MATHEMATICA, Tome 54 (77), N° Spécial, 2012, pp. 60–67

THE DIAGONALIZATION PROCEDURE FOR THE FINITE DIMENSIONAL DIFFERENTIAL OPERATOR EQUATIONS SYSTEM OVER THE m-DIMENSIONAL COMPLEX SPACE

IRINA DMITRIEVA

Abstract. The present article is a generalization of the author's former results concerning the partial differential operator analog to the well known algebraic Gauss method.

MSC 2010. 30E99, 35Q60.

Key words. Diagonalization, complex space, vector function, scalar function, system of operator equations.

1. INTRODUCTION

The present results came up from recent engineering problems that appear in the classical Maxwell field theory and concern signal transmissions in various kinds of media (see [1, 2]).

If the investigator encounters physical or industrial problems that are dealing with the study of some vector field, then in the majority of cases its mathematical model is represented as a finite dimensional system of PDEs, in other words, matrix or vector PDEs. The sought for a vector function which is responsible for all investigated vector field properties is "hidden inside" of the above mentioned system as the corresponding finite set of the unknown co-ordinate components that describe the appropriate scalar fields and form the initially sought for vector field function.

Therefore, the diagonalization problem of the arbitrary finite dimensional system of PDEs appears. As usual, it means the reduction of the original n-dimensional vector/matrix equation to the system of the corresponding n scalar equations where each of them depends on the only one of the studied vector function's components. It is obvious that the final diagonalized matrix equation must be equivalent to the initial one.

These approaches that have become almost classical in the diagonalization problems of the finite dimensional PDEs' systems [3, p. 127-261] have not changed much until now. Mostly, they use the generalized function theory even in the case of the first order linear systems of PDEs over $\mathbb{R}^n$ and in every functional class the respective scalar equations are obtained in their specific

The author is very grateful to the ATF2009 Organizing Committee whose valuable support has allowed to propose present results for open discussion.

different way. Such methods as presented, for instance in [3, 4], though mathematically very elegant, remain difficult and rather often even unattainable for engineers and other non-mathematicians. Moreover, all well-known procedures for the applied diagonalization problem before the reduction to the respective system of scalar equations take into account either the initial and boundary conditions or the original matrix structure and sometimes even the fixed space dimension over which the vector field function is given.

Hence, this paper presents an obvious and simple idea which occurs naturally. We extend the well-known Gauss method to arbitrary finite dimensional systems of PDEs. This was done first for the so called "symmetrical" generalized Maxwell systems in two stages "by blocks and by coordinates" (see [5]). Further, this method was generalized in the case of arbitrary $n \times n$ systems of partial differential operator equations over the finite dimensional real space (see [6]). Here, the additional real parameters in comparison with the classical Maxwell space (x, y, z, t) describe the appropriate physical features of the considered medium (temperature, density etc.). In both papers [5] and [6], all partial differential operators obey only the requirement of the commutativity in pairs. It was proved in these papers that the proposed diagonalization procedure is independent both of the original matrix structure and of the boundary and initial conditions. Thus the vector field function was found in a simple algorithmic and correct mathematical way.

The present paper shows how the partial differential operator analog of the Gauss method [5, 6] can be improved in the case of the finite dimensional complex space and, additionally, when the given matrix elements represent the arbitrary bounded invertible operators that are assumed to be even nonlinear.

2. PRELIMINARIES

We consider the following $n \times n$ system of arbitrary operator equations over the "mixed" $m + 1$-dimensional complex-real space $(z_1, \dots, z_m, t) = (z, t)$, where $z = (z_1, \dots, z_m) \in \mathbb{C}^m$ and t is the time argument

$$\sum_{i=1}^{n} A_{ji} F_i = f_j \quad (j = \overline{1, n}). \tag{1}$$

In (1), the functions $\overrightarrow{F} = \overrightarrow{F}(z, t) = \{F_i(z, t)\}_{i=1}^n$ and $\overrightarrow{f} = \overrightarrow{f}(z, t) = \{f_j(z, t)\}_{j=1}^n$ are, respectively, the unknown and the given vector functions from the same class K. All known matrix operator elements A_{ji} $(j, i = \overline{1, n})$ from (1) can be utterly arbitrary (nonlinear, in general) but obligatory bounded, invertible, and commutative in pairs, i.e.,

$$A_{ji} A_{kl} = A_{kl} A_{ji} \quad (j, i, k, l = \overline{1, n}). \tag{2}$$

The evident drawback of [5] and [6] was the concrete definition of the space and its dimension over which the vector functions $\overrightarrow{F}$ and $\overrightarrow{f}$ from (1) were considered. In reality, from the algorithmic mathematical point of view, the features of the space over which (1) and (2) are given have no significance. The only important requirement is its finite dimensionality.

On the other hand, regarding the physical and industrial applications, the needed space model can be sufficiently accepted as the "mixed" complex-real space whose points have the similar structure to (1). Namely, (z, t) where t is the time variable and $z = (z_1, \dots, z_m) \in \mathbb{R}^m$ or $\mathbb{C}^m$ etc. is the spatial coordinate vector.

It is also obvious that in these circumstances the demands of the study of the mathematical and applied characters in the classical field theory can not and must not be restricted only by the cases of the operator PDEs systems. Even if it does not seem expedient, the author has objective reasons not to refer directly to the majority of current industrial problems that deal with the field theory and various questions of the signal transmissions as its corollary.

Visually, the current diagonalization procedure is completely identical to the case treated in [6]. Nevertheless, it has to be shown that the given process acts in the proposed situation (1), (2) as well. Trying not to be tiresome and avoiding the needless references to [5] and [6], the author reserves the right to neglect all unnecessary insignificant details of the considered diagonalization method and to describe only its main general algorithmic steps.

3. THE "UPWARD" DIAGONALIZATION STAGE

From the very beginning, we raise the problem of obtaining the scalar equation with respect to one of the unknown components $\{F_i\}_{i=1}^n$. Not breaking the common character of our results, we assume that the sought function is F_1.

The following first algorithmic steps reflects clearly the main idea of the given procedure. Therefore, at the **first step** we separate the last equation of system (1) and isolate the item with the scalar F_n in all n equations of the considered system.

$$\tag{3} \begin{cases} \sum\limits_{i=1}^{n-1} A_{ji} F_i + A_{jn} F_n = f_j \quad (j = \overline{1, n-1}) \\ \sum\limits_{i=1}^{n-1} A_{ni} + A_{nn} F_n = f_n. \end{cases}$$

Then we apply to the last equation of (3) the operator

$$\tag{3'} A_{jn} \quad (j = \overline{1, n-1})$$

consistently for all j from (3′) and to the rest $n-1$ equations of the same system the operator

$$\tag{3''} A_{nn}$$

is applied. Afterwards, we summarize consistently the transformed nth equation and the other $n-1$ (also transformed) equations for all $j = \overline{1, n-1}$. We obtain the following system which is equivalent to (3)

$$(4)\qquad \begin{cases} \sum\limits_{i=1}^{n-1}(A_{nn}A_{ji}-A_{jn}A_{ni})F_i = A_{nn}f_j - A_{jn}f_n \quad (j=\overline{1,n-1}) \\ \sum\limits_{i=1}^{n-1} A_{ni}F_i + A_{nn}F_n = f_n, \end{cases}$$

where in all equations, excepting the last one, the scalar function F_n does no longer appear. Such equations that close the appropriate system at each diagonalization step as in [6] are called "single".

Introducing the auxiliary notation for the known operators and functions

$$(5)\qquad A_{nn}A_{ji} - A_{jn}A_{ni} = B_{ji}^{(1)}, \quad A_{nn}f_j - A_{jn}f_n = g_{j1},\ (j,i=\overline{1,n-1}),$$

we consider now the following subsystem of (4)

$$(6)\qquad \sum_{i=1}^{n-1} B_{ji}^{(1)}F_i = g_{j1} \qquad (j,i=\overline{1,n-1}),$$

which consists of all equations from (4) without the single one.

Continuing this process, we can consider the general **step number** $k+1$ $(k=\overline{1,n-1})$ with its respective initial system

$$(7)\qquad \begin{cases} \sum\limits_{i=1}^{n-k-1} B_{ji}^{(k)}F_i + B_{j,n-k}^{(k)}F_{n-k} = g_{jk} \quad (j=\overline{1,n-k-1}) \\ \sum\limits_{i=1}^{n-k-1} B_{n-k,i}^{(k)}F_i + B_{n-k,n-k}^{(k)}F_{n-k}F_{n-k} = g_{n-k,k} \quad (k=\overline{1,n-1}), \end{cases}$$

whose $(n-k)$th equation we transform by the operator

$$(7')\qquad \left(-B_{j,n-k}^{(k)}\right) \qquad (j=\overline{1,n-k-1})$$

for all j from $(7')$ and to the other $n-k-1$ equations of (7) the operator

$$(7'')\qquad B_{n-k,n-k}^{(k)}$$

is applied.

Again we next summarize the $(n-k)$th and the other $n-k-1$ transformed equations from (7) for all $j=\overline{1,n-k-1}$. We get the equivalent system

$$(8)\qquad \begin{cases} \sum\limits_{i=1}^{n-k-1}(B_{n-k,n-k}^{(k)}B_{ji}^{(k)} - B_{j,n-k}^{(k)}B_{n-k,i}^{(k)})F_i = B_{n-k,n-k}^{(k)}g_{jk} \\ \qquad\qquad -B_{j,n-k}^{(k)}g_{n-k,k} \quad (j=\overline{1,n-k-1}) \\ \sum\limits_{i=1}^{n-k-1} B_{n-k,i}^{(k)}F_i + B_{n-k,n-k}^{(k)}F_{n-k} = g_{n-k,k}. \end{cases}$$

The first $n-k-1$ equations of (8) have now more scalar components F_i $(i=\overline{1,n-k-1})$ and no F_i $(i=\overline{n-k,n})$. The last equation in (8) is single.

Simplifying the visual algorithmic features, we put into (7), (8) the following designations

$$B_{n-k,n-k}^{(k)} B_{ji}^{(k)} - B_{j,n-k}^{(k)} B_{n-k,i}^{(k)} = B_{ji}^{(k+1)} \qquad (j, i = \overline{1, n-k-1}) \tag{9}$$
$$B_{n-k,n-k}^{(k)} g_{jk} - B_{j,n-k}^{(k)} g_{n-k,k} = g_{j,k+1}.$$

Taking into account formula (9), the **final step**, for $k = n-1$, leads to the sought for scalar equation with respect to the component F_1

$$B_{11}^{(n-1)} F_1 = g_{1,n-1}, \tag{10}$$

while the other $n-1$ non scalar equations are single

$$\sum_{i=1}^{n-k} B_{ji}^{(k)} F_i = g_{jk} \quad (j = \overline{2, n-k}; \quad k = \overleftarrow{0, n-2}) \tag{11}$$

and

$$B_{ni}^{(0)} = A_{ni} \quad (i = \overline{1, n}), \quad g_{n0} = f_n. \tag{12}$$

The arrow-direction for the index k from (11) will describe in the next section the backward count, from the right to the left.

It is clear that the system (10), (11) is equivalent to the original system (1)≡(3) and completes the “upward” diagonalization stage.

In conclusion, it should be noted that in this section the upper index in the round brackets of the given operator $B\ldots$ and the second lower index of the known function $g\ldots$ mean the step number of the diagonalization procedure in the “upward” direction.

4. THE “DOWNWARD” DIAGONALIZATION STAGE

Now we pass to to the second diagonalization stage that works in the opposite, i.e., “downward” direction.

At the **first step** ($k = n-2$) we isolate the first equation from the subsystem (11) and write it together with the first obtained scalar equation (10) that has the component F_1. Here we neglect the other $k = \overleftarrow{0, n-3}$ equations from (11) considering them as single ones

$$\begin{cases} B_{11}^{(n-1)} F_1 = g_{1,n-1} \\ \sum\limits_{i=1}^{2} B_{ji}^{(n-2)} F_i = g_{j,n-2} \quad (j = 2). \end{cases} \tag{13}$$

We separate the item with the scalar F_2 in the last equation of (13), apply to the second and first system of equations the appropriate operators

$$B_{11}^{(n-1)}, \tag{13'}$$

$$\left(-B_{21}^{(n-2)}\right) \tag{13''}$$

and sum up these both transformed objects. We get the following system that is equivalent to (13) and whose second equation has become scalar with respect to the component F_2

$$\left\{ \begin{array}{l} B_{11}^{(n-1)} F_1 = g_{1,n-1} \\ B_{11}^{(n-1)} B_{22}^{(n-2)} F_2 = h_1, \end{array} \right. \tag{14}$$

where

$$h_1 = B_{11}^{(n-1)} g_{2,n-2} - B_{21}^{(n-2)} g_{1,n-1}. \tag{15}$$

It is evident that at this moment the subsystem (11) has lessened by one equation and $j = \overline{3, n-k}$, $k = \overleftarrow{0, n-3}$.

Further we generalize the "downward" diagonalization stage in the case of the arbitrary **step number** l $(l = \overline{1, n-1})$. To this purpose we take into account the subsystem of the previous step number $l-1$

$$\sum_{i=1}^{n-k} B_{ji}^{(k)} F_i = g_{ik} \quad (j = \overline{l+1, n-k};\;\; k = \overleftarrow{0, n-l-1}), \tag{16}$$

separate there the first equation and attach it to the concluding system of the obtained scalar equations from the preceding step $l-1$. Simultaneously the rest $k = \overleftarrow{0, n-l-2}$ equations from (16) remain single.

Hence, the present system whose last equation must become scalar is written below

$$\left\{ \begin{array}{l} \prod\limits_{q=1}^{p+1} B_{qq}^{(n-q)} F_{p+1} = h_p \quad (p = \overline{0, l-1};\;\; h_0 = g_{1,n-1}) \\ \sum\limits_{i=1}^{l+1} B_{ji}^{(n-l-1)} F_i = g_{j,n-l-1} \quad (j = l) \end{array} \right. \tag{17}$$

and the symbol of the finite operator product here, as later in this section, implies the usual consequent operator application from the inner to the external, i.e., from "the right to the left direction".

Then we isolate the item with the component F_{l+1} in the last equation of (17) and apply to this $(l+1)$th equation the operator

$$\prod_{q=1}^{l} B_{qq}^{(n-q)}. \tag{17$'$}$$

To the other equations in (17), from the first to the $(l-1)$th one, we apply the appropriate operators

$$\left(-B_{l+1,r}^{(n-l-1)} \prod_{q=r+1}^{l} B_{qq}^{(n-q)} \right) \quad (r = \overline{1, l-1}), \tag{17$''$}$$

while the lth equation of the same system is transformed by the operator

$$\left(-B_{l+1,l}^{(n-l-1)}\right). \tag{17'''}$$

Then we sum up all these $l+1$ transformed equations and obtain a system that is equivalent to (17). This will be the final "scalar" system for the arbitrary step l $(l=\overline{1,n-2})$

$$\prod_{q=1}^{p+1} B_{qq}^{(n-q)} F_{p+1} = h_p \quad (p=\overline{0,l};\ \ h_0 = g_{1,n-1}), \tag{18}$$

where

$$\begin{aligned} h_l = \prod_{q=1}^{l} B_{qq}^{(n-q)} g_{l+1,n-l-1} - &\sum_{r=1;\,(l\neq 1)}^{l-1} B_{l+1,r}^{(n-l-1)} \prod_{q=r+1}^{l} B_{qq}^{(n-q)} h_{n-1} \\ &- B_{l+1,l}^{(n-l-1)} h_{l-1} \qquad (l=\overline{1,n-2}) \end{aligned} \tag{19}$$

and the second item from the right-hand side of (19) equals zero when $l=1$.

The recurrent formula (18), (19) can be easily verified even for the simplest step $l=1$.

After continuation of this "downward" diagonalization stage, including the final step $l=n-1$, we get a system of scalar equations with all wanted components F_i $(i=\overline{1,n})$ (see (18) when $l=n-1$)

$$\prod_{q=1}^{p+1} B_{qq}^{(n-q)} F_{p+1} = h_p \qquad (p=\overline{0,n-1};\ \ h_0 = g_{1,n-1}), \tag{20}$$

where the definite operators and functions are described by (9) and (19).

It is not difficult to notice that the final system (20) is equivalent to the interim system (10), (11) and therefore, to the original system (1)≡(3), too. This fact follows directly from the proposed diagonalization procedure and completes it.

Thus we have proved the existence of the solution of the initial system by means of the diagonalization method, so the main purpose of the paper is attained. In other words, we have proved the following theorem by means of the described explicit algorithm.

Theorem 1. *The explicit solution of the system* (1)≡(3) *in terms of the diagonalization procedure exists and can be obtained algorithmically.*

5. REMARKS

In conclusion, it should be noted again that the given diagonalization procedure is irrespective of the initial and boundary conditions which become necessary only after the diagonalization completion, when the corresponding obtained scalar equations have to be solved. Moreover, the proposed method can be applied to the matrices of the arbitrary block structure. In this situation, the diagonalization process acts consistently from the external blocks to the inner ones, until the scalar equations with respect to the original vector function's components are found.

Since the present diagonalizing algorithm is the operator analog to the method for solving algebraic systems, it deals only with the matrix elements whose explicit operator expressions are given and the procedure own course of action "knows" in advance what kind of operator and when it is applied. Additionally, the initial system of operators can be even nonlinear, but it has obligatory to be bounded, invertible, and commutative in pairs (2).

The author presents apologies for the deliberate enumeration of all obvious simple merits of the described method, because we completely understand that its (algorithm) positive feature is more of the applied than of the pure mathematical character.

REFERENCES

[1] Ivanitckiy, A.M. and Dmitrieva, I.Yu., *Technical scientific report "Diagonalization of the "complete" Maxwell system of the differential equations"*, Odessa 2007, ONAT (Odessa National Academy of Transcommunications), state registered number **0105u007232**, 10–41 (in Russian).

[2] Ivanitckiy, A.M. and Dmitrieva, I.Yu., *Technical scientific report "Investigation of the electric circuits and electric fields with the expo-functional influences" (final)*, Odessa 2008, ONAT, state registered number **0108u010946**, 10–23 (in Russian).

[3] Komech, A.I., *Scientific and engineering results. Current mathematical problems. Fundamental trends. PDE's,* Vol. **31**, 127–261, Moscow, Nauka, 1988 (in Russian).

[4] Vladimirov, V.S., *Generalized functions in mathematical physics,* Moscow, Nauka, 1979 (in Russian).

[5] Dmitrieva, I., *On the constructive solution of n-dimensional differential operator equations' system and its application to the classical Maxwell theory*, Proc. of the 6th Congress of Roman. Mathem., Bucharest 2007, Vol. **1**, 241–246, Editura Academiei Române, Bucureşti, 2009.

[6] Dmitrieva, I., *Diagonalization problems in classical Maxwell theory and their industrial applications*, Proc. of the Internat. Conf. on Econophysics etc. (ENEC08), Bucharest 2008, 11–23, Victor Publishing House, Bucharest 2009.

South Ukrainian National Pedagogical University
Institute of Physics and Mathematics
Staroportofrankovskaya Str. 26
65020 Odessa, Ukraine

Odessa National Academy of Telecommunications
Higher Mathematics Department
Kuznechnaya Str. 1
65029 Odessa, Ukraine
E-mail: irina.dm@mail.ru