MATHEMATICA, Tome 54 (77), N° Spécial, 2012, pp. 120–130

# SUBORDINATION PROPERTIES FOR SPECIAL INTEGRAL OPERATORS

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**Abstract.** Applying the Integral Existence Theorem for normalized analytic functions concerning the existence and analyticity of a general integral operator which was proven by S. S. Miller and P. T. Mocanu (J. Math. Anal. Appl. **157** (1991), 147–165), the analyticity and univalency of the functions defined by a certain special integral operator is discussed, and some interesting subordination criteria concerning with several integral operators are obtained.

**MSC 2010.** 34C40.
**Key words.** Differential subordination, integral operator, analyticity and univalency, integral existence theorem, spirallike function, subordination chain.

## 1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}$ denote the class of functions $f(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer $n$ and a complex number $a$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ defined by

$$\mathcal{H}[a,n] = \left\{ f(z) \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \right\}.$$

Also, we define the class $\mathcal{A}_n$ of normalized analytic functions $f(z)$ as

$$\mathcal{A}_n = \left\{ f(z) \in \mathcal{H} : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots \right\}$$

with $\mathcal{A}_1 = \mathcal{A}$. In addition, we need the classes of convex (univalent) and starlike (univalent) functions given respectively by

$$\mathcal{K} = \left\{ f(z) \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{U}) \right\}$$

and

$$\mathcal{S}^* = \left\{ f(z) \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{U}) \right\}.$$

Furthermore, a function $f(z) \in \mathcal{A}$ is said to be $\lambda$-spirallike in $\mathbb{U}$ if it satisfies

$$\operatorname{Re}\left(\mathrm{e}^{\mathrm{i}\lambda} \frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U})$$

for some real number $\lambda$ with $|\lambda| < \dfrac{\pi}{2}$. We denote by $\mathcal{S}^\lambda$ the class of all such functions. And, the class $\widehat{\mathcal{S}}$ is defined by

$$\widehat{\mathcal{S}} = \bigcup \left\{ \mathcal{S}^\lambda : |\lambda| < \frac{\pi}{2} \right\},$$

which implies that $\mathcal{S}^* \subset \widehat{\mathcal{S}}$. Specially, we note that all spirallike functions are univalent in $\mathbb{U}$.

We also introduce the familiar principle of differential subordinations between analytic functions. Let $f(z)$ and $g(z)$ be members of the class $\mathcal{H}$. Then the function $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}$, written by $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a function $w(z)$ analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, and such that $f(z) = g\big(w(z)\big)$ $(z \in \mathbb{U})$. In particular, if $g(z)$ is univalent in $\mathbb{U}$, then $f(z) \prec g(z)$ $(z \in \mathbb{U})$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

For the function $F(z) \in \mathcal{A}_n$, Miller and Mocanu [3] (see also [4]) proved the Integral Existence Theorem concerning with the existence and analyticity of a general integral operator of the form

$$\mathbf{I}[F](z) = \left\{ \frac{\beta+\gamma}{z^{\gamma}\psi(z)} \int_0^z \big(F(t)\big)^{\alpha} \varphi(t) t^{\delta-1}\,\mathrm{d}t \right\}^{\frac{1}{\beta}}, \tag{1}$$

where $\alpha$, $\beta$, $\gamma$ and $\delta$ are complex constants, and $\varphi(z)$, $\psi(z) \in \mathcal{H}[1,n]$. This operator was introduced by Miller, Mocanu and Reade [6].

In the present paper, applying a certain special Integral Existence Theorem which is obtained by giving some conditions, we discuss the analyticity and univalency of the functions defined by the following special integral operator

$$\tilde{\mathbf{I}}[F](z) = \left\{ \beta \int_0^z \big(F(t)\big)^{\alpha} \varphi(t) t^{\delta-1}\,\mathrm{d}t \right\}^{\frac{1}{\beta}}. \tag{2}$$

Further, by making use of the properties of subordination chains [8] (see also [4]) and the lemma given by Miller and Mocanu [2] (see also [4]) often used in the theory of differential subordinations, we deduce some subordination criteria concerning with

$$f(z) \prec \left\{ \beta \int_0^z \big(F(t)\big)^{\alpha} \varphi(t) t^{\delta-1}\,\mathrm{d}t \right\}^{\frac{1}{\beta}} \qquad (z \in \mathbb{U})$$

for analytic functions $f(z)$ with $f(0) = 0$. Moreover, we apply our result to find several subordination criteria for certain analytic functions defined as follows:

If the function $f(z) \in \mathcal{A}$ with $\dfrac{f(z)f'(z)}{z} \neq 0$ in $\mathbb{U}$ satisfies

$$\operatorname{Re}\left\{ (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \right\} > 0 \qquad (z \in \mathbb{U}) \tag{3}$$

for some real constant $\alpha$, then $f(z)$ is said to be $\alpha$-convex in $\mathbb{U}$. We denote this class by $\mathcal{M}_\alpha$. The class of $\alpha$-convex functions in $\mathbb{U}$ was introduced by Mocanu [7], and was studied by Mocanu, Miller and Reade [5]. They proved the following lemma.

Lemma 1. *If $f(z) \in \mathcal{M}_\alpha$, then $f(z) \in \mathcal{S}^*$. Moreover, if $\alpha \geqq 1$, then $f(z) \in \mathcal{K}$.*

Also, Sakaguchi and Fukui [10] proved that, if $f(z)$ satisfies the inequality (3), then $\frac{f(z)f'(z)}{z}$ never vanishes in $\mathbb{U}$. In other words, this fact means below.

Remark 1. A necessary and sufficient condition for $f(z) \in \mathcal{A}$ to be $\alpha$-convex in $\mathbb{U}$ is that $f(z)$ satisfies the inequality (3).

In 1962, Sakaguchi [9] introduced the class of $k$-starlike functions $f(z) \in \mathcal{A}$ which are defined by

$$\operatorname{Re}\left\{k\frac{zf'(z)}{f(z)} + \left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0 \qquad (z \in \mathbb{U}), \tag{4}$$

where $k$ is a complex constant such that $\operatorname{Re} k > -1$. We denote by $\mathcal{S}_k$ the class of $k$-starlike functions. We note that $k$-starlike functions are different from $\alpha$-convex functions.

Finally, we introduce the class of functions well-known as Bazilevič function. A function $f(z) \in \mathcal{A}$ is called Bazilevič of type $(a, b, \lambda)$, if there exists a function $g(z) \in \mathcal{S}^*$ such that

$$\operatorname{Re}\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^a\left(\frac{f(z)}{z}\right)^{ib}\right\} > 0 \qquad (z \in \mathbb{U}) \tag{5}$$

for some real numbers $a$, $b$ ($a \geqq 0$) and $\lambda$ $\left(|\lambda| < \frac{\pi}{2}\right)$. We denote by $\mathcal{B}_\lambda(a, b)$ the class of all such functions. Then, we note that $\mathcal{B}_0(0, 0) = \mathcal{S}^*$ and $\mathcal{B}_\lambda(0, 0) = \mathcal{S}^\lambda$. In 1955, Bazilevič [1] proved that Bazilevič functions are univalent in $\mathbb{U}$.

## 2. NOTE ON THE INTEGRAL EXISTENCE THEOREM

To considering the Integral Existence Theorem, we need to introduce the following open door mapping which is a special mapping from $\mathbb{U}$ onto a slit domain.

Definition 1. (*The Open Door Function*) Let $c$ be a complex number such that $\operatorname{Re} c > 0$, let $n$ be a positive integer, and let

$$C_n = C_n(c) = \frac{n}{\operatorname{Re} c}\left\{|c| \cdot \sqrt{1 + \frac{2\operatorname{Re} c}{n}} + \operatorname{Im} c\right\}. \tag{6}$$

If $R(z)$ is univalent defined by $R(z) = 2C_n\frac{z}{1-z^2}$ $(z \in \mathbb{U})$, then the open door function $R_{c,n}(z)$ is defined by

$$R_{c,n}(z) \equiv R\left(\frac{z+b}{1+\overline{b}z}\right) = 2C_n\frac{(z+b)(1+\overline{b}z)}{(1+\overline{b}z)^2 - (z+b)^2},$$

where $b$ is complex number with $R(b) = c$. If $c > 0$, then since $C_n$ in (6) is simplified to $C_n = C_n(c) = n \cdot \sqrt{1 + \frac{2c}{n}}$, and since $b = \overline{b} > 0$ and $b + \frac{1}{b} = \frac{2(c+n)}{c}$, we obtain $R_{c,n}(z) = c\frac{1+z}{1-z} + \frac{2nz}{1-z^2}$.

Remark 2. From the above definition, we see that $R_{c,n}(z)$ is univalent in $\mathbb{U}$, $R_{c,n}(0) = c$ and $R_{c,n}(\mathbb{U}) = R(\mathbb{U})$ is the complex plane $w$ with slits along the half-lines $\operatorname{Re} w = 0$, $\operatorname{Im} w \geqq C_n$ and $\operatorname{Re} w = 0$, $\operatorname{Im} w \leqq -C_n$. Also note that if $c > 0$, then $C_{n+1} > C_n$ and $\lim_{n\to\infty} C_n = \infty$. This leads us to $R_{c,n}(z) \prec R_{c,n+1}(z)$ and $\lim_{n\to\infty} R_{c,n}(\mathbb{U}) = \mathbb{C}$.

Lemma 2. (Integral Existence Theorem) *Let $\varphi(z)$, $\psi(z) \in \mathcal{H}[1, n]$ with $\varphi(z)\cdot\psi(z) \neq 0$ in $\mathbb{U}$. Also, let $\alpha$, $\beta$, $\gamma$ and $\delta$ be complex numbers with $\beta \neq 0$, $\alpha + \delta = \beta + \gamma$ and $\operatorname{Re}(\alpha + \delta) > 0$. Moreover, let $F(z) \in \mathcal{A}_n$ and suppose that*

$$P(z) \equiv \alpha\frac{zF'(z)}{F(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\alpha+\delta,n}(z) \qquad (z \in \mathbb{U}), \tag{7}$$

*where $R_{\alpha+\delta,n}(z)$ is the open door function. If $g(z) = \mathbf{I}[F](z)$ is defined by (1), then*

$$g(z) \in \mathcal{A}_n,\ \frac{g(z)}{z} \neq 0 \ \text{ and } \ \operatorname{Re}\left\{\beta\frac{zg'(z)}{g(z)} + \frac{z\psi'(z)}{\psi(z)} + \gamma\right\} > 0 \tag{8}$$

*for $z \in \mathbb{U}$, where all powers in (1) are principal ones.*

From the assumptions of Lemma 2, we deduce that $P(z) \in \mathcal{H}[\alpha + \delta, n]$. Also, from Remark 2, the condition $P(z) \prec R_{\alpha+\delta,n}(z)$ in (7) can be replaced by the stronger condition $\operatorname{Re} P(z) > 0$ $(z \in \mathbb{U})$. Hence, using this result in the Integral Existence Theorem, we find the following lemma.

Lemma 3. *Let $\varphi(z)$, $\psi(z) \in \mathcal{H}[1, n]$ with $\varphi(z) \cdot \psi(z) \neq 0$ in $\mathbb{U}$. Also, let $\alpha$, $\beta$, $\gamma$ and $\delta$ be complex numbers with $\beta \neq 0$, $\alpha + \delta = \beta + \gamma$ and $\operatorname{Re}(\alpha + \delta) > 0$. Moreover, let $F(z) \in \mathcal{A}_n$ and suppose that*

$$P(z) \equiv \alpha\frac{zF'(z)}{F(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \in \mathcal{H}[\alpha + \delta, n]$$

*satisfies $\operatorname{Re} P(z) > 0$ $(z \in \mathbb{U})$. If $g(z) = \mathbf{I}[F](z)$ is defined by (1), then $g(z)$ satisfies the conditions (8).*

We next consider a few special cases of Lemma 3. If we let $\psi(z) \equiv 1$ and $\gamma = 0$, then we derive special Integral Existence Theorem below.

Lemma 4. *Let $\varphi(z) \in \mathcal{H}[1, n]$ with $\varphi(z) \neq 0$ in $\mathbb{U}$. Also, let $\alpha$, $\beta$ and $\delta$ be complex numbers with $\beta = \alpha + \delta$ and $\operatorname{Re}(\alpha + \delta) > 0$. Moreover, let $F(z) \in \mathcal{A}_n$ and suppose that*

$$P(z) \equiv \alpha\frac{zF'(z)}{F(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \in \mathcal{H}[\beta, n]$$

*satisfies $\operatorname{Re} P(z) > 0$ $(z \in \mathbb{U})$. If $g(z) = \tilde{\mathbf{I}}[F](z)$ is defined by (2), then*

$$g(z) \in \mathcal{A}_n,\ \frac{g(z)}{z} \neq 0 \ \text{ and } \ \operatorname{Re}\left(\beta\frac{zg'(z)}{g(z)}\right) > 0$$

*for $z \in \mathbb{U}$, all powers in (2) are principal ones.*

REMARK 3. Since $\operatorname{Re}\beta > 0$, the above inequality

$$\operatorname{Re}\left(\beta\frac{zg'(z)}{g(z)}\right) > 0 \qquad (z \in \mathbb{U}) \tag{9}$$

shows that $g(z)$ produces spirallike, which implies that $g(z)$ is univalent, even when $F(z)$ is not univalent. That is, the above lemma provides conditions for which the function $g(z) = \tilde{\mathbf{I}}[F](z)$ defined by (2) will be an analytic and univalent function. In particular, if $\beta > 0$, then $g(z) \in \mathcal{S}^*$.

REMARK 4. By Remark 3, if the function $g(z) \in \mathcal{A}_n$ satisfies (9), then since $g(z)$ is univalent in $\mathbb{U}$ with $g(0) = 0$, we can deduce the condition $\frac{g(z)}{z} \neq 0$ $(z \in \mathbb{U})$, because we know that $\left.\frac{g(z)}{z}\right|_{z=0} = 1 \neq 0$.

## 3. AN APPLICATION OF INTEGRAL EXISTENCE THEOREM CONCERNING WITH DIFFERENTIAL SUBORDINATIONS

Applying special Integral Existence Theorem which was obtained in the previous section, we discuss the following subordination

$$f(z) \prec \left\{\beta\int_0^z \big(F(t)\big)^{\alpha}\varphi(t)t^{\delta-1}\,\mathrm{d}t\right\}^{\frac{1}{\beta}} \qquad (z \in \mathbb{U})$$

for analytic functions $f(z)$ with $f(0) = 0$, and deduced a subordination criterion.

In order to discuss our main result, we need some lemmas for subordination (or Loewner) chains. A function $L(z,t)$, $z \in \mathbb{U}$, $t \geqq 0$, is said to be a subordination chain if $L(\cdot,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geqq 0$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$, and $L(z,s) \prec L(z,t)$, when $0 \leqq s \leqq t$ (Pommerenke [8] or Miller and Mocanu [4]). The following lemma provides a necessary and sufficient condition for $L(z,t)$ to be a subordination chain.

LEMMA 5. (Loewner's Theorem) *The function* $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$, *with* $a_1(t) \neq 0$ *for* $t \geqq 0$, *and* $\lim_{t\to\infty} |a_1(t)| = \infty$, *is a subordination chain if and only if there exist constants* $r \in (0,1]$ *and* $M > 0$ *such that*

(i) $L(z,t)$ *is analytic in* $|z| < r$ *for each* $t \geqq 0$, *locally absolutely continuous in* $t \geqq 0$ *for each* $|z| < r$, *and satisfies*

$$|L(z,t)| \leqq M|a_1(t)|, \text{ for } |z| < r \text{ and } t \geqq 0.$$

(ii) *there exists a function* $p(z,t)$ *analytic in* $\mathbb{U}$ *for all* $t \in [0,\infty)$ *and measurable in* $[0,\infty)$ *for each* $z \in \mathbb{U}$, *such that* $\operatorname{Re} p(z,t) > 0$ *for* $z \in \mathbb{U}$, $t \in [0,\infty)$, *and*

$$\frac{\partial L(z,t)}{\partial t} = z\frac{\partial L(z,t)}{\partial z}p(z,t),$$

*for* $|z| < r$, *and for almost all* $t \in [0,\infty)$.

Note that the univalency of the function $L(z,t)$ can be extended from $|z| < r$ to all of $\mathbb{U}$. This lemma is well-known as the Loewner's theorem (see [8]). In the proof of our main result, the following lemma given by Pommerenke [8] is useful to apply the slight forms of Lemma 5.

LEMMA 6. *The function* $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$, *with* $a_1(t) \neq 0$ *for all* $t \geqq 0$ *and* $\lim_{t\to\infty} |a_1(t)| = \infty$, *is a subordination chain if and only if*

$$\operatorname{Re}\left\{ \frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}} \right\} > 0,$$

*for* $z \in \mathbb{U}$ *and* $t \geqq 0$.

In addition, the next lemma comes from the general theory of differential subordinations.

LEMMA 7. *Let* $g(z)$ *be analytic and univalent on the closed unit disk* $\overline{\mathbb{U}}$ *except for at most one pole on* $\partial\mathbb{U}$, *where* $\partial\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$, $\overline{\mathbb{U}} = \mathbb{U} \cup \partial\mathbb{U}$. *Also, let* $a = g(0)$ *and* $f(z) \in \mathcal{H}[a,n]$ *with* $f(z) \not\equiv a$. *If* $f(z)$ *is not subordinate to* $g(z)$, *then there exist two points* $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ *and* $\zeta_0 \in \partial\mathbb{U}$, *and a real number* $m$ *with* $m \geqq n \geqq 1$ *for which* $f(\mathbb{U}_{r_0}) \subset g(\mathbb{U})$, $f(z_0) = g(\zeta_0)$ *and* $z_0 f'(z_0) = m\zeta_0 g'(\zeta_0)$, *where* $\mathbb{U}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$.

More general forms of this lemma are given by Miller and Mocanu [2] (see also [4]).

Our main theorem is contained in Theorem 1.

THEOREM 1. *Let* $\alpha$, $\beta$ *and* $\delta$ *be complex numbers with* $\beta = \alpha + \delta$ *and* $\operatorname{Re}(\alpha+\delta) > 0$. *Also, let* $F(z) \in \mathcal{A}_n$, $\varphi(z) \in \mathcal{H}[1,n]$ *with* $\varphi(z) \neq 0$ *in* $\mathbb{U}$, *and suppose that*

$$P(z) \equiv \alpha\frac{zF'(z)}{F(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \in \mathcal{H}[\beta,n] \tag{10}$$

*satisfies* $\operatorname{Re} P(z) > 0 \quad (z \in \mathbb{U})$. *If* $f(z)$ *is analytic in* $\mathbb{U}$ *with* $f(0) = 0$ *and satisfies the following differential subordination*

$$\big(f(z)\big)^{\frac{\beta-1}{\beta}}\big(zf'(z)\big)^{\frac{1}{\beta}} \prec \Big\{\big(F(z)\big)^{\alpha}\varphi(z)z^{\delta}\Big\}^{\frac{1}{\beta}} \qquad (z \in \mathbb{U}), \tag{11}$$

*then*

$$f(z) \prec \left\{\beta\int_0^z \big(F(t)\big)^{\alpha}\varphi(t)t^{\delta-1}\,\mathrm{d}t\right\}^{\frac{1}{\beta}} \qquad (z \in \mathbb{U}).$$

*Proof.* From (10), we see that $\dfrac{F(z)}{z} \neq 0$ in $\mathbb{U}$. If we let

$$G(z) = \Big\{\big(F(z)\big)^{\alpha}\varphi(z)z^{\delta}\Big\}^{\frac{1}{\beta}}, \tag{12}$$

then $G(z)$ can be represented by

$$G(z) = z\left(\frac{F(z)}{z}\right)^{\frac{\alpha}{\beta}}\left(\varphi(z)\right)^{\frac{1}{\beta}} = z + A_{n+1}z^{n+1} + \cdots .$$

We note that $G(z) \in \mathcal{A}_n$ and $\dfrac{G(z)}{z} \neq 0$ in $\mathbb{U}$. Moreover, since $\operatorname{Re} P(z) > 0$ $(z \in \mathbb{U})$, we have

$$\operatorname{Re}\left(\beta\frac{zG'(z)}{G(z)}\right) = \operatorname{Re}\left\{\alpha\frac{zF'(z)}{F(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta\right\} > 0 \qquad (z \in \mathbb{U}). \tag{13}$$

Thus, by Remark 3, we deduce that the function $G(z)$ is univalent (spirallike) in $\mathbb{U}$, and hence the subordination (11) is well defined. Also, if we set

$$g(z) = \left\{\beta\int_0^z \left(F(t)\right)^{\alpha}\varphi(t)t^{\delta-1}\,\mathrm{d}t\right\}^{\frac{1}{\beta}}, \tag{14}$$

then from Lemma 4 and Remark 3, we see that the function $g(z)$ is analytic and univalent (spirallike) in $\mathbb{U}$. And, from (12) and (14), we have

$$\left(g(z)\right)^{\frac{\beta-1}{\beta}}\left(zg'(z)\right)^{\frac{1}{\beta}} = \left\{\left(F(z)\right)^{\alpha}\varphi(z)z^{\delta}\right\}^{\frac{1}{\beta}} = G(z).$$

We now show that

$$L(z,t) = \left(g(z)\right)^{\frac{\beta-1}{\beta}}\left\{(1+t)zg'(z)\right\}^{\frac{1}{\beta}} = (1+t)^{\frac{1}{\beta}}G(z) \qquad (t \geqq 0) \tag{15}$$

is a subordination chain. Since $G(z) \in \mathcal{A}_n$, the function

$$L(z,t) = (1+t)^{\frac{1}{\beta}}G(z) = a_1(t)z + a_{n+1}(t)z^{n+1} + \cdots$$

is analytic in $\mathbb{U}$ for all $t \geqq 0$, and is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$. Also, we have

$$\frac{\partial L(z,t)}{\partial z} = (1+t)^{\frac{1}{\beta}}G'(z), \quad \frac{\partial L(z,t)}{\partial t} = \frac{1}{\beta}(1+t)^{\frac{1}{\beta}-1}G(z).$$

Then, since $G(z) \in \mathcal{A}_n$, it is clear that

$$a_1(t) = \left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = (1+t)^{\frac{1}{\beta}}G'(0) = (1+t)^{\frac{1}{\beta}} \neq 0 \qquad (t \geqq 0),$$

and

$$\lim_{t\to\infty}|a_1(t)| = \lim_{t\to\infty}\left|(1+t)^{\frac{1}{\beta}}\right| = \infty.$$

A simple calculation combined with the condition (13) yields

$$\operatorname{Re}\left\{\frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}}\right\} = \operatorname{Re}\left\{\frac{(1+t)^{\frac{1}{\beta}}zG'(z)}{\frac{1}{\beta}(1+t)^{\frac{1}{\beta}-1}G(z)}\right\} = (1+t)\operatorname{Re}\left(\beta\frac{zG'(z)}{G(z)}\right) > 0,$$

for $z \in \mathbb{U}$ and $t \geqq 0$. Hence by Lemma 6, $L(z,t)$ is a subordination chain, and we have $L(z,s) \prec L(z,t)$, when $0 \leqq s \leqq t$. From (15), we obtain $L(z,0) = G(z)$, and hence we must have

$$L(\zeta,t) \notin G(\mathbb{U}), \tag{16}$$

for $|\zeta| = 1$ and $t \geqq 0$.

Next, applying Lemma 7, we will show that

$$\big(f(z)\big)^{\frac{\beta-1}{\beta}}\big(zf'(z)\big)^{\frac{1}{\beta}} \prec G(z) \;\text{ implies }\; f(z) \prec g(z) \qquad (z \in \mathbb{U}).$$

We observed that the function $g(z)$ is univalent in $\mathbb{U}$. Here, without loss of generality, we can assume that $g(z)$ is univalent on $\overline{\mathbb{U}}$, and $g'(\zeta) \neq 0$ for $|\zeta| = 1$. If not, then we can continue the remainder of the proof with the function $g(rz)$ $(0 < r < 1)$ which is univalent on $\overline{\mathbb{U}}$, and obtain our final result by letting $r \to 1^-$.

If we assume that $f(z)$ is not subordinate to $g(z)$, then by Lemma 7, there exist two points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$, and a real number $m \geqq 1$ such that $f(z_0) = g(\zeta_0)$ and $z_0 f'(z_0) = m\zeta_0 g'(\zeta_0)$. Then from (15) and (16), we have

$$\begin{aligned}\big(f(z_0)\big)^{\frac{\beta-1}{\beta}}\big(z_0f'(z_0)\big)^{\frac{1}{\beta}} &= \big(g(\zeta_0)\big)^{\frac{\beta-1}{\beta}}\big(m\zeta_0 g'(\zeta_0)\big)^{\frac{1}{\beta}} = m^{\frac{1}{\beta}}G(\zeta_0)\\ &= L(\zeta_0, m-1) \notin G(\mathbb{U}),\end{aligned}$$

where $z_0 \in \mathbb{U}$, $|\zeta_0| = 1$ and $m \geqq 1$. This contradicts the assumption (11) of the theorem, and hence we must have $f(z) \prec g(z)$. Therefore, we conclude that

$$f(z) \prec \left\{\beta\int_0^z \big(F(t)\big)^{\alpha}\varphi(t)t^{\delta-1}\,\mathrm{d}t\right\}^{\frac{1}{\beta}} \qquad (z \in \mathbb{U}),$$

which completes the proof of Theorem 1. □

As an example of Theorem 1, we give

Example 1. For the following functions

$$F(z) = \frac{z}{(1-z)^2} \in \mathcal{A} \quad \text{and} \quad \varphi(z) = \frac{1}{(1-z)^{2(\mathrm{Re}\,\beta-\alpha)}} \in \mathcal{H}[1,1],$$

since $\beta = \alpha + \delta$, we see that

$$\begin{aligned}\mathrm{Re}\left\{\alpha\frac{zF'(z)}{F(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta\right\} &= \mathrm{Re}\left\{\alpha\frac{1+z}{1-z} + \frac{2(\mathrm{Re}\,\beta-\alpha)z}{1-z} + \delta\right\}\\ &= \mathrm{Re}\left\{\frac{(\alpha+\delta) + \big(2\mathrm{Re}\,\beta - (\alpha+\delta)\big)z}{1-z}\right\}\\ &= \mathrm{Re}\left(\frac{\beta + \overline{\beta}z}{1-z}\right) > 0 \qquad (z \in \mathbb{U}).\end{aligned}$$

Hence, $F(z)$ and $\varphi(z)$ satisfies the assumption of Theorem 1. And, we have

$$\left\{\big(F(z)\big)^{\alpha}\varphi(z)z^{\delta}\right\}^{\frac{1}{\beta}} = \left\{\frac{z^{\alpha+\delta}}{(1-z)^{2\alpha}}\cdot(1-z)^{2(\alpha-\mathrm{Re}\,\beta)}\right\}^{\frac{1}{\beta}} = \frac{z}{(1-z)^{\frac{2\mathrm{Re}\,\beta}{\beta}}} \in \mathcal{A}.$$

Thus, from Theorem 1, we find that

$$\big(f(z)\big)^{\frac{\beta-1}{\beta}}\big(zf'(z)\big)^{\frac{1}{\beta}} \prec \frac{z}{(1-z)^{\frac{2\mathrm{Re}\,\beta}{\beta}}}$$

implies

$$f(z) \prec \left\{\beta\int_0^z \frac{t^{\beta-1}}{(1-t)^{2\mathrm{Re}\,\beta}}\,\mathrm{d}t\right\}^{\frac{1}{\beta}} \qquad (z\in\mathbb{U})$$

for analytic functions $f(z)$ with $f(0)=0$.

Remark 5. For the function in Example 1, we note that

$$\begin{aligned}\left\{\beta\int_0^z \frac{t^{\beta-1}}{(1-t)^{2\mathrm{Re}\,\beta}}\,\mathrm{d}t\right\}^{\frac{1}{\beta}} &= \left[\beta\int_0^z\left\{\sum_{n=0}^{\infty}\frac{(2\mathrm{Re}\,\beta)_n}{(1)_n}t^{\beta+n-1}\right\}\mathrm{d}t\right]^{\frac{1}{\beta}}\\ &= \left\{\sum_{n=0}^{\infty}\frac{\beta}{\beta+n}\cdot\frac{(2\mathrm{Re}\,\beta)_n}{(1)_n}z^{\beta+n}\right\}^{\frac{1}{\beta}}\\ &= \left\{z^{\beta}\sum_{n=0}^{\infty}\frac{(\beta)_n(2\mathrm{Re}\,\beta)_n}{(\beta+1)_n(1)_n}z^n\right\}^{\frac{1}{\beta}}\\ &= z\left\{{}_2F_1(\beta,2\mathrm{Re}\,\beta,\beta+1\,;z)\right\}^{\frac{1}{\beta}},\end{aligned}$$

where ${}_2F_1(a,b,c\,;z)$ represents the hypergeometric function.

## 4. SOME SUBORDINATION CRITERIA RELATED TO SEVERAL INTEGRAL OPERATORS

Let us consider some particular cases of Theorem 1.

Letting $\alpha=a$ $(a>0)$, $\delta=\mathrm{i}b$ $(b\in\mathbb{R})$, namely $\beta=a+\mathrm{i}b$ in Theorem 1, we obtain

Corollary 1. *Let $a$ and $b$ be real numbers with $a>0$. Also, let $F(z)\in\mathcal{A}_n$ with $\frac{F(z)}{z}\neq 0$ in $\mathbb{U}$, $\varphi(z)\in\mathcal{H}[1,n]$ with $\varphi(z)\neq 0$ in $\mathbb{U}$, and suppose that*

$$\mathrm{Re}\left(\frac{zF'(z)}{F(z)}\right) > -\frac{1}{a}\,\mathrm{Re}\left(\frac{z\varphi'(z)}{\varphi(z)}\right) \qquad (z\in\mathbb{U}).$$

*If $f(z)$ is analytic in $\mathbb{U}$ with $f(0)=0$ and satisfies the following differential subordination*

$$\big(f(z)\big)^{\frac{a-1+\mathrm{i}b}{a+\mathrm{i}b}}\big(zf'(z)\big)^{\frac{1}{a+\mathrm{i}b}} \prec \left\{\big(F(z)\big)^{a}\varphi(z)z^{\mathrm{i}b}\right\}^{\frac{1}{a+\mathrm{i}b}} \qquad (z\in\mathbb{U}),$$

*then*

$$f(z) \prec \left\{(a+\mathrm{i}b)\int_0^z \big(F(t)\big)^a \varphi(t) t^{\mathrm{i}b-1}\,\mathrm{d}t\right\}^{\frac{1}{a+\mathrm{i}b}} \qquad (z \in \mathbb{U}).$$

REMARK 6. The function

$$g(z) = \left\{(a+\mathrm{i}b)\int_0^z \big(F(t)\big)^a \varphi(t) t^{\mathrm{i}b-1}\,\mathrm{d}t\right\}^{\frac{1}{a+\mathrm{i}b}} \in \mathcal{A}_n$$

has the same form as the Bazilevič function. A simple calculation yields that

$$\varphi(z) = \frac{zg'(z)}{g(z)}\left(\frac{g(z)}{F(z)}\right)^a \left(\frac{g(z)}{z}\right)^{\mathrm{i}b}.$$

If $F(z) \in \mathcal{S}^*$, and $\varphi(z) \in \mathcal{H}[1,n]$ satisfies $\operatorname{Re} e^{\mathrm{i}\lambda}\varphi(z) > 0 \quad (z \in \mathbb{U})$ for some real $\lambda$ with $|\lambda| < \dfrac{\pi}{2}$, then since $g(z)$ satisfies the inequality (5), we see that $g(z) \in \mathcal{B}_\lambda(a,b)$ (see [1]).

Setting $\varphi(z) \equiv 1$, $\alpha = k+1$ ($\operatorname{Re} k > -1$) and $\delta = 0$ in Theorem 1, we have

COROLLARY 2. *Let $k$ be a complex number with $\operatorname{Re} k > -1$, and let $F(z) \in \mathcal{A}_n$ satisfies*

$$\operatorname{Re}\left\{(k+1)\frac{zF'(z)}{F(z)}\right\} > 0 \qquad (z \in \mathbb{U}). \tag{17}$$

*If $f(z)$ is analytic in $\mathbb{U}$ with $f(0) = 0$ and satisfies the following differential subordination*

$$\big(f(z)\big)^{\frac{k}{k+1}}\big(zf'(z)\big)^{\frac{1}{k+1}} \prec F(z) \qquad (z \in \mathbb{U}),$$

*then*

$$f(z) \prec \left\{(k+1)\int_0^z \frac{\big(F(t)\big)^{k+1}}{t}\,\mathrm{d}t\right\}^{\frac{1}{k+1}} \qquad (z \in \mathbb{U}).$$

REMARK 7. If we let

$$g(z) = \left\{(k+1)\int_0^z \frac{\big(F(t)\big)^{k+1}}{t}\,\mathrm{d}t\right\}^{\frac{1}{k+1}} \in \mathcal{A}_n,$$

then we have

$$k\frac{zg'(z)}{g(z)} + \left(1 + \frac{zg''(z)}{g'(z)}\right) = (k+1)\frac{zF'(z)}{F(z)}.$$

Thus, from (17), we see that $g(z)$ satisfies (4) which implies that $g(z) \in \mathcal{S}_k$.

Taking $k = \dfrac{1}{\alpha} - 1$ ($\alpha > 0$) in Corollary 2, we find

Corollary 3. *Let $\alpha$ be a real number with $\alpha > 0$, and let $F(z) \in \mathcal{A}_n$ be starlike in $\mathbb{U}$. If $f(z)$ is analytic in $\mathbb{U}$ with $f(0) = 0$ and satisfies the following differential subordination*

$$\big(f(z)\big)^{1-\alpha}\big(zf'(z)\big)^{\alpha} \prec F(z) \qquad (z \in \mathbb{U}),$$

*then*

$$f(z) \prec \left\{ \frac{1}{\alpha}\int_0^z \frac{\big(F(t)\big)^{\frac{1}{\alpha}}}{t}\,\mathrm{d}t \right\}^{\alpha} \qquad (z \in \mathbb{U}).$$

Remark 8. Since the function $F(z)$ is starlike and $\alpha > 0$, it is easy to show that

$$g(z) = \left\{ \frac{1}{\alpha}\int_0^z \frac{\big(F(t)\big)^{\frac{1}{\alpha}}}{t}\,\mathrm{d}t \right\}^{\alpha} \in \mathcal{A}_n$$

satisfies the inequality (3). That is, $g(z) \in \mathcal{M}_\alpha$. In addition, according to Lemma 1, we see that $g(z)$ is not only starlike but also convex for $\alpha \geqq 1$ (see [5]).

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