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THE RELATIONSHIP BETWEEN DIFFERENT SEPARATION NOTIONS ON L-TOPOLOGICAL SPACES

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Abstract. In the present paper we study the smooth topology and its equivalent L-topology, the corresponding L-continuous, L-open, and homeomorphism maps. We also study the concept of several separation axioms (like $ST_\circ$, ST_1, ST_2, and their strong and weak forms on the mentioned topology). Finally we investigate some of their properties and the relations between them.

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1. INTRODUCTION

The concept of fuzzy topology, introduced in 1968 by Chang, has been redefined in a somewhat different way by Hutton, Reilly and others. Some new definitions have been proposed by Badard in [1] and Ramadan in [3].

In the present work we study the concept of separation axioms as strong $ST_\circ$, $ST_\circ$, weak $ST_\circ$, strong ST_1, ST_1, weak ST_1, strong ST_2, ST_2, weak ST_2 on an L-topological space that is equal to the smooth topological space. Also we investigate some of their properties and the relations between them in L-topological spaces.

2. PRELIMINARIES

Throughout this paper L and L' represent two completely distributive lattices with smallest element 0 (or $\perp$) and greatest element 1 (or $\top$), where $0 \neq 1$. We define $M(L)$ to be the set of all non-zero $\vee$-irreducible (or coprime) elements in L such that $a \in M(L)$ if and only if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. Let X be a non-empty set, and L^X be the set of all L-fuzzy sets on X. For each $a \in L$, let $\underline{a}$ denote the constant-valued L-fuzzy set with a as its value. Let $\underline{0}$ and $\underline{1}$ be the smallest element and the greatest element in L^X, respectively. For the empty set $\emptyset \subset L$, we define $\wedge\emptyset = 1$ and $\vee\emptyset = 0$.

Definition 1. An *L-fuzzy topology* on X is a map $\tau\colon L^X \to L$ satisfying the following three axioms:
1) $\tau(\underline{\top}) = \top$,
2) $\tau(A \wedge B) \geq \tau(A) \wedge \tau(B)$ for every $A, B \in L^X$,
3) $\tau(\vee_{i\in\triangle} A_i) \geq \bigvee_{i\in\triangle} \tau(A_i)$ for every family $\{A_i \mid i \in \triangle\} \subseteq L^X$.

The pair (X, τ) is called an *L-fuzzy topological space*. For every $A \in L^X, \tau(A)$ is called the *degree of openness* of the fuzzy subset A. For $a \in L$ and a map $\tau\colon L^X \to L$, we define $\tau_{[a]} = \{A \in L^X \mid \tau(A) \geq a\}$.

Definition 2. A *smooth topological space* (sts) (see [3]) is an ordered pair (X, τ), where X is a non-empty set and $\tau\colon L^X \to L'$ is a mapping satisfying the following properties :
(O1) $\tau(\underline{0}) = \tau(\underline{1}) = 1_L$,
(O2) $\tau(A_1 \cap A_2) \geq \tau(A_1) \wedge \tau(A_2)$, for every $A_1, A_2 \in L^X$,
(O3) $\tau(\bigcup_{i\in I} A_i) \geq \bigwedge_{i\in I} \tau(A_i)$, for every family $\{A_i \mid i \in I\} \subseteq L^X$.

Definition 3. A map $f\colon X \to Y$ is called *smoothly continuous with respect to the smooth topologies* τ_1 and τ_2 if for every $A \in L^Y$ we have $\tau_1(f^{-1}(A)) \geq \tau_2(A)$, where $f^{-1}(A)$ is defined by $f^{-1}(A)(x) = A(f(x)), \forall x \in X$.

Definition 4. A map $f\colon X \to Y$ is called *smoothly open with respect to the smooth topologies* τ_1 and τ_2 if for each $A \in L^X$ we have $\tau_1(A) \leq \tau_2(f(A))$.

Definition 5. A map $f\colon X \to Y$ is called a *smooth homeomorphism with respect to the smooth topologies* τ_1 and τ_2 if f is bijective and f and f^{-1} are smoothly continuous.

Definition 6. (See [3]) A map $f\colon X \to Y$ is called *L-preserving* (resp., *strictly L-preserving*) *with respect to the L-topologies* ${\tau_1}_{[a]}$ and ${\tau_2}_{[a]}$, for each $a \in M(L)$, if for every $A, B \in L^Y$ with $\tau_2(A), \tau_2(B) \geq a$, we have

$$\tau_2(A) \geq \tau_2(B) \Rightarrow \tau_1(f^{-1}(A)) \geq \tau_1(f^{-1}(B))$$
$$(\text{resp., } \tau_2(A) > \tau_2(B) \Rightarrow \tau_1(f^{-1}(A)) > \tau_1(f^{-1}(B))\ .$$

Note that if $f\colon X \to Y$ is a strictly L-preserving and continuous map with respect to the L-topologies ${\tau_1}_{[a]}$ and ${\tau_2}_{[a]}$, then for every $A \in L^Y$ with $\tau_2(A) \geq a$ the relation $f^{-1}(\overline{A}) \supseteq \overline{f^{-1}(A)}$ holds.

3. L-HOMEOMORPHISMS ON L-TOPOLOGICAL SPACES

Theorem 7. *Let $\tau\colon L^X \to L$ be a map. Then the following conditions are equivalent:*
(1) *τ is a smooth topology on X.*
(2) *$\tau_{[a]}$ is an L-topology on X, $\forall a \in M(L)$.*

Proof. The inclusion (1) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (1) We have $\underline{1} \in \tau_{[a]}$ and $\tau(\underline{1}) \geq a$, each $a \in M(L)$. Accordingly, $\tau(\underline{1}) \geq \bigvee \{a \mid a \in M(L)\} = 1$. Thus, $\tau(\underline{1}) = 1$. Similarly, $\tau(\underline{0}) = 1$.
Let $A, B \in L^X$. If $\tau(A)\wedge\tau(B) = 0$, then $\tau(A\cap B) \geq \tau(A)\wedge\tau(B)$. Otherwise, if $\tau(A)\wedge\tau(B) > 0$, then, for each $a \leq \tau(A)\wedge\tau(B)$, we have $\tau(A) \geq a, \tau(B) \geq a$, or $A, B \in \tau_{[a]}$. Consequently, $A \cap B \in \tau_{[a]}$, or $\tau(A \cap B) \geq a$. This implies further that

$$\tau(A \cap B) \geq \bigvee \{a \in M(L) \mid a \leq \tau(A) \wedge \tau(B)\} = \tau(A) \wedge \tau(B).$$

Let $\{A_i \mid i \in I\} \subseteq L^X$. Then, for $a \in M(L)$ and $a \leq \bigwedge_{i\in I} \tau(A_i)$, we have $\tau(A_i) \geq a$ and $A_i \in \tau_{[a]}$ for each $i \in I$. The assertion follows since $\bigcup_{i\in I} A_i \in \tau_{[a]}$, $\tau(\bigcup_{i\in I} A_i) \geq a$, and $\tau(\bigcup_{i\in I} A_i) \geq \bigvee \left\{a \in M(L) \mid a \leq \bigwedge_{i\in I} \tau(A_i)\right\}$. $\square$

THEOREM 8. *Let (X,τ_1) and (Y,τ_2) be smooth topological spaces and $f\colon X \to Y$ be a map. Then the following conditions are equivalent:*
(1) $f\colon (X,\tau_1) \to (Y,\tau_2)$ *is a smoothly continuous map.*
(2) $f(X,\tau_{1_{[a]}}) \to (Y,\tau_{2_{[a]}})$ *is an L-continuous map, $\forall a \in M(L)$.*

Proof. The inclusion (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) For all $A \in L^Y$, $a \in M(L)$, such that $a \le \tau_2(A)$ we have $(A) \in \tau_{2_{[a]}}$ and $f^{-1}(A) \in \tau_{1_{[a]}}$, by the continuity of $f\colon (X,\tau_{1_{[a]}}) \to (Y,\tau_{2_{[a]}})$. Accordingly, $\tau_1(f^{-1}(A)) \ge a$ for each $a \in M(L) \bigcap M(\tau_2(A))$, where $M(\tau_2(A)) = \{a \in M(L) \mid a \le \tau_2(A)\}$. It follows that $\tau_1(f^{-1}(A)) \ge \bigvee M(\tau_2(A)) = \tau_2(A)$. Thus $f : (X,\tau_1) \to (Y,\tau_2)$ is a smoothly continuous map. □

THEOREM 9. *Let (X,τ_1) and (Y,τ_2) be smooth topological spaces, and $f\colon X \to Y$ be a map. Then the following conditions are equivalent:*
(1) $f\colon (X,\tau_1) \to (Y,\tau_2)$ *is a smoothly open map.*
(2) $f : (X,\tau_{1_{[a]}}) \to (Y,\tau_{2_{[a]}})$ *is L-open for each $a \in M(L)$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from the definition of L-open maps.

(2) $\Rightarrow$ (1) Let $A \in L^X$. If $\tau_1(A) = 0$, then clearly, $\tau_1(A) \le \tau_2(f^{-1}(A))$. If $\tau_1(A) > 0$, then since $\tau_1(A) = \bigvee M(\tau_1(A))$, we have $\tau_2(f^{-1}(A)) \ge a$ for each $a \in M(\tau_1(A))$. Hence $\tau_2(f^{-1}(A)) \ge \bigvee \{a \mid a \in M(\tau_1(A))\} = \tau_1(A)$. □

THEOREM 10. *Let (X,τ_1) and (X,τ_2) be two smooth topological spaces, and let $f\colon X \to Y$ be a bijective map. The following conditions are equivalent:*
(1) $f\colon (X,\tau_1) \to (Y,\tau_2)$ *is a smooth homeomorphism.*
(2) $f\colon (X,\tau_{1_{[a]}}) \to (Y,\tau_{2_{[a]}})$ *is a homeomorphism for each $a \in M(L)$.*

Proof. The proof is straightforward. □

4. SEPARATION AXIOMS

DEFINITION 11. Let $a \in M(L)$. The L-topological space $(X,\tau_{[a]})$ is called:

(a) a *strong $ST_\circ$ space* if for each $x,y \in X$, $x \ne y$, there exists $A \in L^X$ with $\tau(A) \ge a$ such that one of the conditions ($x \in \mathrm{supp}A$, $y \notin \mathrm{supp}\overline{A}$ and $\tau(A) \ge A(x)$) or ($y \in \mathrm{supp}A$, $x \notin \mathrm{supp}\overline{A}$ and $\tau(A) \ge A(y)$) holds,

(b) an *$ST_\circ$ space* if for each $x,y \in X$, $x \ne y$, there exists $A \in L^X$ with $\tau(A) \ge a$ such that ($x \in \mathrm{supp}A$, $y \notin \mathrm{supp}A$ and $\tau(A) \ge A(x)$) or ($y \in \mathrm{supp}A$, $x \notin \mathrm{supp}A$ and $\tau(A) \ge A(y)$),

(c) a *weak $ST_\circ$ space* if for each $x,y \in X$, $x \ne y$, there exists $A \in L^X$ with $\tau(A) \ge a$ such that ($x \in \mathrm{supp}A$, $y \notin \mathrm{supp}A^\circ$ and $\tau(A) \ge A(x)$) or ($y \in \mathrm{supp}A$, $x \notin \mathrm{supp}A^\circ$ and $\tau(A) \ge A(y)$).

DEFINITION 12. Let $a \in M(L)$. The L-topological space $(X,\tau_{[a]})$ is called:

(a) a *strong ST_1* (resp., *strong ST_1'*) *space* if for each $x,y \in X$, $x \ne y$, there exist $A,B \in L^X$ with $\tau(A),\tau(B) \ge a$ such that $[x \in \mathrm{supp}A \setminus \mathrm{supp}\overline{B}$

(resp., $x \in \text{supp}(A \setminus \overline{B})$ and $\tau(A) \geq A(x))$] or [$y \in \text{supp}B \setminus \text{supp}\overline{A}$ (resp., $y \in \text{supp}(B \setminus \overline{A})$) and $\tau(B) \geq B(y)$],

(b) an ST_1 (resp., ST_1') *space* if for each $x, y \in X$, $x \neq y$, there exist $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$ such that [$x \in \text{supp}A \setminus \text{supp}B$ (resp., $x \in \text{supp}(A \setminus B)$) and $\tau(A) \geq A(x)$] or [$y \in \text{supp}B \setminus \text{supp}A$ (resp., $y \in \text{supp}(B \setminus A)$) and $\tau(B) \geq B(y)$],

(c) a *weak* ST_1 (resp., *weak* ST_1') *space* if for each $x, y \in X$, $x \neq y$, there exist $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$ such that [$x \in \text{supp}A \setminus \text{supp}B^\circ$) (resp., $x \in \text{supp}(A \setminus B^\circ)$) and $\tau(A) \geq A(x)$] or [$y \in \text{supp}B \setminus \text{supp}A^\circ$ (resp., $y \in \text{supp}(B \setminus A^\circ)$) and $\tau(B) \geq B(y)$].

Definition 13. Let $a \in M(L)$. The L-topology $(X, \tau_{[a]})$ is called:

(a) a *strong* ST_2 (resp., *strong* ST_2') *space* if for each $x, y \in X$, $x \neq y$, there exist $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$ such that $x \in \text{supp}A$ (resp., $x \in \text{supp}(A \setminus \overline{B})$), $\tau(A) \geq A(x)$, $y \in \text{supp}B$ (resp., $y \in \text{supp}(B \setminus \overline{A})$), $\tau(B) \geq B(y)$, $\overline{A} \cap \overline{B} = \underline{0}$ (resp., $\overline{A} \subseteq (\overline{B})^c$),

(b) an ST_2 (resp., ST_2') *space* if for each $x, y \in X$, $x \neq y$, there exist $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$ such that $x \in \text{supp}A$ (resp., $x \in \text{supp}(A\ B)$), $\tau(A) \geq A(x)$, $y \in \text{supp}B$ (resp., $y \in \text{supp}(B \setminus A)$), $\tau(B) \geq B(y)$, $A \cap B = \underline{0}$ (resp., $A \subseteq B^c$),

(c) a *weak* ST_2 (resp., *weak* ST_2') *space* if for each $x, y \in X$, $x \neq y$, there exist $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$ such that $x \in \text{supp}A \setminus \text{supp}B^\circ$ (resp., $x \in \text{supp}(A \setminus B^\circ)$), $\tau(A) \geq A(x)$, $y \in \text{supp}B \setminus \text{supp}A^\circ$ (resp., $y \in \text{supp}(B \setminus A^\circ)$), $\tau(B) \geq B(y)$, $A^\circ \cap B^\circ = \underline{0}$ (resp., $A^\circ \subseteq (B^\circ)^c$).

Lemma 14. *Let $(X, \tau_{[a]})$ be an L-topological space for each $a \in M(L)$, let $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$. Then the following properties hold:*
(i) $\text{supp}A \setminus \text{supp}B \subseteq \text{supp}(A \setminus B)$,
(ii) $\text{supp}A \setminus \text{supp}\overline{B} \subseteq \text{supp}A \setminus \text{supp}B \subseteq \text{supp}A \setminus \text{supp}B^\circ$,
(iii) $A \setminus \overline{B} \subseteq A \setminus B \subseteq A \setminus B^\circ$,
(iv) $A \cap B = \underline{0} \Rightarrow A \subseteq B^c$.

Proof. (i) Let $x \in \text{supp}A \setminus \text{supp}B$. Then $A(x) > 0$ and $B(x) = 0$. Hence $\min\{A(x), 1 - B(x)\} = A(x) > 0$, i.e., $x \in \text{supp}(A \setminus B)$. Note that the reverse inclusion in (i) is not true as it can be seen from the following counterexample: Let $X = \{x_1, x_2\}$, $A(x_1) = 0.5$, $B(x_1) = 0.3$. Then we have $x_1 \in \text{supp}(A \setminus B)$ and $x_1 \notin \text{supp}A \setminus \text{supp}B$. The properties (ii) and (iii) follow from $B^\circ \subseteq B \subseteq \overline{B}$. For (iv) we refer to [4]. □

Corollary 15. *Let $(X, \tau_{[a]})$ be an L-topological space for each $a \in M(L)$. Then all relations between the separation notions listed in* Figure 1 *hold.*

Proof. All implications in Figure 1 are straightforward consequences of Lemma 14. As an example we prove that strong ST_2 implies strong ST_2'. Suppose that the space $(X, \tau_{[a]})$ is strong ST_2 and let $x, y \in X$, $x \neq y$. Since

$$\begin{array}{ccccc}
\text{strong } ST_1 \Rightarrow ST_1 \Rightarrow \text{weak } ST_1 & & \text{strong } ST_\circ \Rightarrow ST_\circ \Rightarrow \text{weak } ST_\circ \\
\Downarrow \quad\quad \Downarrow \quad\quad \Downarrow & & \Uparrow \quad\quad \Uparrow \quad\quad \Uparrow \\
\text{strong } ST_1' \Rightarrow ST_1' \Rightarrow \text{weak } ST_1' & & \text{strong } ST_1 \Rightarrow ST_1 \Rightarrow \text{weak } ST_1 \\
\Uparrow \quad\quad \Uparrow \quad\quad \Uparrow & & \Uparrow \quad\quad \Uparrow \quad\quad \Uparrow \\
\text{strong } ST_2 \Rightarrow ST_2 \Rightarrow \text{weak } ST_2 & & \text{strong } ST_2 \Rightarrow ST_2 \Rightarrow \text{weak } ST_2 \\
\Downarrow \quad\quad \Downarrow \quad\quad \Downarrow & & \\
\text{strong} ST_2' \Rightarrow ST_2' \Rightarrow \text{weak} ST_2' & &
\end{array}$$

$$\begin{array}{c}
\text{strong } ST_1' \Rightarrow ST_1' \Rightarrow \text{weak } ST_1' \\
\Uparrow \quad\quad \Uparrow \quad\quad \Uparrow \\
\text{strong } ST_2' \Rightarrow ST_2' \Rightarrow \text{weak } ST_2'
\end{array}$$

Fig. 1. Relationship between the different separation notions.

$(X, \tau_{[a]})$ is strong ST_2, it follows that there exist $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$ such that $x \in \text{supp}A$, $\tau(A) \geq A(x)$, $y \in \text{supp}B$, $\tau(B) \geq B(y)$ and $\overline{A} \cap \overline{B} = \underline{0}$. From the assertions (i) and (iv) of Lemma 14 it follows that $x \in \text{supp}(A \backslash \overline{B})$, $y \in \text{supp}(B \setminus \overline{A})$ and $\overline{A} \subseteq (\overline{B})^c$, hence $(X, \tau_{[a]})$ is strong ST_2'. □

Corollary 16. *The ST_i $(i = 0, 1, 2)$ (resp., $ST_i'(i = 1, 2)$) property is a topological property, when $f\colon (X, \tau_1) \to (Y, \tau_2)$ is a smooth homeomorphism or $f\colon (X, \tau_{1_{[a]}}) \to (Y, \tau_{1_{[a]}})$ is a homeomorphism for each $a \in M(L)$.*

Proof. As an example we give the proof for ST_2', when $(X, \tau_{1_{[a]}})$, $(Y, \tau_{2_{[a]}})$ are two L-topological spaces for for all $a \in M(L)$. Let $f\colon X \to Y$ be a homeomorphism from the ST_2' space $(X, \tau_{1_{[a]}})$ onto a space $(Y, \tau_{2_{[a]}})$, and let $x, y \in Y$ with $x \neq y$. We have $f^{-1}(x) \neq f^{-1}(y)$, because f is bijective. Since $(X, \tau_{1_{[a]}})$ is ST_2', there exist $A, B \in L^X$ with $\tau(A) \geq a$ and $\tau(B) \geq a$ such that $f^{-1}(x) \in \text{supp}(A \setminus B)$, $\tau_1(A) \geq A(f^{-1}(x))$, $f^{-1}(y) \in \text{supp}(B \setminus A)$, $\tau_1(A) \geq B(f^{-1}(y))$ and $A \subseteq B^c$. Since f is L-open it follows that for each $A \in \tau_{1_{[a]}}$ we have $f(A) \in \tau_{2_{[a]}}$, i.e., $\tau_1(A) \geq a$ yields $\tau_2(f(A)) \geq a$. Consequently, $\tau_1(A) \leq \tau_2(f(A))$. Similarly, $\tau_1(B) \leq \tau_2(f(B))$. As f is bijective we have $A(f^{-1}(x)) = f(A)(x)$, $B(f^{-1}(y)) = f(B)(y)$, $(A \setminus B)(f^{-1}(x)) = f(A \setminus B)(x) = (f(A) \cap f(B^c))(x) = (f(A) \cap (f(B^c))(x) = (f(A) \cap (f(B))^c)(x) = (f(A) \setminus (f(B)))(x)$ and similarly, $(B \setminus A)(f^{-1}(y)) = (f(B) \setminus f(A))(y)$. Moreover $f(A) \subseteq f(B^c) = (f(B))^c$. Hence $(Y, \tau_{2_{[a]}})$ is ST_2'. □

Corollary 17. *Let $f\colon (X, \tau_{1_{[a]}}) \to (Y, \tau_{2_{[a]}})$ be an injective, L-continuous map for each $a \in M(L)$. If $(Y, \tau_{2_{[a]}})$ is ST_i $(i = 0, 1, 2)$ (resp., ST_i' $(i = 1, 2)$), then so is $(X, \tau_{1_{[a]}})$.*

Proof. As an example we give the proof for ST_2'. Pick $x, y \in X$ with $x \neq y$. We have $f(x) \neq f(y)$, because f is injective. Since $(Y, \tau_{2_{[a]}})$ is ST_2', there exist

$A, B \in L^Y$ with $\tau_2(A), \tau_2(B) \geq a$ such that $f(x) \in \text{supp}(A \setminus B)$, $\tau_2(A) \geq A(f(x))$, $f(y) \in \text{supp}(B \setminus A)$, $\tau_2(B) \geq B(f(y))$ and $A \subseteq B^c$. Since f is injective and L-continuous, it follows that $f^{-1}(A) \in \tau_{1_{[a]}}$, for $A \in \tau_{2_{[a]}}$. Thus $\tau_2(A) \geq a$ yields $\tau_1(f^{-1}(A)) \geq a$. It follows that $\tau_1(f^{-1}(A)) \geq \tau_2(A)$. Since $\tau_2(A) \geq A(f(x))$, we obtain that $\tau_1(f^{-1}(A)) \geq \tau_2(A) \geq A(f(x)) = f^{-1}(A)(x)$, and, similarly, $\tau_1(f^{-1}(B)) \geq \tau_2(B) \geq B(f(y)) = f^{-1}(B)(y)$. Also,

$$(A \setminus B)f(x) = f^{-1}(A \setminus B)(x) = [f^{-1}(A) \setminus (f^{-1}(B))](x) > 0,$$

i.e., $x \in \text{supp}(f^{-1}(A) \setminus (f^{-1}(B)))$. Similarly, $y \in \text{supp}(f^{-1}(B) \setminus (f^{-1}(A)))$ and $f^{-1}(A) \subseteq f^{-1}(B^c) = (f^{-1}(B))^c$. Hence $(X, \tau_{1_{[a]}})$ is ST_2'. □

COROLLARY 18. *Let $f\colon X \to Y$ be a strict L-preserving, injective and L-continuous map with respect to the L-topologies $\tau_{1_{[a]}}$ and $\tau_{2_{[a]}}$ for each $a \in M(L)$. If $(Y, \tau_{2_{[a]}})$ is strong ST_i $(i = 0, 1, 2)$ (resp., strong ST_i' $(i = 1, 2)$), then so is $(X, \tau_{1_{[a]}})$.*

Proof. As an example we give the proof for strong ST_2. Pick $x, y \in X$ with $x \neq y$. We have $f(x) \neq f(y)$, because f is injective. Since $(Y, \tau_{2_{[a]}})$ is strong ST_2, there exist $A, B \in L^Y$ with $\tau(A), \tau(B) \geq a$ such that $f(x) \in \text{supp}A$, $\tau_2(A) \geq A(f(x))$, $f(y) \in \text{supp}(B)$, $\tau_2(B) \geq B(f(y))$ and $\overline{A} \cap \overline{B} = \underline{0}$. Since f is injective and L-continuous, from $A \in \tau_{2_{[a]}}$ we get $f^{-1}(A) \in \tau_{1_{[a]}}$, i.e., $\tau_2(A) \geq a$ yields $\tau_1(f^{-1}(A)) \geq a$. Thus $\tau_1(f^{-1}(A) \geq \tau_2(A) \geq A(f(x)) = f^{-1}(A)(x)$. Similarly, $\tau_1(f^{-1}(B)) \geq \tau_2(B) \geq B(f(y)) = f^{-1}(B)(y)$. From $f(x) \in \text{supp}(A)$ we have $x \in \text{supp}(f^{-1}(A))$, and $f(y) \in \text{supp}(B)$ implies $y \in \text{supp}(f^{-1}(B)$. Since f is injective and $\overline{A} \cap \overline{B} = \underline{0}$, we get $f^{-1}(\overline{A}) \cap f^{-1}(\overline{B}) = f^{-1}(\underline{0}) = \underline{0}$. Since f is L-preserving and L-continuous, $f^{-1}(\overline{A}) \cap f^{-1}(\overline{B}) \supseteq \overline{f^{-1}(A)} \cap \overline{f^{-1}(B)}$. So $\overline{f^{-1}(A)} \cap \overline{f^{-1}(B)} = \underline{0}$. Hence $(X, \tau_{1_{[a]}})$ is strong ST_2. □

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