frac{1}{2}(1-j)a_k+(1+j)b_k)]\\
&=\frac{1}{2}[\sum_{n=2}^{\infty}(1+(k-1)\lambda)^{n+1}(k^2\lambda+k(1-\lambda)-1+2\alpha\gamma)(1-j)a_k\\
&+\sum_{n=2}^{\infty}(1+(k-1)\lambda)^{n+1}(k^2\lambda+k(1-\lambda)-1+2\alpha\gamma)(1+j)b_k]\\
&\le\frac{1}{2}[(1-j)\alpha\gamma(2-\beta(1+\theta))+(1+j)\alpha\gamma(2-\beta(1+\theta))]=\alpha\gamma(2-\beta(1+\theta)).
\end{aligned}$$

Hence by Theorem 1 $h(z)\in S_n^\lambda(\alpha,\beta,\gamma,\theta)$. □

THEOREM 3. *If $f_j(z)$ $(j=1,2,\cdots,m)$ defined by $f_j(z)=z-\sum_{n=2}^{\infty}a_{k,j}z^k$ be in the class $S_n^\lambda(\alpha,\beta,\gamma,\theta)$, then the aritheoremetic mean of $f_j(z)$ is also in the same class.*

Proof. By a direct calculation we obtain $h(z)=z-\sum_{n=2}^{\infty}\left(\frac{1}{m}\sum_{n=1}^{m}a_{k,j}\right)z^k$. Since $f_j(z)\in S_n^\lambda(\alpha,\beta,\gamma,\theta)$ for every $j=1,2,\cdots,m$, by using Theorem 1 we get

$$\begin{aligned}
&\sum_{n=2}^{\infty}(1+(k-1)\lambda)^{n+1}(k^2\lambda+k(1-\lambda)-1+2\alpha\gamma)\left(\frac{1}{m}\sum_{n=1}^{m}a_{k,j}\right)z^k\\
&\le\frac{1}{m}\sum_{n=1}^{\infty}\alpha\gamma(2-\beta(1+\theta))
\end{aligned}$$

which in view of Theorem 2.1, yields the proof of Theorem 3. □

THEOREM 4. *Let $f(z)$ of the from (2) be in $S_n^\lambda(\alpha,\beta,\gamma,\theta)$, then the Komato operator* [2] *of f defined by*

$$\mathrm{K}(f(z)) = \int_0^1 \frac{(c+1)^\delta}{\Gamma(\delta)} t^c \left(\log\frac{1}{t}\right)^{\delta-1} \frac{f(z)}{t}\mathrm{d}t,\ \ c>-1, \delta\geq 0,$$

belongs to $S_n^\lambda(\alpha,\beta,\gamma,\theta)$.

Proof. Since $\int_0^1 t^c(\log\frac{1}{t})^{\delta-1}\mathrm{d}t = \frac{\Gamma(\delta)}{(c+1)^\delta}$ and $\int_0^1 t^{+kc-1}(\log\frac{1}{t})^{\delta-1}\mathrm{d}t = \frac{\Gamma(\delta)}{(c+k)^\delta}$, $k\geq 2$, we obtain

$$\mathrm{K}(f(z)) = z - \sum_{n=2}^{\infty}\left(\frac{c+1}{c+k}\right)^\delta a_k z^k.$$

Since $(\frac{c+1}{c+k})^\delta < 1$ for $k\geq 2$ and (5) holds, we conclude,

$$\sum_{n=2}^{\infty}\left[(1+(k-1)\lambda)^{n+1}(k^2\lambda + k(1-\lambda) - 1 + 2\alpha\gamma)\left(\frac{c+1}{c+k}\right)^\delta a_k\right]$$
$$< \alpha\gamma(2-\beta(1+\theta)),$$

and this completes the proof. □

THEOREM 5. *Let $f(z)\in S_n^\lambda(\alpha,\beta,\gamma,\theta)$, then the function $J_\mu(f(z))$ defined by $J_\mu(f(z)) = (1-\mu)z + \mu\int_0^z \frac{f(z)}{t}\mathrm{d}t$ where $\mu\geq 0$ and $z\in\Delta$ is in $S_n^\lambda(\alpha,\beta,\gamma,\theta)$ if $0\leq\mu\leq 2$.*

Proof. By a simple calculation, we obtain $J_\mu(f(z)) = z - \sum_2^\infty \frac{\mu}{k}a_k z^k$, and $\frac{\mu}{2}\leq 1$ But since $f(z)\in S_n^\lambda(\alpha,\beta,\gamma,\theta)$, so by Theorem 1 we have

$$\sum_2^\infty[(1+(k-1)\lambda)^{n+1}(k^2\lambda + k(1-\lambda) - 1 + 2\alpha\gamma)\frac{\mu}{2}a_k \leq \alpha\gamma(2-\beta(1+\theta)).$$

This inequality completes the proof. □

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